Study on Intuitionistic Fuzzy Abundant Semigroup

Ancy Varghese[a], Paul Isaac[b]

[a]Research Scholar,Department of Mathematics, Bharata Mata College, Thrikkakara, Kerala,8281357721
[b]Associate Prof.,Department of Mathematics, Bharata Mata College, Thrikkakara, Kerala

Abstract

We present some features of intuitionistic fuzzy subsets on an abundant semi-group in this study. Also presented the concept of intuitionistic fuzzy abundant semigroups, as well as some of its features.We also define homomorphism and investigate the image and inverse image of an intuitionistic fuzzy abundant semigroup when homomorphism is applied.

Keywords: Abundant semigroup, Intuitionistic fuzzy set, Intuitionistic fuzzy abundant semigroup, Homomorphism of intutionstic fuzzy abundant semigroup.
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1. Introduction

Lofti A. Zadeh [1] proposed the concept of fuzzy sets. Azriel Rosenfeld [2] ap- plied it to the fundamentals of group theory. He started the fuzzification of some algebraic structures using the concept of characteristics function. Various types of fuzzy semigroups have been presented by a number of writers. J B Fountain[3]studied the Abundant Semigroup, while Chunhua Li [4] extended the Abundant Semigroup to the Fuzzy Abundant Semigroup. Intuitionistic fuzzy sets, which are a generalisation of fuzzy sets, were developed by K Atanassov [5]. The idea of an intuitionistic fuzzy abundant semigroup is introduced, and the image and inverse image of intuitionistic fuzzy abundant semigroups are examined under homomorphism.

Email addresses: annaancy.09@gmail.com (Ancy Varghese), pibmct@gmail.com (Paul Isaac)

2. Preliminaries

Definition 2.1. *[6] A semigroup* $(S, .)$ *is a set* S *together with an associative binary operation* $(.)$ *on* S*. If the semigroup* S *does not have an identity then it is easy to adjoin an element* 1 *to* S *to form a monoid and is denoted as* S^1*. We define* $1.s = s = s.1$ *for all* $s \in S$*, and* $1.1 = 1$.

Definition 2.2. *[6] A non-empty subset T of a semigroup S is called a sub-semigroup if* $T^2 \subseteq T$.
That is $\forall x, y \in T, xy \in T$.

Definition 2.3. *[6] A non-empty subset A of a semigroup S is called a left idealif* $SA \subseteq A$ *and a right ideal if* $AS \subseteq A$.
A is said to be an ideal if it is both left ideal and right ideal. Every ideal is a subsemigroup but the converse is not true.

The notion of ideal leads to the consideration of certain equivalence relations on a semigroup.These equivalences first studied by James Alexander Green[1951] have played a fundamental role in the development of semigroup theory.

If a is an element of a semigroup S, the smallest left ideal of S containing a is $Sa \cup a$ and we denote it by S^1a and call it as principal left ideal of Sgenerated by a. Similarly aS^1 is the principal right ideal of S generated by a. The principal two-sided ideal of S generated by a is denoted by S^1aS^1.

Definition 2.4. *[6] Green's Equivalence relations on a semigroup S is definedas follows:*

$\forall a, b \in S.$

(1) $aLb \Leftrightarrow S^1a = S^1b$

(2) $aRb \Leftrightarrow aS^1 = bS^1$

(3) $aJb \Leftrightarrow S^1aS^1 = S^1bS^1$

If $L \circ R = R \circ L$, then we have the equivalence relations $D = L \circ R = R \circ L = L \vee R$ and $H = L \cap R$

The relations L^* and R^* defined below are the generalization of Green's relations L and R. They are known as Generalized Green's relations.

Definition 2.5. *[3] For* $a, b \in S$ $aL^*b \Leftrightarrow \forall x, y \in S^1, ax = ay \Leftrightarrow bx = by aR^*b$
$\Leftrightarrow \forall x, y \in S^1, xa = ya \Leftrightarrow xb = yb$
$aJ^*b \Leftrightarrow \forall x, y \in S^1, axa = aya \Leftrightarrow bxb = byb$

The relations L^* and R^* are equivalence relations, L^*_a is the L^*-class containing a and R^*_a is the R^*-class containing a. L^* is a right congruence and R^* is a left congruence relation on a semigroup.

Lemma 2.6. *[3] If e is an idempotent in a semigroup S for* $a \in S$. *Then*

(1) $aL^*e \Leftrightarrow ae = a$ $\forall x, y \in S^1$, $ax = ay \Rightarrow ex = ey$

(2) $aR^*e \Leftrightarrow ea = a$ $\forall x, y \in S^1$,$xa = ya \Rightarrow xe = ye$

An idempotent element in an abundant semigroup act as a right identity within L^*-class and left identity within R^*-class. We denote by a^* and a^+ the idempotents in L^*_a and R^*_a respectively.

We have intersection of two equivalence relation is an equivalence relation. Hence $H^* = L^* \cap R^*$ is an equivalence relation and H^*_a is the H^*-class containing a .

Definition 2.7. *[3] A semigroup in which each L^*-class and R^*-class contains atleast one idempotent is called an abundant semigroup.*

Definition 2.8. *[7] A map $\varphi : S \to T$ where $(S, .)$ and $(T, *)$ are semigroups is called a homomorphism if for all $x, y \in S$*
$\varphi(x.y) = \varphi(x) * \varphi(y)$

Definition 2.9. *[7] A homomorphism φ from an abundant semigroup S into a semigroup T is good if for all $x, y \in S$*
aL^*b *in* $S \Rightarrow \varphi(a)L^*\varphi(b)$ *in* T
aR^*b *in* $S \Rightarrow \varphi(a)R^*\varphi(b)$ *in* T

Definition 2.10. *[7] Let S be a semigroup $E(S)$ denotes the set of idempotent in S. A semigroup S is said to satisfy regularity condition if $E(S)$ is a subsemi-group of S.*

Lemma 2.11. *[7] Let S be an abundant semigroup satisfying the regularity condition, let φ be a good homomorphism from S into a semigroup T and let $\varphi(a)$ be an idempotent of T for some a of S. Then there is an idempotent $e \in E(S)$ such that $\varphi(a) = \varphi(e)$.*

Definition 2.12. *[1] Let X be a non-empty set. A fuzzy subset A of X is defined as $A = \{(x, \mu_A) : x \in X\}$ where μ_A is the membership function from X into* $[0, 1]$.

Definition 2.13. *[1] Let X be a non-empty set, and let A be a fuzzy subset of X, $t \in [0, 1]$, $A^t = \{x \in X : \mu_A(x) \geq t\}$. Then A^t is called a $t-$ cut. Also $A^{t+} = \{x \in X : \mu_A(x) > t\}$ is called a strong $t-$ cut.*

Definition 2.14. *[8] Let A, B be two fuzzy subsets of a set X with membership functions μ_A and μ_B respectively. We have the following definitions:*

(1) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)\ \forall x \in S$,

(2) $(A \cap B)(x) = \mu_A(x) \wedge \mu_B(x)$; $(A \cup B)(x) = \mu_A(x) \vee \mu_B(x)$

Definition 2.15. *[8] Let A, B be two fuzzy subsets of a semigroup S with membership functions μ_A and μ_B respectively. We define*

$$(A \circ B)(x) = \begin{cases} \bigvee_{x=yz} \{\mu_A(y) \wedge \mu_B(z)\} & \text{if } \exists\, y, z \in S, x = yz \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.16. *[8] A fuzzy subset* $A = \{(x, \mu_A): x \in S\}$ *of a semigroup S is called a fuzzy subsemigroup of S, if* $\mu_A(ab) \geq min(\mu_A(a), \mu_A(b))\ \forall\ a, b \in S$.

Definition 2.17. *[4] Let* $A = \{(x, \mu_A): x \in S\}$ *be a fuzzy subsemigroup of an abundant semigroup* S. *If there are* $a^+ \in R^*_a, a^* \in L^*_a$ *such that* $\mu_A(a^+) \geq \mu_A(a), \mu_A(a^*) \geq \mu_A(a)\ \forall a \in S$ *then* A *is called a fuzzy abundant subsemigroup of S.*

Theorem 2.18. *[4] Let* $A = \{(x, \mu_A): x \in S\}$ *be a fuzzy subset of an abundantsemigroup S then following properties hold*

(1) $A = \{(x, \mu_A) : x \in S\}$ *is a fuzzy abundant subsemigroup of S if and only if* A^t *is an abundant subsemigroup of S* $\forall t \in [0, 1]$

(2) $A = \{(x, \mu_A) : x \in S\}$ *is a fuzzy abundant subsemigroup of S if and only if* A^{t+} *is an abundant subsemigroup of S* $\forall t \in [0, 1]$

Definition 2.19. *[5] An intuitionistic fuzzy set A defined on a non-empty set* X *is given by* $A = \{(x, \mu_A, \nu_A): x \in X\}$. *The functions* $\mu_A : X \to [0, 1]$, $\nu_A : X \to [0, 1]$ *denote degree of membership and degree of non-membership of theelement* $x \in X$ *to the set* A *respectively and* $0 \leq \mu_A(x) + \nu_A(x) \leq 1\ \forall x \in X$. *Simply we denote an intuitionistic fuzzy subset A of X by* (μ_A, ν_A).

Definition 2.20. *[9] Let A be an intuitionistic fuzzy set of a universe set X. Then* (α, β)*-cut of A is denoted by* $C_{\alpha,\beta}(A)$ *and is given by*
$C_{\alpha,\beta}(A) = \{x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$, *where* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$

Definition 2.21. *[10] Let X and Y be two non-empty sets and* $\varphi : X \to Y$ *be a mapping. Let A and B be intuitionistic fuzzy subsets of X and Y respectively. Then the image of A under the map* φ *is denoted by* $\varphi(A)$ *and is defined as*
$\varphi(A)(y) = (\mu_{\varphi(A)}(y), \nu_{\varphi(A)}(y))$

where

$$\mu_{\varphi(A)}(y) = \begin{cases} \vee \{\mu_A(x) \mid x \in X, \varphi(x) = y\}, & \text{if } \varphi^{-1}(y) \,/\!= 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{\varphi(A)}(y) = \begin{cases} \wedge \{\nu_A(x) \mid x \in X, \varphi(x) = y\}, & \text{if } \varphi^{-1}(y) \,/\!= 0 \\ 1 & \text{otherwise} \end{cases}$$

Also the pre-image of B under φ is denoted by $\varphi^{-1}(B)(x)$ and is given by $\varphi^{-1}(B)(x) = (\mu_{\varphi^{-1}(B)}(x), \nu_{\varphi^{-1}(B)}(x))$
where $\mu_{\varphi^{-1}(B)}(x) = \mu_B(\varphi(x)),\ \nu_{\varphi^{-1}(B)}(x) = \nu_B(\varphi(x))$

3. Intuitionistic Fuzzy subset of an Abundant Semigroup

In this section we deal with intuitionistic fuzzy subsets of an abundant semi-group. We provide definitions and results for intuitionistic fuzzy subsets of an abundant semigroup that are similar to those presented for regular semigroups in [11].

Definition 3.1. *Let S be an semigroup and let A* $=(\mu_A, \nu_A)$ *be an intuitionistic fuzzy subset of S. Then A is called an intuitionistic fuzzy subsemi- group of S if for any* $x, y \in S$, $\mu_A(xy) \geq min(\mu_A(x), \mu_A(y))$ *and* $\nu_A(xy) \leq max(\nu_A(x), \nu_A(y))$.

Remark 3.2. *Note that an semigroup S can be considered as an intuitionistic fuzzy subset of itself.* $S = (\mu_S, \nu_S)$ *where* $\mu_S(x) = 1, \nu_S(x) = 0 \forall x \in S$.

Definition 3.3. *Let A be an intuitionistic fuzzy subset of an abundant semi-group S. Then A is called an*

(1) *intuitionistic fuzzy left ideal of of S, if* $\mu_A(ab) \geq \mu_A(b)$ *and* $\nu_A(ab) \leq \nu_A(b)$ *For all* $a, b \in S$.

(2) *intuitionistic fuzzy right ideal of of S, if* $\mu_A(ab) \geq \mu_A(a)$ *and* $\nu_A(ab) \leq \nu_A(a)$ *for all* $a, b \in S$.

(3) *Intuitionistic fuzzy ideal of of S, if it is both intuitionistic fuzzy left idealand intuitionistic fuzzy right ideal of S.*

Proposition 3.4. *Let A be a non-empty subset of an abundant semigroup S and let* χ_A *be the characteristic function of A.*

(1) *A is a left ideal of S if and only if* (χ_A, χ_{A^c}) *is an intuitionistic fuzzy leftideal of S.*

(2) *A is a right ideal of S if and only if* (χ_A, χ_{A^c}) *is an intuitionistic fuzzy rightideal of S.*

(3) *A is an ideal of S if and only if* (χ_A, χ_{A^c}) *is an intuitionistic fuzzy ideal of S.*

Proposition 3.5. *Let A* $= (\mu_A, \nu_A)$ *be an intuitionistic fuzzy subset of an abundant semigroup S. Then*

(1) $A = (\mu_A, \nu_A)$ *is an intuitionistic fuzzy left ideal of* S *if and only if* $S \circ A \subseteq A$ *i,e* $(\mu_S \circ \mu_A \subseteq \mu_A, \nu_S \circ \nu_A \geq \nu_A)$

(2) $A = (\mu_A, \nu_A)$ *is an intuitionistic fuzzy right ideal of* S *if and only if* $A \circ S \subseteq A$ *i,e* $(\mu_A \circ \mu_S \subseteq \mu_A, \nu_A \circ \nu_S \geq \nu_A)$

(3) $A = (\mu_A, \nu_A)$ *is an intuitionistic fuzzy ideal of* S *if and only if* $A \circ S \subseteq A$ *and* $S \circ A \subseteq A$

4. Intuitionistic Fuzzy Abundant Subsemigroup

In this section we define intuitionistic fuzzy abundant subsemigroup of an abundant subsemigroup and derive some results in this direction.

Definition 4.1. *An intuitionistic fuzzy subset* $A = \{(x, \mu_A, \nu_A) : x \in S\}$ *of an abundant semigroup* S *is called an intuitionistic fuzzy abundant subsemigroup of* S*, if there are* $a^+ \in R^*_a, a^* \in L^*_a$ *such that* $\mu_A(a^+) \geq \mu_A(a), \mu_A(a^*) \geq \mu_A(a)$ *and* $\nu_A(a^+) \leq \nu_A(a), \nu_A(a^*) \leq \nu_A(a)\ \forall a \in S$.

Proposition 4.2. *Let* A *be a subset of a semigroup* S *and let* χ_A *be the char-acteristic function of* A*.* A *is an Abundant subsemigroup of* S *if and only if* (χ_A, χ_{A^c}) *is an intuitionistic fuzzy abundant subsemigroup of* S*.*

Definition 4.3. *Let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *are two intuitionistic fuzzy subset of a semigroup* S*. The product of* A *and* B *is an intuitionistic fuzzy subset* $A \circ B$ *defined as follows:* $A \circ B = \{(x, (\mu_A \circ \mu_B), (\nu_A \circ \nu_B)) : x \in S\}$ *where*

$$(\mu_A \circ \mu_B)(x) = \begin{cases} \vee_{x=yz}\{\mu_A(y) \wedge \mu_B(z)\} & \text{if } \exists\, y, z \in S, x = yz \\ 0 & \text{otherwise} \end{cases}$$

$$(\nu_A \circ \nu_B)(x) = \begin{cases} \wedge_{x=yz}\{\nu_A(y) \vee \nu_B(z)\} & \text{if } \exists\, y, z \in S, x = yz \\ 1 & \text{otherwise} \end{cases}$$

Lemma 4.4. *Let* $A = (\mu_A, \nu_A)$ *be an intuitionistic fuzzy subset of a semigroup* S*.* A *is an intuitionistic fuzzy subsemigroup if and only if* $\mu_A \circ \mu_A \subseteq \mu_A$ *and* $\nu_A \circ \nu_A \geq \nu_A$*. i.e. if and only if* $A \circ A \subseteq A$*.*

Proof. First we prove if $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subsemigroup ofa semigroup S then $\mu_A \circ \mu_A \subseteq \mu_A$ and $\nu_A \circ \nu_A \geq \nu_A$.

Suppose A is an intuitionistic fuzzy subsemigroup of a semigroup S. For $y, z \in S$ with $x = yz$, we have
$\mu_A(x) = \mu_A(yz) \geq \mu_A(y) \wedge \mu_A(z)$. $\therefore \mu_A(x) \geq \vee_{x=yz}\{\mu_A(y) \wedge \mu_A(z)\}$. i.e. $\mu_A(x) \geq (\mu_A \circ \mu_A)(x)$. Hence $\mu_A \geq (\mu_A \circ \mu_A)$.
Similarly we can prove $\nu_A \circ \nu_A \geq \nu_A$.

Conversely suppose $\mu_A \circ \mu_A \subseteq \mu_A$ and $\nu_A \circ \nu_A \geq \nu_A$. To prove that A is an intuitionistic fuzzy subsemigroup of a semigroup S. As $(\mu_A \circ \mu_A)(x) \leq \mu_A(x)$, for $x = yz$ with $y, z \in S$ we get
$\mu_A(x) \geq \vee_{x=yz}\{\mu_A(y) \wedge \mu_A(z)\} \geq \mu_A(y) \wedge \mu_A(z)$. That is $\mu_A(yz) \geq \mu_A(y) \wedge \mu_A(z)$. Similarly $\nu_A(yz) \leq (\nu_A(y) \vee \nu_A(z))$. Hence A is an intuitionistic fuzzy subsemigroup of the semigroup S.

■

Proposition 4.5. *Let S be an abundant semigroup. If A is an intuitionistic fuzzy abundant subsemigroup of S then* $\mu_A \circ \mu_A = \mu_A$ *and* $\nu_A \circ \nu_A = \nu_A$.

Proof. From lemma 4.4 we get $\mu_A \circ \mu_A \subseteq \mu_A$ and $\nu_A \circ \nu_A \geq \nu_A$. It is enoughto show $\mu_A \circ \mu_A \geq \mu_A$ and $\nu_A \circ \nu_A \subseteq \nu_A$. Let $a \in S$, if $\mu_A(a) = 0$, then $\mu_A \circ \mu_A(a) \geq \mu_A(a)$. If $\mu_A(a) \neq 0$, then there is $a^* \in S$ such that $\mu_A(a^*) \geq \mu_A(a)$. Since S is abundant, here $a^* \in L_a^*$ act as right identity and $a.a^* = a$. Thus $(\mu_A \circ \mu_A)(a) = \vee_{a=xy}\{\mu_A(x) \wedge \mu_A(y)\} \geq (\mu_A(a) \wedge \mu_A(a^*)) = \mu_A(a)$. Therefore $\mu_A \circ \mu_A = \mu_A$. Similarly we can prove $\nu_A \circ \nu_A = \nu_A$.

■

Proposition 4.6. *Let S be an abundant semigroup and A be an intuitionistic fuzzy abundant subsemigroup of S. Then A is an intuitionistic fuzzy left ideal of S if and only if* $S \circ A = A$, *i.e.* $\mu_S \circ \mu_A = \mu_A$ *and* $\nu_S \circ \nu_A = \nu_A$.

Proof. From proposition 3.5 we get $\mu_S \circ \mu_A \subseteq \mu_A$ and $\nu_S \circ \nu_A \geq \nu_A$. It is enough to show $\mu_S \circ \mu_A \geq \mu_A$ and $\nu_S \circ \nu_A \subseteq \nu_A$. Let $a \in S$, if $\mu_A(a) = 0$, then $\mu_S \circ \mu_A(a) \geq \mu_A(a)$. If $\mu_A(a)$ 0, then there are two cases

Case 1: $a^* \in L_a^*$

We have A is an intuitionistic fuzzy abundant subsemigroup of S such that $\mu_A(a^*) \geq \mu_A(a)$. Since S is abundant, here $a^* \in L_a^*$ act as right identity. Thus $\mu_S \circ \mu_A(a) = \vee_{z=xy}\{\mu_S(x) \wedge \mu_A(y)\} \geq (\mu_S(a) \wedge \mu_A(a^*)) = (1 \wedge \mu_A(a^*)) = \mu_A(a^*) \geq \mu_A(a)$.

Case 2: $a^{+} \in \boldsymbol{R}_a^{*}$

We have A is an intuitionistic fuzzy abundant subsemigroup of S such that $\mu_A(a^{+}) \geq \mu_A(a)$. Since S is abundant, here $a^{+} \in \boldsymbol{R}_a^{*}$ act as left identity. Thus $\mu_S \circ \mu_A(a) = \vee_{z=xy}\{\mu_S(x) \wedge \mu_A(y)\} \geq (\mu_S(a^{+}) \wedge \mu_A(a)) = (1 \wedge \mu_A(a)) = \mu_A(a)$. Therefore $\mu_S \circ \mu_A = \mu_A$. Similarly we can prove $\nu_S \circ \nu_A = \nu_A$.

■

Now we state the following result, the proof of which is similar to the above.

Proposition 4.7. *Let S be an abundant semigroup and A be an intuitionistic fuzzy abundant subsemigroup of S. Then A is an intuitionistic fuzzy right ideal of S if and only if* $A \circ S = A$, *i.e.* $\mu_A \circ \mu_S = \mu_A$ *and* $\nu_A \circ \nu_S = \nu_A$.

Now the following proposition is an immediate consequence of the above two results.

Proposition 4.8. *Let S be an abundant semigroup and A be an intuitionistic fuzzy abundant subsemigroup of S. Then A is an intuitionistic fuzzy ideal of S if and only if* $A \circ S = S \circ A$

5. (α, β)-cut of Intuitionistic Fuzzy set

Definition 5.1. *Let* $A = (\mu_A, \nu_A)$ *be an intuitionistic fuzzy subsemigroup of asemigroup S. Then* (α, β)*-cut of* A *is denoted by* $C_{\alpha,\beta}(A)$ *and is given by* $C_{\alpha,\beta}(A) = \{x \in S : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$, *where* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$

Proposition 5.2. *Let S be an abundant semigroup and let A be an intuitionistic fuzzy abundant subsemigroup of S if and only if for any* $\alpha, \beta \in [0, 1]$, (α, β)*-cut* $C_{\alpha,\beta}(A)$ *is an abundant subsemigroup of S.*

Proof. Given that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy abundant subsemigroup of S. To prove $C_{\alpha,\beta}(A)$ is a subsemigroup of S. Let $x, y \in C_{\alpha,\beta}(A)$ then $\mu_A(x) \geq \alpha$, $\mu_A(y) \geq \alpha$, $\nu_A(x) \leq \beta$ and $\nu_A(y) \leq \beta$. Since $A = (\mu_A, \nu_A)$is an intuitionistic fuzzy subsemigroup of S $\mu_A(xy) \geq (\mu_A(x) \wedge \mu_A(y))$ and $\nu_A(xy) \leq (\nu_A(x) \vee \nu_A(y))$. This implies $\mu_A(xy) \geq \alpha$ and $\nu_A(xy) \leq \beta$. Thus $xy \in C_{\alpha,\beta}(A)$. Hence $C_{\alpha,\beta}(A)$ is a subsemigroup of S.

Now it is enough to prove $C_{\alpha,\beta}(A)$ is abundant. Let $a \in C_{\alpha,\beta}(A)$ then $\mu_A(a) \geq \alpha$, $\nu_A(a) \leq \beta$. As A is intuitionistic fuzzy abundant subsemigroup of S, we have $\mu_A(a^{+}) \geq \mu_A(a) \geq \alpha$ $\mu_A(a^{*}) \geq \mu_A(a) \geq \alpha$ and $\nu_A(a^{+}) \leq \nu_A(a) \leq \beta$ $\nu_A(a^{*}) \leq \nu_A(a) \leq \beta$, for $a^{+} \in \boldsymbol{R}_a^{*}$, $a^{*} \in \boldsymbol{L}_a^{*}$ That is $a^{+}, a^{*} \in C_{\alpha,\beta}(A)$ and hence

$C_{\alpha,\beta}(A)$ is abundant subsemigroup of S for all $\alpha, \beta \in [0, 1]$with $\alpha + \beta \leq 1$.
Conversely assume that $C_{\alpha,\beta}(A)$ is a non-empty abundant subsemigroup of S, to prove A is an intuitionistic abundant subsemigroup of S.

We first prove A is a subsemigroup of S. Let $a, b \in S$ with $\mu_A(a) \geq \alpha$ $\mu_A(b) \geq \alpha$ $\nu_A(a) \leq \beta$ $\nu_A(b) \leq \beta$
ie $a, b \in C_{\alpha,\beta}(A)$, as $C_{\alpha,\beta}(A)$ is subsemigroup $ab \in C_{\alpha,\beta}(A)$
ie $\mu_A(ab) \geq min(\mu_A(a), \mu_A(b)$ also $\nu_A(ab) \leq max(\nu_A(a), \nu_A(b)$
Hence A is an intuitionistic fuzzy subsemigroup of S. Now to prove A is abun-dant. Let $a, b \in S$, set $\alpha = \mu_A(a), \beta = \nu_A(a)$.
Since $C_{\alpha,\beta}(A)$ is an abundant subsemigroup of S there are $a^+ \in R^*_a, a^* \in L^*_a$ such that
$\mu_A(a^+) \geq \alpha$, $\mu_A(a^*) \geq \alpha$,$\nu_A(a^+) \leq \beta$, $\nu_A(a^*) \leq \beta$
$\mu_A(a^+) \geq \mu_A(a)$, $\mu_A(a^*) \geq \mu_A(a)$, $\nu_A(a^+) \leq \mu_A(a)$, $\nu_A(a^*) \leq \mu_A(a)$ for all $a \in S$
Hence A is an intuitionistic fuzzy abundant subsemigroup of S. ■

6. Homomorphism of intuitionistic fuzzy abundant subsemigroup

Definition 6.1. *Let X and Y be two non-empty sets and φ: $X \to Y$ be a mapping. Let A and B be intuitionistic fuzzy abundant subsemigroup of X and Y respectively. Then the image of A under the map φ is denoted by $\varphi(A)$ and is defined as*
$\varphi(A)(y) = (\mu_{\varphi(A)}(y), \nu_{\varphi(A)}(y))$
where

$$\mu_{\varphi(A)}(y) = \begin{cases} \vee\{\mu_A(x) \mid x \in X, \varphi(x) = y\}, & if \varphi^{-1}(y) /= 0 \\ 0 & , otherwise \end{cases}$$

and

$$\nu_{\varphi(A)}(y) = \begin{cases} \wedge\{\nu_A(x) \mid x \in X, \varphi(x) = y\}, & if \varphi^{-1}(y) /= 0 \\ 1 & , otherwise \end{cases}$$

Also the pre-image of B under φ is denoted by $\varphi^{-1}(B)(x)$ and is given by
$\varphi^{-1}(B)(x) = (\mu_{\varphi^{-1}(B)}(x), \nu_{\varphi^{-1}(B)}(x))$
where $\mu_{\varphi^{-1}(B)}(x) = \mu_B(\varphi(x))$, $\nu_{\varphi^{-1}(B)}(x) = \nu_B(\varphi(x))$

Theorem 6.2. *Let φ be a surjective good homomorphism from an abundant semiroup S satisfying regularity condition onto a semigroup T.Then the following statements are true:*

1. *If A is an intuitionistic fuzzy abundant subsemigroup of S then $\varphi(A)$ is an intuitionistic fuzzy abundant subsemigroup of T.*
2. *If B is an intuitionistic fuzzy abundant subsmigroup of T then $\varphi^{-1}(B)$ is an intuitionistic fuzzy abundant subsemigroupof S.*

Proof. (1) Given that A is an intuitionistic fuzzy abundant subsemigroup of S to prove $\varphi(A)$ is an intuitionistic fuzzy abundant subsemigroup of T . From Proposition 5.2 it is enough to prove $C_{\alpha,\beta}\varphi(A)$ is an abundant subsemigroup of T , for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.
Suppose that there is $\alpha_0, \beta_0 \in [0, 1]$ such that $C_{\alpha_0,\beta_0}\varphi(A)$ is not anabundant subsemigroup of T.
Then there is $x \in C_{\alpha_0,\beta_0}(\varphi(A))$ such that $x^+ \in T$, but $x^+ \notin C_{\alpha_0,\beta_0}(\varphi(A))$
ie $\mu_{\varphi(A)}(x) \geq \alpha_0$, $\nu_{\varphi(A)}(x) \leq \beta_0$
but $\mu_{\varphi(A)}(x^+) < \alpha_0, \nu_{\varphi(A)}(x^+) > \beta_0$ for all $x^+ \in R^*_x$.
Similarly for $x^* \in L^*_x$.
Since φ is surjective then $\varphi(S) = T$
For $x \in T \Rightarrow x = \varphi(a)$ for some $a \in S$ and $x^+ \in T \Rightarrow x = \varphi(b)$ for some $b \in S$

Hence $\mu_{\varphi(A)}(x) = \vee_{a\in\varphi^{-1}(x)}\{\mu_A(a)\} \geq \alpha_0$ and
$\nu_{\varphi(A)}(x) = \wedge_{a\in\varphi^{-1}(x)}\{\nu_A(a)\} \leq \beta_0$
Also $\mu_{\varphi(A)}(x^+) = \vee_{b\in\varphi^{-1}(x^+)}\{\mu_A(b)\} < \alpha_0$ and $\nu_{\varphi(A)}(x^+) = \wedge_{b\in\varphi^{-1}(x^+)}\{\nu_A(x)\} > \beta_0$
Choose an element $\alpha, \beta \in [0, 1]$ such that $0 < \alpha < \alpha_0, 0 < \beta_0 < \beta$. We have
$\vee_{a\in\varphi^{-1}(x)}\{\mu_A(a)\} \geq \alpha_0$
$\wedge_{a\in\varphi^{-1}(x)}\{\nu_A(a)\} \leq \beta_0$
$\vee_{b\in\varphi^{-1}(x^+)}\{\mu_A(b)\} < \alpha_0$
$\wedge_{b\in\varphi^{-1}(x^+)}\{\nu_A(b)\} > \beta_0$.
Thus there is $a \in S$ such that $\varphi(a) = x$ and $\mu_A(a) \geq \alpha$, $\nu_A(a) \leq \beta$
$\Rightarrow a \in C_{\alpha,\beta}(A)$ and $C_{\alpha,\beta}(A)$ non empty.
Also for all $x^+ \in R^*_x$ we have $x^+ = \varphi(b)$ and $\mu_A(b) < \alpha$, $\nu_A(b) > \beta$
$\Rightarrow b \notin C_{\alpha,\beta}(A)$
Since φ is a surjective homomorphism from an abundant semiroup S satisfying

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regularity condition onto a semigroup T, by Lemma 2.11 there is an idempotent $a^+ \in E(S)$ such that $\varphi(a^+) = \varphi(b) = x^+$

Hence for all $a^+ \in \boldsymbol{R}^*_a \mu_A(a^+)$

$< \alpha, \nu_A(a^+) > \beta$

$\Rightarrow a^+ \notin C_{\alpha,\beta}(A)$

Thus $C_{\alpha,\beta}(A)$ is not an abundant semigroup of S. By Poposition 5.2 A is not an intuitionistic abundant subsemigroup of S which is a contradiction.

Therefore for all $\alpha, \beta \in [0, 1], \alpha + \beta \leq 1, C_{\alpha,\beta}(\varphi(A))$ is an intuitionistic fuzzy abundant subsemigroup of T.

Hence $\varphi(A)$ is an abundant subsemigroup of S.

(2) Now assume that B is an intuitionistic fuzzy abundant subsemigroup of T to prove $\varphi^{-1}(B)$ is an intuitionstic fuzzy abundant subsemigroup S.

As B is an intuitionistic fuzzy abundant subsemigroup of T then for all $y \in T$ with $\mu_B(y) /= 0$ there are y^+, y^* such that $\mu_B(y^+) \geq \mu_B(y), \nu_B(y^+) \leq \nu_B(y) \mu_{\boldsymbol{B}}(y^*) \geq \mu_{\boldsymbol{B}}(y), \nu_{\boldsymbol{B}}(y^*) \leq \nu_{\boldsymbol{B}}(y)$

Given φ is surjective good homomorphism for all $y \in T$ there is $a \in S$ such that $\varphi(a) = y$.

For all $a \in S$ $\mu_B[\varphi(a)] /= 0$ then there are $[\varphi(a)]^+, [\varphi(a)]^* \in T$ such that

$\mu_{\boldsymbol{B}}[\varphi(a)]^+ \geq \mu_{\boldsymbol{B}}[\varphi(a)], \mu_{\boldsymbol{B}}[\varphi(a)]^* \geq \mu_{\boldsymbol{B}}[\varphi(a)]$

$\nu_{\boldsymbol{B}}[\varphi(a)]^+ \leq \nu_{\boldsymbol{B}}[\varphi(a)], \nu_{\boldsymbol{B}}[\varphi(a)]^* \leq \nu_{\boldsymbol{B}}[\varphi(a)]$

Since φ is a good homomorphism by Lemma2.11 for all $[\varphi(a)]^+ \in \boldsymbol{R}^*_{\varphi(a)}$ there is $a^+ \in \boldsymbol{R}^*_a$ such that $[\varphi(a)]^+ = \varphi(a^+)$ and $a^* \in \boldsymbol{L}^*_a$ such that $[\varphi(a)]^* = \varphi(a^*)$ $\mu_B[\varphi(a)]^+$

$= \mu_B[\varphi(a^+)] \geq \mu_B[\varphi(a)]$

$\mu_{\boldsymbol{B}}[\varphi(a)]^* = \mu_{\boldsymbol{B}}[\varphi(a^*)] \geq \mu_{\boldsymbol{B}}[\varphi(a)]$

$\nu_B[\varphi(a)]^+ = \nu_B[\varphi(a^+)] \leq \nu_B[\varphi(a)]$

$\nu_{\boldsymbol{B}}[\varphi(a)]^* = \nu_{\boldsymbol{B}}[\varphi(a^*)] \leq \nu_{\boldsymbol{B}}[\varphi(a)]$

But $\mu_{\varphi^{-1}(B)}(a^+) = \mu_B[\varphi(a^+)]$ and $\nu_{\varphi^{-1}(B)}(a^+) = \nu_B[\varphi(a^+)]$

ie $\mu_{\varphi^{-1}(B)}(a^+) \geq \mu_{\varphi^{-1}(B)}(a)$

$\mu_{\varphi^{-1}(\boldsymbol{B})}(a^*) \geq \mu_{\varphi^{-1}(\boldsymbol{B})}(a)$

$\nu_{\varphi^{-1}(B)}(a^+) \leq \nu_{\varphi^{-1}(B)}(a)$

$\nu_{\varphi^{-1}(B)}(a^+) \leq \nu_{\varphi^{-1}(B)}(a)$

$\Rightarrow \varphi^{-1}(B)$ is an intuitionistic fuzzy abundant subsemigroup of S.

Hence the theorem.

■

Conclusion

In this article we tried to introduce some concepts about intuitionistic fuzzy abundant subsemigroups and also study the image and inverse image of intuitionistic fuzzy abundant subsemigroups under homomorphism.There are much more to be done.

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