Residual Decomposition of the Green's Function of the Dirichlet problem for a Differential Operator on a star-graph for m=2

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Abstract: In this paper, we have researched a system of second-order differential equations, which is a model of oscillatory systems with a rod structure. Problems for differential operators on graphs are currently being actively studied by mathematicians and have applications in quantum mechanics, organic chemistry, nanotechnology, waveguide theory, and other areas of natural science. A graph is a structure consisting of "abstract" segments and vertexes, the adjacency of which to each other is described by some relation. To define an operator on a given graph, it is necessary to select a set of boundary vertexes. Vertexes that are not boundary are called internal vertexes. A differential operator on a given graph is determined not only by given differential expressions on edges, but also by Kirchhoff-type conditions at internal vertexes of the graph. In this article, the Dirichlet problem for a differential operator on a star graph is solved. We have used standard gluing conditions at internal vertexes and Dirichlet boundary conditions at boundary vertexes. Also in this paper, the Green's function of a differential operator on a star graph is presented. Questions from the spectral theory, such as the construction of the Green's function and the expansion in terms of eigenfunctions for models of connected rods, have been little studied. Spectral analysis of differential operators on graphs is the main mathematical tool for solving modern problems of quantum mechanics.

Key words: directed graph, graph vertexes, Kirchhoff conditions, elastic network oscillations, Dirichlet problem, eigenfunction expansion, Residual expansion.

1. Introduction

The paper investigates a system of second-order differential equations, which is a model of oscillatory systems with a rod structure. The focus of this article is on the spectrum of differential operators second order on graphs. Various functional spaces on graphs are defined, and we define, in terms of both differential systems and the aforementioned function spaces, boundary value problems on graphs. Shown that a boundary value problem on a graph is spectrally equivalent to a system with a separated boundary condition. The main goal of this article is to solve the Dirichlet problem and construct its Green's function for the star graph and Residual decomposition. A star graph is a connected graph in which at most one vertex has degree greater than one. A vertex with a degree greater than one is called an internal vertex of a star graph. Vertexes that are not internal are called boundary vertexes.

Let a directed star graph be given $G=\{\nu,\varepsilon\}$ Where ν,ε - two sets. Set elements ν -are called the vertexes of the graph, through ε the set of its edges is indicated. The number of edges will be denoted by m. Let $\Gamma=\{0\}$ - inner vertex, $\Gamma=\{1,...,\mathrm{m}\}-$ boundary vertexes. At $j=\overline{1,m}$ emanating from the vertexes j denote the edges e_j In what follows, we assume that the length of each edge $|e_j|=a_j$. On every edge e_j introduce a variable $x_j\in\left[0,a_j\right]$. For convenience, we denote by the value $x_j=a_j$ corresponds to the boundary vertex of the edges e_j, and while the value $x_j=0$ corresponds to the inner vertex.

In the proposed paper, we study the properties of the Green's functions of a boundary value problem for second-order differential equations on a star graph.

2 Literature review

In the last 25-30 years, the theory of differential equations and boundary value problems on geometric graphs (spatial networks) has been intensively developed, as evidenced by numerous scientific papers. The beginning of research was laid in the works (B.S. Pavlov [17], Yu.V. Pokorny, O.M. Penkin ([16], [3]) and others) and foreign (Aimal Rasa.G.H. ([4,5,6]), A.M. Akhtyamov.[3], Б.Е Кангужин [11]) and other[10-19] mathematicians and dealt with problems describing various models: diffusion, oscillations of elastic nets, nerve impulse propagation, etc. Works of foreign mathematicians, are mainly devoted to the justification of the solvability of boundary value problems on graphs, the study of the structure of the spectrum of these problems, the asymptotics of the spectrum, and obtaining estimates of the resolvent. Currently, the most active research is being carried out by the creative group of Yu.V. Pokorny (Б.Е Кангужин, Даирбаева Г.,Мадибайулы Ж), Aimal Rasa Ghulam Hazrat. and Gaukhar Auzerkhan , the main results of which are reflected in [5,11] (see also the bibliography in [5,11]).

3. Materials and methods

Let us dwell in more detail on the results concerning the Green's function for second-order differential operators on manifolds of network type. In this paper, for a complete description and solution of the Dirichlet problem for a second-order differential operator on a star graph, a synthetic approach is used.

Methodology Information has been collected in the form of libraries, websites, domestic and foreign scientific articles, undergraduate and doctoral research dissertations.

3.1 Definition of a differential operator on a star graph

In what follows, it is useful to introduce the space

$$L_2(G)=\prod_{e\in\varepsilon}L_2(e)$$

with elements

$$\vec{Y}(\vec{x})=\left[y_e(x_e),e\in\varepsilon\right]^T$$

(where $\vec{x}=(x_e,e\in\varepsilon)$ and $\prod_{e\in\varepsilon}$ - is the Cartesian product of subspaces) and with finite

the norm

$$\left\|\vec{Y}\right\|_{L_2(G)}=\sqrt{\sum_{e\in\varepsilon}\int_e\left|y_e(x_e)\right|^2dx_e}.$$

In the same way, the space is introduced in the standard space

$$W_2^2(G)=\prod_{e\in\varepsilon}W_2^2(e).$$

Let us introduce a set of functions $D(\wedge)\subset W_2^2(G)$, whose elements at each internal vertex satisfy the Kirchhoff conditions (Afanas'eva, 2010).

$$\begin{cases}y_1(0)=y_j(0) & j=2,...,m\\ \sum_{j=1}^m y'(0)=0.\end{cases} \tag{1}$$

In electrical networks, they express Kirchhoff's law, with oscillations of elastic networks - the balance of voltages. $L_2(G)$ consider the differential operator $\wedge$, given linear differential expressions

$$-y_j''(x_j)=\lambda y_j(x_j)+f_j(x_j),\ e_j\in\varepsilon,\ 0<x_j<a_j,\quad j=1,\ldots,m. \qquad (2)$$

Whit region definitions $D(\wedge)$. λ - Spectral parameter, $\{f_j(x_j),0<x_j<a_j\}-$ distribution density of the external force.

In this paper, the Green's function of problem (1), (2) is constructed constructively with the Dirichlet conditions at the boundary vertexes

$$y_j(a_j)=...=y_m(a_m)=0. \qquad (3)$$

3.2 Construction of the Green's function of the Dirichlet problem

In this subsection, we study the question of the existence of the Green's function for the Dirichlet problem

$$-y''(x)=\lambda y(x)+F(x),\ \ 0<x<a, \qquad (4)$$

$$y(a)=0\ \ y(0)=0 \qquad (5)$$

By the Green's function we mean the matrix function of two variables $G(\vec{x},\vec{t},\lambda)$, at every $\vec{F}(\cdot)$ continuous on the graph G and given by the formula

$$y(\vec{x},\lambda)=\int_G G(\vec{x},t,\lambda)\vec{F}(t)\,dt.$$

4 Green's function of the Dirichlet problem (1),(2),(3)

we solve problems of a differential operator on a star graph for m=2

$$\begin{cases}-y_1''(x_1)=\lambda y_1(x_1)+f_1(x_1)\,;x_1\in e_1\\ -y_2''(x_2)=\lambda y_2(x_2)+f_2(x_2);x_2\in e_2\end{cases} \qquad (6)$$

In the first equation $y_1(x_1),\ y_2(x_2)$ unknown, known features $f_1(x_1),f_2(x_2),\lambda.$

Also the Dirichlet conditions:

$$y_1(a_1)=y_2(a_2)=0 \qquad (7)$$

and Kirchhoff conditions:

$$y_1(0)=y_2(0),\ y_1'(0)+y_2'(0)=0 \qquad (8)$$

Theorem 1. If $f_1(x_1)\neq 0, f_2(x_2)\neq 0$ then the solution of the Green's functions is the problem (6),(7),(8) Dirichlet can be written as:

$$y_1(x_1)=\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)}\begin{cases}\int_0^{a_2}\sin\sqrt{\lambda}(a_2-t_2)\sin\sqrt{\lambda}(a_1-x_1)f_2(t_2)dt_2+\\+\int_0^{x_1}\sin\sqrt{\lambda}(a_1-x_1)\sin\sqrt{\lambda}(t_1+a_2)f_1(t_1)dt_1+\int_{x_1}^{a_1}\sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}(x_1+a_2)f_1(t_1)dt_1\end{cases}$$

$$\left\{ y_2(x_2)=\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)}\begin{cases}\int_0^{a_1}\sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}(a_2-x_2)f_1(t_1)dt_1+\\+\int_0^{x_2}\sin\sqrt{\lambda}(a_2-x_2)\sin\sqrt{\lambda}(t_2+a_1)f_2(t_2)dt_2+\int_{x_2}^{a_2}\sin\sqrt{\lambda}(a_2-t_2)\sin\sqrt{\lambda}(x_2+a_1)f_2(t_2)dt_2\end{cases}\right.$$

This theorem is a general solution of a linear differential equation and a problem for $m=2$ has the form.

Proof. We need to show the theorem, first we calculate the first derivative

$$y'_{12}(x_1)=\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)}\begin{cases}-\sqrt{\lambda}\int_0^{a_2}\sin\sqrt{\lambda}(a_2-t_2)\cos\sqrt{\lambda}(a_1-x_1)f_2(t_2)dt_2+\\-\sqrt{\lambda}\int_0^{x_1}\cos\sqrt{\lambda}(a_1-x_1)\sin\sqrt{\lambda}(t_1+a_2)f_1(t_1)dt_1+\sqrt{\lambda}\int_{x_1}^{a_1}\sin\sqrt{\lambda}(a_1-t_1)\cos\sqrt{\lambda}(x_1+a_2)f_1(t_1)dt_1\end{cases}$$

Now we calculate the second derivative

$$y''_{12}(x_1)=\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)}\begin{cases}-\lambda\int_0^{a_2}\sin\sqrt{\lambda}(a_2-t_2)\cos\sqrt{\lambda}(a_1-x_1)f_2(t_2)dt_2-\lambda\int_0^{x_1}\cos\sqrt{\lambda}(a_1-x_1)\sin\sqrt{\lambda}(t_1+a_2)f_1(t_1)dt_1\\-\lambda\int_{x_1}^{a_1}\sin\sqrt{\lambda}(a_1-t_1)\cos\sqrt{\lambda}(x_1+a_2)f_1(t_1)dt_1\end{cases}+f_1(x_1)$$

$$-y''_{12}(x_1)=\lambda y_1(x_1)+f_1(x_1).$$

Let us show that the given functions satisfy equations (2), boundary conditions (3), conditions for $m=2$.

First, we check the Dirichlet conditions from equation (2) $y_1(a_1)=y_2(a_2)=0$.

$$\begin{cases} y_1(a_1) = \dfrac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)} \left\{ \begin{aligned} &\int_0^{a_2} \sin\sqrt{\lambda}(a_2-t_2)\sin\sqrt{\lambda}(a_1-a_1) f_2(t_2)dt_2 + \\ &+\int_0^{a_1} \sin\sqrt{\lambda}(a_1-a_1)\sin\sqrt{\lambda}(t_1+a_2) f_1(t_1)dt_1 + \int_{a_1}^{a_1} \sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}(a_1+a_2) f_1(t_1)dt_1 = 0 \end{aligned} \right\} \\ y_2(a_2) = \dfrac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)} \left\{ \begin{aligned} &\int_0^{a_1} \sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}(a_2-a_2) f_1(t_1)dt_1 + \\ &+\int_0^{a_2} \sin\sqrt{\lambda}(a_2-a_2)\sin\sqrt{\lambda}(t_2+a_1) f_2(t_2)dt_2 + \int_{a_2}^{a_2} \sin\sqrt{\lambda}(a_2-t_2)\sin\sqrt{\lambda}(a_2+a_1) f_2(t_2)dt_2 = 0 \end{aligned} \right\} \end{cases}$$

Now we check the Kirchhoff conditions for $m=2: y_1(0)=y_2(0),\ y_1'(0)+y_2'(0)=0$,

$$\begin{cases} y_1(0) = \dfrac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)} \left\{ \int_0^{a_2} \sin\sqrt{\lambda}(a_2-t_2)\sin\sqrt{\lambda}a_1 f_2(t_2)dt_2 + \int_0^{a_1} \sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}a_2 f_1(t_1)dt_1 \right\} \\ y_2(0) = \dfrac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)} \left\{ \int_0^{a_1} \sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}a_2 f_1(t_1)dt_1 + \int_0^{a_2} \sin\sqrt{\lambda}(a_2-t_2)\sin\sqrt{\lambda}a_1 f_2(t_2)dt_2 \right\} \end{cases}$$

from here we calculate the first derivative of the expression

$$y_1'(0) = \frac{1}{\sin\sqrt{\lambda}(a_1+a_2)} \left\{ -\int_0^{a_2} \sin\sqrt{\lambda}(a_2-t_2)\cos\sqrt{\lambda}a_1 f_2(t_2)dt_2 + \int_0^{a_1} \sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}a_2 f_1(t_1)dt_1 \right\}$$

$$y_2'(0) = \frac{1}{\sin\sqrt{\lambda}(a_1+a_2)} \left\{ -\int_0^{a_1} \sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}a_2 f_1(t_1)dt_1 + \int_0^{a_2} \sin\sqrt{\lambda}(a_2-t_2)\sin\sqrt{\lambda}a_1 f_2(t_2)dt_2 \right\}$$

5. Residual Fourier expansion of the Green's function of the Dirichlet problem

In this subsection, we study the question of the Fourier series expansion of the Green's function of problem (6), (7), (8) in terms of eigenfunctions of the corresponding spectral problem.

Theorem 2. (Naimark, 1969:90) Any function from the domain of definition of a self-adjoint differential operator expands into a Fourier series in terms of the eigenfunctions of this operator. Let us prove the following intermediate result.

$$\underset{\lambda_0}{res}\, y_1(x_1,\lambda)=\underset{\lambda_0}{res}\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)}\int_0^{a_2}\sin\sqrt{\lambda}(a_2-t_2)\sin\sqrt{\lambda}(a_1-x_1)\,f_2(t_2)dt_2+$$

$$+\underset{\lambda_0}{res}\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)}\int_0^{x_1}\sin\sqrt{\lambda}(a_1-x_1)\sin\sqrt{\lambda}(t_1+a_2)\,f_1(t_1)dt_1+$$

$$+\underset{\lambda_0}{res}\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)}\int_{x_1}^{a_1}\sin\sqrt{\lambda}(a_1-t_1)\sin\sqrt{\lambda}(x_1+a_2)\,f_1(t_1)dt_1$$

Theorem 3. Must know $\sqrt{\lambda}\sin\sqrt{\lambda}(a_1+a_2)=0$ then we can write $\lambda_0=0$ and $\sin\sqrt{\lambda}(a_1+a_2)=\sin k\pi$, then root $\lambda_0=\left(\frac{k\pi}{a_1+a_2}\right)^2$.

$$\underset{\left(\frac{k\pi}{a_1+a_2}\right)^2}{res} y_1(x_1,\lambda)=\int_0^{a_2}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_2-t_2)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(a_1-x_1)\right\}}{\sin\left\{\frac{k\pi}{a_1+a_2}(a_1+a_2)\right\}+\frac{k\pi}{a_1+a_2}(a_1+a_2)\cos\left\{\frac{k\pi}{a_1+a_2}(a_1+a_2)\right\}}f_2(t_2)dt_2+$$

$$+\int_0^{x_1}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_1-x_1)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(t_1+a_2)\right\}}{\sin\left\{\frac{k\pi}{a_1+a_2}(a_1+a_2)\right\}+\frac{k\pi}{a_1+a_2}(a_1+a_2)\cos\left\{\frac{k\pi}{a_1+a_2}(a_1+a_2)\right\}}f_1(t_1)dt_1+$$

$$+\int_{x_1}^{a_1}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_1-t_1)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(x_1+a_2)\right\}}{\sin\left\{\frac{k\pi}{a_1+a_2}(a_1+a_2)\right\}+\frac{k\pi}{a_1+a_2}(a_1+a_2)\cos\left\{\frac{k\pi}{a_1+a_2}(a_1+a_2)\right\}}f_1(t_1)dt_1$$

now we can as it is written

$$\underset{\left(\frac{k\pi}{a_1+a_2}\right)^2}{res} y_1(x_1,\lambda)=\int_0^{a_2}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_2-t_2)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(a_1-x_1)\right\}}{k\pi\cos k\pi}f_2(t_2)dt_2+$$

$$+\int_0^{x_1}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_1-x_1)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(t_1+a_2)\right\}}{k\pi\cos k\pi}f_1(t_1)dt_1+$$

$$+\int_{x_1}^{a_1}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_1-t_1)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(x_1+a_2)\right\}}{k\pi\cos k\pi}f_1(t_1)dt_1.$$

Hence the residue expansion of the Green's function, we calculate $y_2(x_2,\lambda)$

$$\underset{\left(\frac{k\pi}{a_1+a_2}\right)^2}{res}\, y_2(x_2,\lambda)=\int_0^{a_1}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_1-t_1)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(a_2-x_2)\right\}}{k\pi\cos k\pi}f_1(t_1)dt_1+$$

$$+\int_0^{x_2}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_2-x_2)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(t_2+a_1)\right\}}{k\pi\cos k\pi}f_2(t_2)dt_2+$$

$$+\int_{x_2}^{a_2}\frac{\sin\left\{\frac{k\pi}{a_1+a_2}(a_2-t_2)\right\}\sin\left\{\frac{k\pi}{a_1+a_2}(x_2+a_1)\right\}}{k\pi\cos k\pi}f_2(t_2)dt_2.$$

6. Conclusion

On a plane graph consisting of several arcs with one common end, the Green's function of the boundary value problem for the Sturm-Liouville equation is constructed. The problem is a model of oscillation of a simple system of several rods with an adjoining end. In this paper, we derive a formula for the function of the Dirichlet problem for a second-order equation on a directed graph. The existence of an expansion of an arbitrary function defined on a graph in terms of eigenfunctions is proved. Questions from spectral theory, such as the construction of the Green's function and the Residual expansion**.**

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