A new approach on some operator theory in certain semi-inner-product space

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Abstract:

In this note, we assume that F is a linear space over the real or complex number field. The concept of semi-inner product space was introduced in 1961 by G. Lumer [8] but the main properties of it were discovered by J.R. Giles [9], P. L. Papini [10], P. M. Milicic [11], I. Rosca [12], B. Nath [13] and others. In this paper, we give the definition of this concept and point out the main facts which are derived directly from the definition.

Keywords: Linear space, 2-normed spaces, semi-inner product, etc.

Introduction:

In the theory of operators on a Hilbert space, the later actually doesnot function as a particular Banach space, but rather as an inner-product space. It is in terms of the inner product space structure that most of the terminology and techniques are developed. On other hand, this type of Hilbert space considerations find no real parallel in the general Banach space setting (see [1-3]). Some time ago, while trying to carry over a Hilbert space argument to a general Banach space situation, we were led to use a suitable mapping from Banach space to its dual in order to make up for the lack of an inner product. Our procedure suggested the existence of a general theory which it seemed should be useful in the study of operator algebras by providing better insight on known facts, a more adequate language to classify special types of operators, as well as new techniques. These ideas evolved into a theory of semi-inner product space which is presented in this paper.

Semi-inner product spaces:

Let F be a complex (real) vector space. We shall say that a complex (real) semi-inner product is defined on F, if to any $a,b \in F$ there corresponds a complex (real) $[a,b]$ and the following properties holds:

(i) $[a+b,c]=[a,c]+[b,c]$

$[\alpha a,b]=\alpha[a,b]$ for $a,b,c \in F$; α complex (or real).

(ii) $[a,a]>0$ for $a \neq 0$

(iii) $|[a,b]|^2 \leq [a,a][b,b]$

In the same way, we may write

(i) $\left[a_1+a_2,\frac{b}{c}\right]=\left[a_1,\frac{b}{c}\right]+\left[a_2,\frac{b}{c}\right]$

(ii) $\left[\alpha a,\frac{b}{c}\right]=\alpha\left[a,\frac{b}{c}\right]$

(iii) $\left[a,\frac{a}{b}\right]>0$ if a and b are linearly independent, and

(iv) $\left\|\left[a,\frac{b}{c}\right]\right\|\le\left[a,\frac{a}{c}\right]^{\frac{1}{p}}.\left[b,\frac{b}{c}\right]^{1-\frac{1}{p}},1<p<\infty$

Then we say that $[\cdot,\div]$ is generalized 2-semi-inner product on F.

Definition: Let F be a linear space of dimension greater than $1$and $\|\cdot,\cdot\|$be a real valued function on $F\times F$ which satisfies the following conditions:

(i) $\|a,b\|=0\Leftrightarrow a$ and b are linearly dependent

(ii) $\|a,b\|=\|b,a\|$

(iii) $\|\delta a,b\|=|\delta|\;\|a,b\|$ where δ is real

(iv) $\|a+b,c\|\le\|a,c\|+\|b,c\|$ then we say that $\|\cdot,\cdot\|$ is 2-norm on F and $(F,\|\cdot,\cdot\|)$is called a 2-normed space (see [4, 6]).

Main theorem:

Theorem 1: Show that every generalized 2-semi-inner product space is a 2-normed linear space and every 2-normed linear space be made into a generalized 2-semi-inner product space where $\|a,b\|=\left[a,\frac{a}{b}\right]^{\frac{1}{\ell}},1<\ell<\infty$.

To prove our theorem, we need the following lemmas:

Lemma 1: Every generalized 2-semi-inner product space is a 2-normed linear space with $\|a,b\|=\left[a,\frac{a}{b}\right]^{\frac{1}{\ell}},1<\ell<\infty$, where $\left[a,\frac{a}{b}\right]=\left[b,\frac{b}{a}\right]$.

For the proof the lemma, we must show the following conditions:

(i) $\left[\alpha a,\frac{\lambda a}{b}\right]^{\frac{1}{\ell}}=|\alpha|\left[a,\frac{a}{b}\right]^{\frac{1}{\ell}}$ and

(ii) $\left[(a+b),\frac{(a+b)}{c}\right]^{\frac{1}{\ell}} \le \left[a,\frac{a}{c}\right]^{\frac{1}{\ell}} + \left[b,\frac{b}{c}\right]^{\frac{1}{\ell}}$

where the vectors a, b and c are linearly independent.

Proof: By the given definition,

$$\left[\alpha a,\frac{\alpha a}{b}\right] = \alpha\left[a,\frac{\alpha a}{b}\right]$$

$$=) \left\|\left[\alpha a,\frac{\lambda a}{b}\right]\right\| = |\alpha| \left\|\left[a,\frac{\alpha a}{b}\right]\right\|$$

and hence

$$\left[\alpha a,\frac{\lambda a}{b}\right] = |\alpha| \left\|\left[a,\frac{\alpha a}{b}\right]\right\|$$

$$\therefore \left\|\left[a,\frac{\alpha a}{b}\right]\right\| \le \left[a,\frac{a}{b}\right]^{\frac{1}{\ell}} . \left[\alpha a,\frac{a}{b}\right]^{1-\frac{1}{\ell}}$$

thus,

$$\left[\alpha a,\frac{\lambda a}{b}\right] \le |\alpha| \left[a,\frac{a}{b}\right]^{\frac{1}{\ell}} \left[\alpha a,\frac{\alpha a}{b}\right]^{1-\frac{1}{\ell}}$$

or,

$$\left[\alpha a,\frac{\lambda a}{b}\right]^{\frac{1}{\ell}} \le |\alpha| \left[a,\frac{a}{b}\right]^{\frac{1}{\ell}} \tag{1}$$

Since $\left[a,\frac{a}{b}\right]^{\frac{1}{\ell}} = \left[\frac{1}{\alpha}\alpha a,\frac{1}{\alpha}\frac{\alpha a}{b}\right], \alpha \neq 0$ then from equation (1),

$$|\alpha| \left[a,\frac{a}{b}\right]^{\frac{1}{\ell}} \le \left[\alpha a,\frac{\alpha a}{b}\right]^{\frac{1}{\ell}} \tag{2}$$

from equations (1) and (2), we obtain

$$\left[\alpha a,\frac{\alpha a}{b}\right]^{\frac{1}{\ell}} = |\alpha| \left[a,\frac{a}{b}\right]^{\frac{1}{\ell}}$$

again, by definition, we may write

$$\left[(a+b),\frac{a+b}{c}\right]=\left[\frac{a(a+b)}{c}\right]+\left[b,\frac{a+b}{c}\right]$$

$$\therefore \qquad \left[a+b,\frac{a+b}{c}\right]=\left\|\left[a,\frac{a+b}{c}\right]+\left[b,\frac{a+b}{c}\right]\right\|$$

$$\leq\left\|\left[a,\frac{a+b}{c}\right]\right\|+\left\|\left[b,\frac{a+b}{c}\right]\right\| \tag{3}$$

and

$$\left\|\left[a,\frac{a+b}{c}\right]\right\|\leq\left[a,\frac{a}{c}\right]^{\frac{1}{\ell}}.\left[(a+b),\frac{a+b}{c}\right]^{1-\frac{1}{\ell}} \tag{4}$$

similarly,

$$\left\|\left[b,\frac{a+b}{c}\right]\right\|\leq\left[b,\frac{b}{c}\right]^{\frac{1}{\ell}}\left[(a+b),\frac{a+b}{c}\right]^{1-\frac{1}{\ell}} \tag{5}$$

thus by using (3), (4) and (5),

$$\left[a+b,\frac{a+b}{c}\right]^{\frac{1}{\ell}}\leq\left[a,\frac{a}{c}\right]^{\frac{1}{\ell}}+\left[b,\frac{b}{c}\right]^{\frac{1}{\ell}}.$$

Lemma 2: Let a and b are linearly independent vectors of the linear space F and $(F,\|\cdot,\cdot\|)$ is a 2-normed linear space.

Let K and $[\{b\}]$ are linear manifolds in $(F,\|\cdot,\cdot\|)$ then there exists a bounded linear 2-functional R with $F\times[\{b\}]$ such that

(i) $\|R\|=\|a,b\|^{\ell-1}$

(ii) $R(a,b)=\|a,b\|^{\ell},1<\ell<\infty$.

Proof: Let $k=[\{a\}]$

and define L on $K\times[\{b\}]$ as follows:

$$L(\alpha a,\beta b)=\alpha\beta\|a,b\|^{\ell}.$$

We have to show that L is a linear z-functional with the property

$$L(a,b)=\|a,b\|^{\ell}$$

For $\left(\alpha_1 a, b_1\beta\right)$ and $\left(\alpha_2 a, b_2\beta\right) \in K \times \left[\{b\}\right]$.

We have

$L\left(\alpha_1 a + \alpha_2 a, \beta_1 b + \beta_2 b\right) = L\left(\left(\alpha_1 + \alpha_2\right)a, \left(\beta_1 + \beta_2\right)b\right)$

$= \left(\alpha_1 + \alpha_2\right)\left(\beta_1 + \beta_2\right)\left\|a, b\right\|^{\ell}$

$= L\left(\alpha_1 a, \beta_1 b\right) + L\left(\alpha_1 a, \beta_2 b\right) + L\left(\alpha_2 a, \beta_1 b\right) + L\left(\alpha_2 a, \beta_2 b\right)$ (6)

and

$L\left(\Delta\alpha a, \Delta_1\beta b\right) = \left(\Delta\alpha\right)\left(\alpha\beta\right)\left\|a, b\right\|^{\ell}$

$= \left(\Delta\alpha\right) L\left(\alpha a, \beta b\right)$ (7)

from equation (6) and (7), we get the required result.

Proof of (ii)

According to the definition of L, $L\left(\alpha a, \beta b\right) = \alpha\beta\left\|a, b\right\|^{\ell}$

=) $L\left(a, b\right) = \left\|a, b\right\|^{\ell}$ (8)

also, for $\left(\alpha a, \beta b\right) \in K \times \left[\{b\}\right]$

$\left|L\left(\alpha a, \beta b\right)\right| = \left|\alpha\right|\left|\beta\right|\left\|a, b\right\|^{\ell}$

$= \left\|a, b\right\|^{\ell-1}.\left\|a, b\right\|$ (9)

=) L is a bounded linear 2-functional.

Thus $\left\|L\right\| = \inf \{\subset \left|L\left(\alpha a, \beta b\right)\right| \leq$

$\subset \left\|\alpha a, \beta b\right\|$

So, $\left\|a, b\right\|^{\ell-1} \in \{< \left\|L\left(\alpha a, \beta b\right)\right\| <$

$< \left\|\alpha a, \beta b\right\|\}$

=) $\left\|L\right\| \leq \left\|a, b\right\|^{\ell-1}$ (10)

Since L is bounded linear z-functional

$\left|L\left(\alpha a, \beta b\right)\right| \leq \left\|L\right\|\left\|\alpha a, \beta b\right\|$

$\Rightarrow \|a,b\|^{\ell-1} \leq \|L\|$ (11)

hence from (10) and (11),

$\|L\| = \|a,b\|^{\ell-1}$

thus, by Hahn-Banach theorem;

using the result of [5],

$\|L\| = \|P\|$ (by result of [5])

$\Rightarrow \|P\| = \|a,b\|^{\ell-1}$.

Also, $L(\alpha a, \beta b) = P(\alpha a, \beta b)$

from relation (8), we conclude that

$P(a,b) = \|a,b\|^{\ell}$.

Lemma 3: Every 2-normed linear space can be made into a g. 2-s-i.p.s.

Using the lemmas 1, 2 and 3, we get the required result of our theorem.

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