# The Existence of Characters on relations between certain intrinsic topologies in certain partially ordered sets.

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**Abstract:** In this paper, we discuss the existence of characters on the relation between certain intrinsic topologies in certain partially ordered sets. We almost discuss the transformation of a normed vector lattice definition into a 2-normed vector lattice definition. This work is motivated by the works of [9, 11-12].

**Keywords: Normal vector lattice, Complete lattice, Self-mapping, etc.**

**Introduction:**

A variety of ways have been suggested for defining topologies from the algebraic structure of a lattice (see [1-2]). If one is given a topological lattice, a natural question is whether the given topology agrees with one or more of these intrinsic topologies. Some results of this nature may be found in (see [2-5]). The theory of 2-normed space was first introduced by Gähler [6-10] as an interesting linear generalization of the theory of the normed linear spaces which was subsequently studied by many authors.

**Definition 1:** Let E be a complete lattice. The elements of E are generally denoted by $a_1, b_1, c_1, \ldots$. The subset of E are generally denoted $A_1, B_1, C_1, \ldots$.

Now, we define a new function $\zeta : E \to E$ with the property $(\beta)$ satisfy the following conditions: $A \leq E$,

$\zeta(\vee A) = \wedge \zeta A$,

$\zeta(A) = \{\zeta(A) : a \in A\}$ and denoted it by $\zeta^2$.

Thus for $a \leq b \Rightarrow \zeta(a) \geq \zeta(b)$

$\Rightarrow \quad \zeta^2(a) \leq \zeta^2(b)$.

**Known theorems:**

In 1965, Broder [12] has established the following theorem:

**Theorem 1:**

Let E be a complete lattice and $b = \zeta(a)$ be isotone signals from E to E. Then we find $r = \zeta(r)$ for some $r \in E$.

In 1975, author [12] has generalized the result of [11] and established the following theorem:

**Theorem 2:**

If a signal $\zeta$ from E to E with the property, $(\beta)$ then there exist a unique element $a \in E$ such that $a = \zeta^2(a)$ and $a \le \zeta(a)$.

Now, we define a new signal $G : E \to E$ with $(\delta)$:

$A \in E$,

$G(\vee A) = \wedge G(A)$

$G(A) = \{G(A) : a \in A\}$,

the composition of $G$ with $G^2$.

**Lemmas:**

**Lemma 1:** The function G satisfies $(\delta)$ then

(i) $a \le b \Rightarrow G(a) \ge G(b)$

$a \le b \Rightarrow a \wedge b = a$,

$G(a \wedge b) = G(a)$

$G(a) \vee G(b) = G(a)$

(ii) $a \le b \Rightarrow G^2(a) \le G^2(b)$.

**Proof:** The $a \le b \Rightarrow G(a) \ge G(b)$ (by given condition)

$\Rightarrow G(a) \wedge G(b) = G(b)$

$\Rightarrow G(G(a) \wedge G(b)) = G(G(b))$

$\Rightarrow G^2(a) \vee G^2(b) = G^2(b)$

$\Rightarrow G^2(a) \le G^2(b)$.

Now, $E_D(G) = \{a \in E : a \ge G(a)\}$

$E_D(G^2) = \{a \in E : a \ge G^2(a)\}$.

**Lemma 2:**

(i) $G^2 : E_D(G) \to E_D(G)$

suppose $a \in E_D(G)$

$$\Rightarrow a \geq G(a)$$

$$\Rightarrow G^2(a) \geq G^2(G(a)) = G(G^2(a))$$

$$\Rightarrow G^2(a) \in E_D(G).$$

(ii) $G^2 : E_D(G^2) \to E_D(G^2)$

consider $a \in E_D(G^2)$

$$\Rightarrow a \geq G^2(a)$$

$$\Rightarrow G^2(a) \geq G^2(G^2(a))$$

$$\Rightarrow G^2(a) \in E_D(G^2).$$

**Lemma 3:**

$E_D(G^2)$ is complete lattice.

**Proof:** Suppose $a,b \in E_D(G^2)$

$$\Rightarrow a \geq G^2(a) \text{ and } b \geq G^2(b).$$

Put $a \wedge b = m$

$$\Rightarrow a \geq m, \quad b \geq m$$

if $a \geq m \Rightarrow G^2(a) \geq G^2(m)$

and if $b \geq m \to G^2(b) \geq G^2(m)$

thus $G^2(a) \wedge G^2(b) \geq G^2(m)$

i.e. $\left.\begin{array}{l} a \geq G^2(a) \\ b \geq G^2(b) \end{array}\right\} \Rightarrow a \wedge b \Rightarrow G^2(a) \wedge G^2(b)$

$$\Rightarrow G^2(m)$$

$$\Rightarrow G^2(a \wedge b).$$

So, $a \wedge b \in E_d(G^2)$.

This completes the proof of the theorem.

Now, we prove the following theorems:

**Theorem 3:** Let E be a complete lattice and $a \in E$ then $a = G^2(a)$ and $a \geq G(a)$.

**Proof:** Let $M = E_M(G) \cap E_M(G^2)$

Put $a = \wedge N$ where N is maximal chain in D.

$\Rightarrow a \in E_M(G^2)$ (by lemma 3).

Let $a < G(a)$ (by property $(\delta)$)

then $G(a) = \vee G(N)$

so, $a < G(a) \Rightarrow n \in N$.

Such that $a < G(n)$.

Thus there exist $n_1 \in N$ such that $n_1 < G(n)$.

Now, $\because N \leq E_M(G) \Rightarrow n \geq G(n)$.

If M is chain, $n \leq n_1$ or $n_1 \leq n$.

Since $n_1 < G(n)$ then $n_1 < n$.

By using lemma 1,

$n \geq n_1 \Rightarrow G(n) \leq G(n_1)$

which gives a contraction.

Thus $a \geq G(a)$ i.e., $a \in E_M(G)$.

Again by using lemma 2,

$G^2(a) \in M$

$\Rightarrow a \geq G^2(a)$.

If $a > G(a)$ then N is not maximal, which again gives a contradiction.

So, $a = G^2(a)$.

**Theorem 4:** Let E be a complete lattice and $2^E$ be the complete lattice subset of E.

Let $\zeta : E \rightarrow 2^E$ be a multivalued mapping, then

$c = \sup \zeta(c)$ for some $c \in E$.

**Proof:** Let c be the l.u.b. of R of $a \in E$ such that $a \le \sup\ \zeta(a)$.

It is clear that R is non-empty.

Since $\zeta$ is multi-valued isotone and $a \le c$ for all $a \in R$,

$a \le \sup\ \zeta(a) \le \sup\ \zeta(c)$
Thus $c = \sup\ R \le \sup\ \zeta(c)$.

If $\sup\ \zeta(c) = d$ then $c \le d$.

Thus by hypothesis,

$d = \sup\ \zeta(c) \le \sup\ \zeta(d)$,

where $\zeta(c) \in R$

$\Rightarrow \sup \zeta(c) \le c \quad (\because \sup\ R)$

So, $\sup \zeta(c) = c$.

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