Odd Exponential Half Logistic Distribution

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Abstract: In this work, we introduce a new distribution, the Odd Exponential Half Logistic Distribution(OEHLD), by expanding the half logistic distribution to boost its adaptability.The survival function, the hazard rate function, the order statistics, the quantiles, the median, and the asymptotic confidence bounds are all derived.After obtaining a new distribution. The new obtained parameters are estimated by using the maximum likelihood estimation (MLE) method to fit the new distribution.To choose the most appropriate distribution for the data, a comparison was performed using goodness-of-fit metrics.

Keywords: Half Logistic Distribution, Maximum Likelihood Estimation, Simulation, Entropy, Order Statistics.

1. Introduction

The half-logistic distribution (HLD) is one of the probability distributions which is a member of the family of logistic distribution. Its probability density function (pdf) is given by

$$f(t) = \frac{2\alpha e^{-\alpha t}}{(1+e^{-\alpha t})^2} \quad t>0, \alpha>0 \tag{1.1}$$

The corresponding cumulative distribution function (cdf) is given by

$$F(t) = 2\,(1+e^{-\alpha t})^{-1} - 1 \quad t>0,\ \alpha>0 \tag{1.2}$$

The survival function of the HLD distribution is given by

$$\overline{F} = 2 - 2(1+e^{-\alpha t})^{-1} \tag{1.3}$$

The HLD distribution's ordering statistics was analyzed by Balakrishnan (1985).Olapade (2003) gave various theorems to characterize the distribution, whereas Balakrishnan and Puthenpura (1986) achieved the best unbiased estimates of the location and scale parameter of the distribution.Both the mean and standard deviation of the HLD distribution were estimated using maximum likelihood methods by Balakrishnan and Wong (1991).Using both complete and censored data, Torabi and Bagheri (2010) developed a generalized HLD distribution and investigated several strategies for predicting its parameters.Researching a lifetime requires additional distributions that are adaptable enough to work with a wide variety of data types.

In light of these issues, a number of authors have recently paid considerable attention to the HLD distribution, proposing a number of extensions and new forms of the HLD, including the generalized HLD (GHLD), the power HLD (PWHLD) by Krishnarani (2016), the generalized HLD (OGHLD) by Olapade (2014), the exponentiated HLD family of distributions (EHLD-G) by Cordeiro et al. (2014), and the type I half-log (2016).

The formula[6] below can be used to calculate the odd ratio, which stands in for the random variable of the producing odd distribution.

$$T = \frac{F(t)}{\overline{F}(t)} \tag{1.4}$$

Then if X is distributed from exponential with parameter, then its pdf is given by

$$f(t) = \vartheta e^{-\vartheta t} \quad , t>0, \vartheta>0 \tag{1.5}$$

While its cdf is given by

$$F(t) = 1 - e^{-\vartheta t} \tag{1.6}$$

One of the essential properties that we need to recall is related to the gamma pdf, that is

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} t^{\alpha-1} e^{-\frac{t}{\beta}} dt \tag{1.7}$$

2. Odd Exponential half-logistic Distribution

We present OEHL distribution and obtain the cdf, and pdf, respectively by using the formula of odd ratio in (1.4) to obtain the following new random variable

$$T = \frac{2\left(1+e^{-\alpha t}\right)^{-1}-1}{2-2(1+e^{-\alpha t})^{-1}} = \frac{1-e^{-\alpha t}}{2e^{-\alpha t}} \tag{2.1}$$

Then

$$F_{OEHL}(t;\vartheta,\alpha) = 1 - e^{\frac{-\vartheta(e^{\alpha t}-1)}{2}} \text{ }, t > 0,\ \vartheta,\alpha > 0 \tag{2.2}$$

$$f_{OEHL}(t;\vartheta,\alpha) = \frac{\alpha\vartheta}{2} e^{\alpha t}\, e^{\frac{-\vartheta(e^{\alpha t}-1)}{2}}, t > 0,\ \vartheta,\alpha > 0, \text{ respectively} \tag{2.3}$$

is the cdf and the corresponding pdf has the following formula

The survival function of the OEHLD is given as follows

$$S_{OEHL}(t) = 1 - F(t)$$

$$= e^{\frac{-\vartheta(e^{\alpha t}-1)}{2}} \text{ , } t>0,\ \vartheta,\alpha>0 \tag{2.4}$$

The plots of the cdf, pdf and suf of the OEHL are given by the following figures

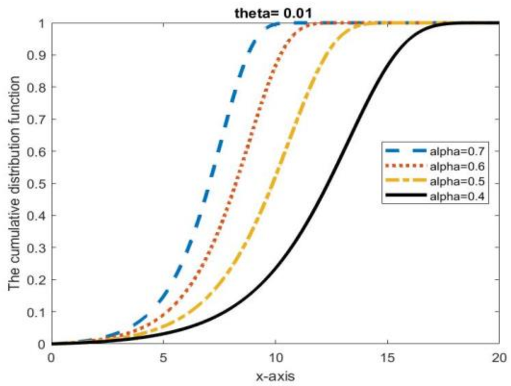

Figure (2.1): the cdf of OEHLD with the parameters $\vartheta = 0.01$, $\alpha = (0.7,0.6,0.5,0.4)$.

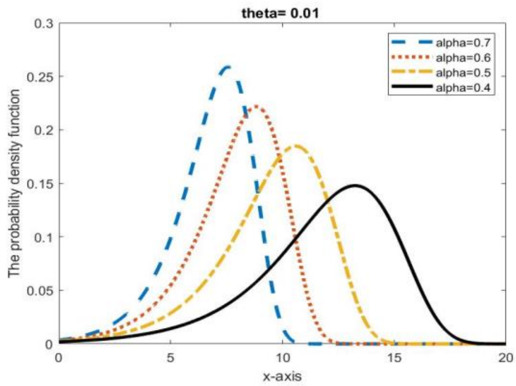

Figure (2.2): the p.d.f of OEHLD with the parameters $\vartheta = 0.01$, $\alpha = (0.7,0.6,0.5,0.4)$

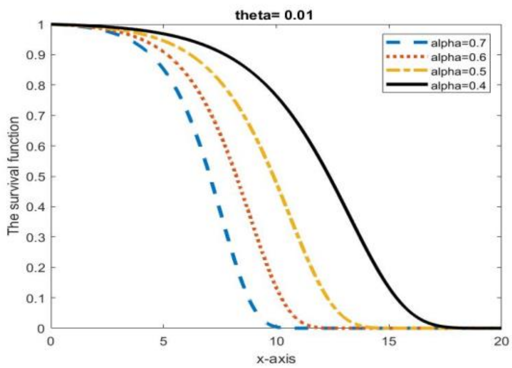

Figure (2.3): the sf of OEHLD with the parameters $\vartheta = 0.01$, $\alpha = (0.7,0.6,0.5,0.4)$

The OEHL distribution can be applied in survival analysis, hydrology, economics. The other characteristic of the random variable is the hazard function. It is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to time t. Then, the hazard function is given by

$$h_{OEHL} = \frac{\alpha\vartheta}{2} e^{\alpha t} \ , \quad t > 0, \ \vartheta, \alpha > 0 \tag{2.5}$$

and the corresponding cumulative hazard function is given by

$$H_{OEHL}(t) = \int_0^t h_{OEHL}(t)\, dt$$

$$= \frac{\vartheta}{2}(e^{\alpha t} - 1), \qquad t > 0, \ \vartheta, \alpha > 0. \tag{2.6}$$

3. Properties of OEHL Distribution

3.1. Moments: In this section, we discuss the moment for OEHLD. Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g. tendency, dispersion, skewness and kurtosis).

Theorem 3.1. If a random variable T from OEHLD , then the r^{th} moments of random variable T, is given by:

$$\mu_r' = E(T^r) = \frac{\alpha\vartheta}{2}\, e^{\frac{\vartheta}{2}} \sum_{i=1}^{\infty} \left(\frac{\vartheta}{2}\right)^i \frac{(-1)^i}{i!} \Gamma(r+1)\varphi^{r+1}, t > 0, \ \vartheta, \alpha > 0, \varphi < 0$$

Proof. The well-known distribution of the r^{th} moments of the random variable T with probability density function f(T) given by

$$E(T^r) = \int_0^\infty t^r \ f(t; \vartheta, \alpha), t > 0, \ \vartheta, \alpha > 0 \tag{3.1}$$

Substituting Eq. (2.3) in Eq. (3.1) obtaining the following equation

$$E(T^r) = \int_0^\infty t^r \frac{\alpha\vartheta}{2} e^{\alpha t} e^{\frac{-\vartheta(e^{\alpha t}-1)}{2}} \ dt \tag{3.2}$$

$$= \frac{\alpha\vartheta}{2}\, e^{\frac{\vartheta}{2}} \int_0^\infty t^r e^{\alpha t} e^{\frac{-\vartheta e^{\alpha t}}{2}} \ dt$$

By using the exponential function, where

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

we obtaining the following solution

$$e^{\frac{-\vartheta e^{\alpha t}}{2}} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\vartheta}{2}\right)^i e^{\alpha i t}$$

Thus

$$E(T^r) = \frac{\alpha\vartheta}{2}\ e^{\frac{\vartheta}{2}} \sum_{i=1}^{\infty} \left(\frac{\vartheta}{2}\right)^i \frac{(-1)^i}{i!} \int_0^{\infty} t^r\ \ e^{\alpha(i+1)t}\, dt \tag{3.3}$$

Let $\varphi = -[\alpha(i+1)]^{-1}$

Then

$$\mu_r' = \mathrm{E}(\mathrm{T}^r) = \frac{\alpha\vartheta}{2}\ e^{\frac{\vartheta}{2}} \sum_{i=1}^{\infty} \left(\frac{\vartheta}{2}\right)^i \frac{(-1)^i}{i!} \int_0^{\infty} t^{r+1-1}\ \ e^{\frac{-t}{\varphi}}\, dt,\ t\ > 0,\ \ \vartheta, \alpha\ \ >\ 0, \varphi < 0 \tag{3.4}$$

Therefore, and by using the property of gamma distribution , where

$$\Gamma(\varphi) = \int_0^{\infty} t^{\varphi-1}\, e^{-t} dt,\ \ \varphi > 0, \qquad 0 < \varphi < \infty.$$

Then

$$\mu_r' = E(t^r) = \frac{\alpha\vartheta}{2}\ e^{\frac{\vartheta}{2}} \sum_{i=1}^{\infty} \left(\frac{\vartheta}{2}\right)^i \frac{(-1)^i}{i!} \Gamma(r+1)\varphi^{r+1}, t\ > 0,\ \ \vartheta, \alpha\ \ >\ 0, \varphi < 0 \tag{3.5}$$

3.2. Order Statistics

We can derive and obtain the order statistics of the OEHL distribution as follows. Let $T_{1:n} \leq T_{2:n} \leq \cdots \leq T_{n:n}$be the ordered sample from a population. The cdf of $T_{n:n}$, the n^{th} order statistics, is given by

$$F_{n:n}(t) = [F(t)]^n = \left[1 - e^{-\vartheta \frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^n \tag{3.6}$$

with F(T) is the cdf of the OEHL distribution. Then, the pdf the n^{th}order statistics for the OEHL random variable $t_{n:n}$ can be obtained by using Eqs.(2.1) and (2.2) in above equation to be

$$f_{n:n}(t) = n[F(t)]^{n-1} f(t)$$

$$= n\left[1 - e^{-\vartheta \frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^{n-1} \frac{\alpha\vartheta}{2} e^{\alpha t}\, e^{\frac{-\vartheta}{2} \cdot \frac{1-e^{-\alpha t}}{e^{-\alpha t}}} \tag{3.7}$$

with F(T) and f(T) are the cdf and pdf of the OEHL distribution, respectively.

The cdf of $t_{1:n}$, the first order statistics, is given by

$$f_{1:n}(t) = [1 - F(t)]^n = \left[e^{-\vartheta \frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^n \tag{3.8}$$

and the pdf of $t_{1:n}$, the first order statistics, is obtained as

$$f_{1:n}(t) = n[1 - F(t)]^{n-1} - f(t)$$

$$= n\left[e^{-\vartheta \frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^{n-1} \frac{-\alpha\vartheta}{2} e^{\alpha t}\, e^{\frac{-\vartheta}{2} \cdot \frac{1-e^{-\alpha t}}{e^{-\alpha t}}} \tag{3.9}$$

Then, the pdf of $t_{k:n}$, the k^{th} order statistics, is given by

$$f_{t_{k:n}}(t) = \frac{n!}{(k-1)!\,(n-k)!} f(t)[F(t)]^{k-1}[1 - F(t)]^{n-k} \tag{3.10}$$

Using this formula, we obtain the pdf of the k^{th} order statistics as follows:

$$f_{k:n}(t) = \frac{n!}{(k-1)!\,(n-k)!}\left[1 - e^{-\vartheta \frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^{k-1} \left[e^{\frac{-\vartheta}{2} \cdot \frac{1-e^{-\alpha t}}{e^{-\alpha t}}}\right]^{n-k} \tag{3.11}$$

3.3 Quantile and Median

In this section, we determine the explicit formulas of the quantile and the median of OEHL distribution. The quantile t of the OEHLD is given by

$$F(t) = q, \qquad 0 < q < 1 \tag{3.12}$$

From Eq. (2.1), $t$can be obtained as follows.

$$t = \frac{1}{\alpha} ln\left[\frac{\vartheta - 2ln(1-q)}{\vartheta}\right] \tag{3.13}$$

Setting q = 0.5 in Eq. (3.13), we get the median of OEHLD as follows.

$$t = \frac{1}{\alpha} ln\left[1 + \frac{2\ln(2)}{\vartheta}\right] \tag{3.14}$$

3.4. Maximum Likelihood Estimates

Many estimation methods have argued in literature but the MLE method provides maximum information about the properties of estimated parameters and mostly used. Moreover, normal approximation of these estimators can frankly be managed systematically and mathematically for large sample theory. The sample likelihood function is given by:

$$\prod_{i=1}^{n} f(t_i) = \frac{\alpha^n \vartheta^n}{2^n} e^{\alpha \sum_{i=1}^{n} t_i} e^{\frac{-\vartheta}{2}\sum_{i=1}^{n}(e^{\alpha t_i - 1})} \tag{3.15}$$

Then, the log-likelihood function is obtained as

$$l = nlog\alpha + nlog\vartheta - nlog2 + \alpha\sum_{i=1}^{n} t_i - \frac{\vartheta}{2}\sum_{i=1}^{n}(e^{\alpha t_i} - 1) \tag{3.16}$$

Therefore, the MLE's of parameters which maximize the above log-likelihood function must satisfy the normal equations. We take the first derivative of the above log-likelihood equation with respect to parameters and equate to zero respectively.

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} t_i - \frac{\vartheta}{2}\sum_{i=1}^{n} t_i e^{\alpha t_i} = o \tag{3.17}$$

$$\frac{\partial l}{\partial \vartheta} = \frac{n}{\vartheta} - \frac{1}{2}\sum_{i=1}^{n}(e^{\alpha t_i} - 1) = 0 \tag{3.18}$$

Assuming that α is known, we can obtain the MLE of ϑ as

$$\hat{\vartheta} = \frac{2n}{\sum_{i=1}^{n}(e^{\alpha t_i} - 1)}$$

3.5. Asymptotic confidence bounds

In this section, we derive the asymptotic confidence intervals of these parameters when $\alpha > 0,\ \vartheta > 0$ as the MLEs of the unknown parameters $\alpha > 0$ .and $\vartheta > 0$ can not be obtained in closed forms, by using variance covariance matrix I^{-1} , see [7], where I^{-1} is the inverse of the observed information matrix which defined as follows

$$I^{-1} = \begin{pmatrix} -\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \vartheta} \\ -\frac{\partial^2 l}{\partial \vartheta \partial \alpha} & -\frac{\partial^2 l}{\partial \vartheta^2} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & -E\left(\frac{\partial^2 l}{\partial \alpha \partial \vartheta}\right) \\ -E\left(\frac{\partial^2 l}{\partial \vartheta \partial \alpha}\right) & -E\left(\frac{\partial^2 l}{\partial \vartheta^2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\vartheta}) \\ cov(\hat{\vartheta},\hat{\alpha}) & var(\hat{\vartheta}) \end{pmatrix}. \tag{3.19}$$

The second partial derivatives included in I are given as follows:

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \frac{\vartheta}{2}\sum_{i=1}^{n} t_i{}^2 e^{\alpha t_i} \tag{3.20}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \vartheta} = -\frac{\vartheta}{2}\sum_{i=1}^{n} t_i e^{\alpha t_i} \tag{3.21}$$

$$\frac{\partial^2 l}{\partial \vartheta^2} = -\frac{n}{\vartheta^2} \tag{3.22}$$

$$\frac{\partial^2 l}{\partial \vartheta \partial \alpha} = -\frac{1}{2}\sum_{i=1}^{n} \frac{t_i}{e^{-\alpha t_i}} \tag{3.23}$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters $\alpha$and ϑ by using variance matrix as in the following forms

$$\left(\hat{\alpha} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\alpha})}\,, \hat{\vartheta} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\vartheta})}\right)$$

By using gamma function in (1.7) we have

$$var(\hat{\alpha}) = -E\left(\frac{\partial^2 l}{\partial \alpha^2}\right)$$

$$= -E\left(-\frac{n}{\alpha^2} - \frac{\vartheta}{2}\sum_{i=1}^{n} \frac{t_i{}^2}{e^{-\alpha t_i}}\right)$$

$$= \frac{n}{\alpha^2} + \frac{\vartheta}{2}\sum_{i=1}^{n} E\left(\frac{t_i{}^2}{e^{-\alpha t_i}}\right)$$

$$= \frac{n}{\alpha^2} + \frac{\vartheta}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{z=0}^{\infty}(-1)^{i+j}\binom{i}{j}\left(\frac{-\vartheta}{2}\right)^{i}\frac{\alpha^z}{z!}\Gamma(z+3)[\alpha(j-i)]^{-(z+3)}$$

where $E\left(\frac{t^2}{e^{-\alpha t}}\right) = \sum_{z=0}^{\infty}\frac{\alpha^z}{z!}\int_0^{\infty} t^{z+2+1-1}\ \frac{\alpha\vartheta}{2}e^{\alpha t}\, e^{\frac{-\vartheta(1-e^{-\alpha t})}{2e^{-\alpha t}}}dt$, and $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2}\right) - th$ percentile of the standard normal distribution.

4. Application

In this section, we present the analysis of a real data set using the OEHLD (α, ϑ) model and compare it with the half-logistic Distribution (HLD), using Akaike information criterion (AIC), Akaike, Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). The data set is obtained from the data in [7]. This data set is in Table (4.1)

Table 4.1. Goodness of fit criteria AIC, BIC, CAIC, HQIC.

Model	$\hat{l}$	AIC	BIC	CAIC	HQIC
HLD	-92.50209	189.00418	189.97399	190.33751	188.64512
OEHLD	-77.21517	158.43034	159.40015	159.76367	158.07128

4. Conclusion

In this paper, we obtain a new distribution namely Odd Exponential half-logistic Distribution (OEHL) distribution by extending the half logistic distribution. Some statistical properties of that distribution are investigated including survival, hazard rate and order statistics. The parameters of the OEHL distribution are estimated by using the maximum likelihood method. The results of goodness-of-fit show that the proposed distribution is flexible and can be a good alternative to existing distributions.

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