Research Article

Transmuted Generalized Half-Logistic Distribution with Applications

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Abstract

Researchers have been focusing on the family of distributions that includes the baseline distribution for some time now. In this paper, we use the transmutation method proposed by Shaw and Buckley (2007), to introduce an improved version of the Generalized Half-Logistic distribution Moments, moment generating function, quantile function, reliability analysis, order statistics, and Renyi entropy are just some of the distributional properties discussed here.The standard maximum likelihood method is employed to determine the distribution's parameters.procedure.In the end, real-world examples demonstrate the distribution's value sets.

Keywords: Transmutation technique, Generalized Half Logistic Distribution, Maximum Likelihood Estimation, Simulation, Entropy, Order Statistics.

1.Introduction

In probability theory, a number of different univariate standard distributions have been used to analyze data from many different fields, including engineering, biomedicine, actuarial science, finance, and economics. It has been found through a number of studies that information gathered in fields like economics, ecology, and a number of others does not adhere to these norms.There is a clear need to improve these distributions' usefulness by extending and generalizing them so that they are suitable for such data. Statisticians have put in a lot of effort in this direction, developing and introducing new methods and techniques for building new models on top of existing distributions .As expected, the adjusted models allowed for more variation than the normative distributions. By applying the quadratic rank transmutation map to the original distribution proposed by Shaw and Buckley (2007), we hope to further generalize the Generalized Half-Logistic. Quadratic transmutation can be found if we assume that G(x) is the cdf of the standard distribution, let

$$F(x) = \int_0^{G(x)} q(u)du$$

Where $$q(u) = (1+\lambda) - 2\lambda u$$

$$F(t) = (1+\lambda)G(t) - \lambda G^2(t);\ |\lambda| \le 1 \tag{1.1}$$

The pdf corresponding (1.1) is

$$f(t) = g(t)[1+\lambda-2\lambda G(x)];\ t>0\ ,|\lambda| \le 1 \tag{1.2}$$

The widespread transformations of probability models have been the focus of many academic investigations, for example, in(2011) Aryal and Tsokos are studied various structural properties of transmuted Weibull distribution. In (2013) Faton Mervoci introduced transmuted Rayleigh distribution. In (2013) Ashour et al. presented the transmuted exponential lomax distribution. In (2015) Ahmad et al. studied transmuted Kumaraswamy and shows its performance through real data sets. . In (2020) Aijaz et al. formulated the transmuted inverse Lindley distribution and through examples they performed the versatility of the model. In the present research, we investigated the Transformed Generalized Half-Logistic distribution and analyzed its many structural features.

in this study we used some functions like

The negative binomial theorem :

$$(1-\mathrm{a})^{-\mathrm{r}} = \sum_{\mathrm{t}=0}^{\infty} \binom{\mathrm{t}+\mathrm{r}-1}{\mathrm{t}} (-\mathrm{a})^{\mathrm{t}} \tag{1.3}$$

The gamma function:

$$\Gamma(\varphi) = \int_0^{\infty} \mathrm{t}^{\varphi-1}\, \mathrm{e}^{-\mathrm{t}} \mathrm{dt}, \quad \varphi > 0, \qquad 0 < \varphi < \infty. \tag{1.4}$$

2. Transmuted Generalized Half-Logistic Distribution(TGHLD)

Let $T\sim$half logistic distribution, the cumulative distribution function (cdf) of T is given by

$$G(t) = \left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}, \quad t,\alpha,\theta > 0 \tag{2.1}$$

And the density function (pdf) is

$$g(t) = \frac{2\theta\alpha e^{-\theta t}\left(1-e^{-\theta t}\right)^{\alpha-1}}{(1+e^{-\theta t})^{\alpha+1}} \quad ,t,\alpha,\theta > 0 \tag{2.2}$$

by substituting equation (2.1) in equation (1.1) we obtain the (cdf) of the transmuted generalized half logistic distribution

$$F(t;\theta,\alpha,\lambda) = \left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\left[1+\lambda-\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right], t>0,\alpha,\theta>0,|\lambda|\le 1 \tag{2.3}$$

The (pdf) of the transmuted generalized half logistic distribution is given by

$$f(t;\theta,\alpha,\lambda) = \frac{2\theta\alpha e^{-\theta t}\left(1-e^{-\theta t}\right)^{\alpha-1}}{(1+e^{-\theta t})^{\alpha+1}}\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right], t>0,\alpha,\theta>0,|\lambda|\le 1 \tag{2.4}$$

The survival function of transmuted generalized half-logistic distribution is defined by using (2.3)

$$S(t;\theta,\alpha,\lambda) = 1 - F(t;\theta,\alpha,\lambda)$$

$$= 1 - \left[\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\left[1+\lambda-\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right]\right], \tag{2.5}$$

$$t>0,\alpha,\theta>0,|\lambda|\le 1$$

The plots of the c.d.f, p.d.f and suf of the TGHLD for different parameters are given by

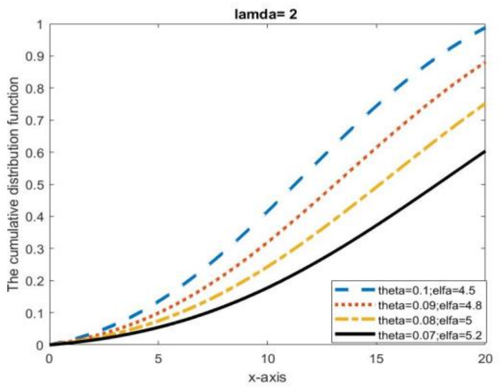

Figure (4.1): the cdf of TGHLD with the parameters $\lambda = 2;\ \theta = (0.1, 0.09, 0.08, 0.07)\ and\ \alpha = (4.5,4.8,5,5.2)$.

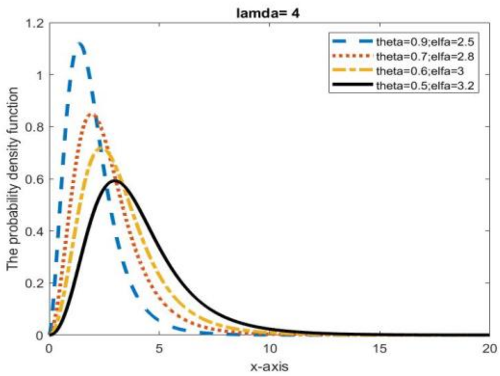

Figure (4.2): the pdf of TGHLD with the parameters $\lambda = 4;\ \theta = (0.9, 0.7, 0.6, 0.5,)\ and\ \alpha = (2.5,2.8,3,3.2)$.

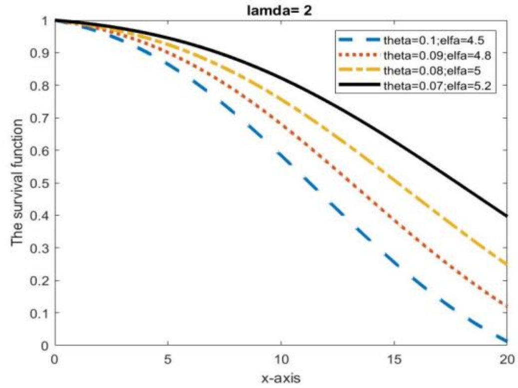

Figure (4.3): the sf of TGHLD with the parameters $\lambda = 2;\ \theta = (0.1\ ,0.09\ ,0.08\ ,0.07)\ and\ \alpha = (4.5,4.8,5,5.2).$

The cumulative hazard functi

$$H(t) = -ln\left[1 - \left[\left(\frac{1}{1} \right. \right. \right. \quad (3.1)$$

The reverse hazard rate function of transmuted generalized half-logistic distribution is defined by using (2.4) and (2.3) respectively

$$r(t;\theta,\alpha,\lambda) = \frac{f(t;\theta,\lambda)}{F(t;\theta,\lambda)} \quad (3.2)$$

$$= \frac{2\theta\alpha e^{-\theta t}\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right]}{(1+e^{-\theta t})(1-e^{-\theta t})\left[1+\lambda-\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right]} \quad (3.3)$$

4. Properties of OEHL Distribution

4.1. Moments: In this section, we discuss the moment for TGHLD. Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution .

Theorem 4.1.1. If a random variable T from TGHLD , then the r^{th} moments of random variable T, is given by:

$$\mu_r' = E(T^r) = 2\theta\alpha(1+\lambda)\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}(-1)^i\binom{i-\alpha}{i}\binom{n+\alpha}{n}\Gamma(r+1)\varphi^{r+1}$$
$$-4\theta\alpha\lambda\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^j\binom{j-2\alpha}{j}\binom{m+2\alpha}{m}\Gamma(r+1)\vartheta^{r+1}\ ;\ t>0\ ,|\lambda|\le 1,\varphi,\vartheta<0, \quad (4.1)$$
$$r = 1,2,3,\ldots$$

Proof: By the probability density function of T in (2.4), so the r^{th}moment distribution of T is given by

$$\mu_r' = E(T^r) = \int_0^\infty t^r f(t;\theta,\alpha,\lambda)dt,\ \ ,t>0,\alpha,\theta>0,|\lambda|\le 1 \quad (4.2)$$
$$r = 1,2,3,\ldots$$

By Substitute Equation (2.4) into Equation (4.2), then equation (4.2) is of the form

$$\mu_r' = E(T) = \int_0^\infty t^r \frac{2\theta\alpha e^{-\theta t}\left(1-e^{-\theta t}\right)^{\alpha-1}}{(1+e^{-\theta t})^{\alpha+1}} \left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right] dt, \quad t>0, \alpha, \theta>0, |\lambda|\le 1, \quad r=1,2,3,\dots \tag{4.3}$$

$$= 2\theta\alpha(1+\lambda)\int_0^\infty t^r e^{-\theta t}\left(1-e^{-\theta t}\right)^{\alpha-1}\left(1+e^{-\theta t}\right)^{-(\alpha+1)} dt - 4\theta\alpha\lambda\int_0^\infty t^r e^{-\theta t}\left(1-e^{-\theta t}\right)^{2\alpha-1}\left(1+e^{-\theta t}\right)^{-(2\alpha+1)} dt; \quad t>0, \theta, \alpha>0, |\lambda|\le 1, \quad r=1,2,3,\dots \tag{4.4}$$

by using the negative binomial theorem in (1.3)

$$\left(1-e^{-\theta t}\right)^{-(1-\alpha)} = \sum_{i=0}^{\infty}\binom{i-\alpha}{i}(-1)^i e^{-\theta i t} \tag{4.5}$$

Similarly, by using the negative binomial theorem in (1.3)

$$\left(1-e^{-\theta t}\right)^{-(1-2\alpha)} = \sum_{j=0}^{\infty}\binom{j-2\alpha}{j}(-1)^j e^{-\theta j t} \tag{4.6}$$

and, by using the negative binomial theorem in (1.3)

$$\left(1+e^{-\theta t}\right)^{-(\alpha+1)} = \sum_{n=0}^{\infty}\binom{n+\alpha}{n} e^{-\theta n t} \tag{4.7}$$

Similarly, by using the negative binomial theorem in (1.3)

$$\left(1+e^{-\theta t}\right)^{-(2\alpha+1)} = \sum_{m=0}^{\infty}\binom{m+2\alpha}{m} e^{-\theta m t} \tag{4.8}$$

By substituting Equations (4.5), (4.6), (4.7) and (4.8) in Equation (4.4) we have

$$\mu_r' = E(T^r) = 2\theta\alpha(1+\lambda)\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}(-1)^i\binom{i-\alpha}{i}\binom{n+\alpha}{n}\int_0^\infty t^r e^{-\theta(1+i+n)t} dt - 4\theta\alpha\lambda\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^j\binom{j-2\alpha}{j}\binom{m+2\alpha}{m}\int_0^\infty t^r e^{-\theta(1+j+m)t} dt; \; t>0, |\lambda|\le 1, \quad r=1,2,3,\dots \tag{4.9}$$

Let $\varphi = [\theta(1+i+n)]^{-1}, \; \varphi < 0$

Then by using the gamma distribution in (1.4), then r^{th} moment becomes

$$\mu_r' = E(T^r) = 2\theta\alpha(1+\lambda)\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}(-1)^i\binom{i-\alpha}{i}\binom{n+\alpha}{n}\Gamma(r+1)\varphi^{r+1} - 4\theta\alpha\lambda\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^j\binom{j-2\alpha}{j}\binom{m+2\alpha}{m}\Gamma(r+1)\vartheta^{r+1}; \; t>0, |\lambda|\le 1, \varphi, \; \vartheta<0, r=1,2,3,\dots$$

4.2 Order Statistics

Let $T_{1:h} \le T_{2:h} \le \cdots \le T_{h:h}$be random samples , we can derive the ordered sample. The cdf of $T_{h:h}$, the n^{th} order statistics, obtaining by

$$F_{h:h}(t) = [F(\text{t})]^h = \left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha h}\left[1+\lambda-\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right]^h \tag{4.10}$$

the F(T) is cdf of the TGHLD.

The h^{th} order statistics of the pdf for the TGHLD, of random variable $\text{t}_{\text{h:h}}$ given by used the equations (2.3) and (2.4)

$$f_{h:h}(t) = h[F(t)]^{h-1}f(t) \tag{4.11}$$

$$= \frac{2\theta\alpha h e^{-\theta t}\left(1-e^{-\theta t}\right)^{\alpha h-1}}{(1+e^{-\theta t})^{\alpha h+1}} \left[1+\lambda-\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right]^{h-1} \left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right];\ t>0,\alpha,\theta>0\,,|\lambda|\le 1 \quad (4.12)$$

Where $f(T)$ is the pdf of *TGHLD*. The cumulative distribution function of the first order statistics $t_{1:\mathrm{h}}$, is

$$F_{1:h}(t) = [1-F(t)]^h = \left[1-\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\left[1+\lambda-\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right]\right]^h \quad (4.13)$$

and the pdf of t, is given by

$$f_{1:h}(t) = -h[1-F(t)]^{h-1}f(t) \quad (4.14)$$

$$= -h\frac{2\theta\alpha e^{-\theta t}\left(1-e^{-\theta t}\right)^{\alpha-1}}{(1+e^{-\theta t})^{\alpha+1}}\left[1-\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\left[1+\lambda-\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right]\right]^{h-1} \left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta t}}{1+e^{-\theta t}}\right)^{\alpha}\right];\ t>0,\theta,\alpha>0\,,|\lambda|\le 1 \quad (4.15)$$

4.3 Quantile and Median

The quantile of the TGHLD are determined.

$$\mathrm{F}(t_q) = \mathrm{p}(\mathrm{t}_q \le \mathrm{q}) = \mathrm{q}, \ (0 < \mathrm{q} < 1). \quad (4.16)$$

From Equation (2.3), the following Equation can be obtained

$$\mathrm{F}(t_q) = \mathrm{p}(\mathrm{t}_q \le \mathrm{q}) = \left(\frac{1-e^{-\theta \mathrm{t}_q}}{1+e^{-\theta \mathrm{t}_q}}\right)^{\alpha}\left[1+\lambda-\lambda\left(\frac{1-e^{-\theta \mathrm{t}_q}}{1+e^{-\theta \mathrm{t}_q}}\right)^{\alpha}\right] = q \Rightarrow$$

$$\lambda\left(\left(\frac{1-e^{-\theta \mathrm{t}_q}}{1+e^{-\theta \mathrm{t}_q}}\right)^{\alpha}\right)^2 - (1+\lambda)\left(\frac{1-e^{-\theta \mathrm{t}_q}}{1+e^{-\theta \mathrm{t}_q}}\right)^{\alpha} + q = 0$$

$$\frac{1-e^{-\theta \mathrm{t}_q}}{1+e^{-\theta \mathrm{t}_q}} = \frac{\left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha}}{(2\lambda)^{1/\alpha}}$$

$$(2\lambda)^{1/\alpha} - (2\lambda)^{1/\alpha}e^{-\theta \mathrm{t}_q} = \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha} + \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha}e^{-\theta \mathrm{t}_q}$$

$$\left[\left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha} + (2\lambda)^{1/\alpha}\right]e^{-\theta \mathrm{t}_q} = (2\lambda)^{1/\alpha} - \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha} \quad (4.17)$$

$$e^{-\theta \mathrm{t}_q} = \frac{(2\lambda)^{1/\alpha} - \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha}}{(2\lambda)^{1/\alpha} + \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha}}$$

$$-\theta \mathrm{t}_q = ln\left[\frac{(2\lambda)^{1/\alpha} - \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha}}{(2\lambda)^{1/\alpha} + \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha}}\right]$$

$$\mathrm{t}_q = \frac{-1}{\theta}ln\left[\frac{(2\lambda)^{1/\alpha} - \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha}}{(2\lambda)^{1/\alpha} + \left(1+\lambda+\sqrt{(1+\lambda)^2-4\lambda q}\right)^{1/\alpha}}\right]$$

Then, by substituting $q = 0.5$ into Equation (4.17), obtain the median of the random variable $T from\ TGHLD,$ and then solve this Equation numerically.

4.4 Maximum Likelihood Estimates

This section, investigates the estimation of the parameters model of the TGHLD by the MLE. If $t_1, t_2, \dots, t_n$ be a random sample from TGHLD, then the likelihood function of those distribution is given by

$$L(\theta,\alpha,\lambda;t_1,t_2,\ldots,t_n)=\prod_{i=1}^{n} f(t_i,\theta,\alpha,\lambda),\quad t>0,\theta,\alpha>0,|\lambda|\le 1 \tag{4.48}$$

by substituting Equation (4.7) into Equation (4.48)

$$L=\prod_{i=1}^{n}\frac{2\theta\alpha e^{-\theta t_i}\left(1-e^{-\theta t_i}\right)^{\alpha-1}}{(1+e^{-\theta t_i})^{\alpha+1}}\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta}t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right]. \tag{4.49}$$

Then the log-likelihood function in (4.49) will have the following

$$\ell(\varphi)=nlog2+nlog\theta+nlog\alpha-\theta\sum_{i=1}^{n}t_i+(\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\theta t_i}\right)-(\alpha+1)\sum_{i=1}^{n}\log\left(1+e^{-\theta t_i}\right)+\sum_{i=1}^{n}\log\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta}t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right] \tag{4.50}$$

The MLEs for the parameters $\theta,\alpha\, and\, \lambda$ can be obtaind by maximizing the log-likelihood function or by

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1-e^{-\theta t_i}\right)-\sum_{i=1}^{n}\log\left(1+e^{-\theta t_i}\right)-2\lambda\sum_{i=1}^{n}\frac{\left(\frac{1-e^{-\theta t_i}}{1+e^{-\theta t_i}}\right)^{\alpha}log\left(\frac{1-e^{-\theta t_i}}{1+e^{-\theta t_i}}\right)}{\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta t_i}}{1+e^{-\theta t_i}}\right)^{\alpha}\right]} \tag{4.52}$$

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}t_i+(\alpha-1)\sum_{i=1}^{n}\frac{t_ie^{-\theta t_i}}{\left(1-e^{-\theta t_i}\right)}+(\alpha+1)\sum_{i=1}^{n}\frac{t_ie^{-\theta t_i}}{\left(1+e^{-\theta t_i}\right)}+4\alpha\lambda\sum_{i=1}^{n}\frac{t_ie^{-\theta t_i}\left(1-e^{-\theta t_i}\right)^{\alpha-1}}{\left\{1+\lambda-2\lambda\left(\frac{1-e^{-\theta t_i}}{1+e^{-\theta t_i}}\right)^{\alpha}\right\}\left(1+e^{-\theta t_i}\right)^{\alpha+1}} \tag{4.51}$$

solving the following differential equations with respect to $\theta,\alpha\, and\, \lambda$

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{n}\frac{1-2\left(\frac{1-e^{-\theta}t_i}{1+e^{-\theta t_i}}\right)^{\alpha}}{1+\lambda-2\lambda\left(\frac{1-e^{-\theta}t_i}{1+e^{-\theta t_i}}\right)^{\alpha}} \tag{4.53}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\theta^2}=&\frac{-n}{\theta^2}-(\alpha-1)\sum_{i=1}^{n}\frac{t_i^2e^{-\theta t_i}}{(1-e^{-\theta t_i})^2}-(\alpha+1)\sum_{i=1}^{n}\frac{t_i^2e^{-\theta t_i}}{(1+e^{-\theta t_i})^2}\\&+4\alpha\lambda\sum_{i=1}^{n}\frac{t_i^2e^{-\theta t_i}\left(1-e^{-\theta t_i}\right)^{\alpha-1}}{\left\{1+\lambda-2\lambda\left(\frac{1-e^{-\theta t_i}}{1+e^{-\theta t_i}}\right)^{\alpha}\right\}(1+e^{-\theta t_i})^{\alpha+1}}\\&-4\alpha(\alpha-1)\lambda\sum_{i=1}^{n}\frac{t_i^2e^{-2\theta t_i}\left(1-e^{-\theta t_i}\right)^{\alpha-2}}{\left\{1+\lambda-2\lambda\left(\frac{1-e^{-\theta t_i}}{1+e^{-\theta t_i}}\right)^{\alpha}\right\}(1+e^{-\theta t_i})^{\alpha+1}}\\&-16\alpha^2\lambda^2\sum_{i=1}^{n}\frac{t_i^2e^{-2\theta t_i}\left(1-e^{-\theta t_i}\right)^{2\alpha-2}}{\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta}t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right]^2(1+e^{-\theta t_i})^{2\alpha+2}}\\&-4\alpha(\alpha+1)\lambda\sum_{i=1}^{n}\frac{t_i^2e^{-2\theta t_i}\left(1-e^{-\theta t_i}\right)^{\alpha-1}}{\left\{1+\lambda-2\lambda\left(\frac{1-e^{-\theta t_i}}{1+e^{-\theta t_i}}\right)^{\alpha}\right\}(1+e^{-\theta t_i})^{\alpha+2}}\end{aligned} \tag{4.54}$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2} - 2\lambda \sum_{i=1}^{n} \frac{\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha} \left(log\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)\right)^2}{\left\{1+\lambda-2\lambda\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right\}} - 4\lambda^2 \sum_{i=1}^{n} \frac{\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{2\alpha} \left(log\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)\right)^2}{\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right]^2} \qquad (4.55)$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\sum_{i=1}^{n} \frac{\left[1-2\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right]^2}{\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right]^2} \qquad (4.56)$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \alpha} = \sum_{i=1}^{n} \frac{t_i e^{-\theta t_i}}{1-e^{-\theta t_i}} + \sum_{i=1}^{n} \frac{t_i e^{-\theta t_i}}{1+e^{-\theta t_i}}$$

$$-4\lambda \sum_{i=1}^{n} \frac{t_i e^{-\theta t_i}\left(1-e^{-\theta t_i}\right)^{\alpha-1}}{\left\{1+\lambda-2\lambda\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right\}(1+e^{-\theta t_i})^{\alpha+1}}$$

$$-4\alpha\lambda \sum_{i=1}^{n} \frac{t_i e^{-\theta t_i}\left(1-e^{-\theta t_i}\right)^{\alpha-1} log\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)}{\left\{1+\lambda-2\lambda\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right\}(1+e^{-\theta t_i})^{\alpha+1}} \qquad (4.57)$$

$$-8\alpha\lambda^2 \sum_{i=1}^{n} \frac{t_i e^{-\theta t_i}\left(1-e^{-\theta t_i}\right)^{2\alpha-1} log\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)}{\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right]^2(1+e^{-\theta t_i})^{2\alpha+1}}$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \lambda} = -4\alpha \sum_{i=1}^{n} \frac{t e^{-\theta t_i}\left(1-e^{-\theta t_i}\right)^{\alpha-1}}{\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right]^2\left(1+e^{-\theta t_i}\right)^{\alpha+1}} \qquad (4.58)$$

Note

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = -2\sum_{i=1}^{n} \frac{\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha} log\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)}{\left[1+\lambda-2\lambda\left(\frac{1-e^{-\theta} t_i}{1+e^{-\theta t_i}}\right)^{\alpha}\right]^2} \qquad (4.59)$$

$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = \frac{\partial^2 \ell}{\partial \theta \partial \alpha}$, $\quad \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = \frac{\partial^2 \ell}{\partial \alpha \partial \lambda}$, $\quad \frac{\partial^2 \ell}{\partial \lambda \partial \theta} = \frac{\partial^2 \ell}{\partial \theta \partial \lambda}$,

and

$$I = \begin{vmatrix} \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \alpha} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \alpha \partial \theta} & \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda^2} \end{vmatrix} ; \quad I \neq 0$$

5. Application

In this section, we present the analysis of Financial Data using the TGHLD model and compare it with GHLD, using Akaike information criterion (AIC), Akaike, Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). The data set is obtained from the data in [7]. This data set is in Table (5.1)

Table 5.1. Goodness of fit criteria AIC, BIC, CAIC, HQIC

Model	$\hat{\ell}$	AIC	BIC	CAIC	HQIC
GHLD	-210.587	425.174	430.847	425.206	427.479
TGHLD	-51.229	108.459	116.968	108.556	111.916

6. Conclusion

In this paper, we obtain a new distribution by extending the Generalized Half Logistic Distribution, we create a new distribution called Transmeted Generalized Half Logistic Distribution (TGHLD). Investigations are conducted into the survival, hazard rate, and order statistics of that distribution. The maximum likelihood method is used to estimate the TGHLD's parameters. The proposed distribution is adaptable and can be a good substitute for existing distributions, according to the goodness-of-fit results.

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