A collection of visual trigonometric proofs

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Abstract: We present a collection of new visual proofs of the principal trigonometric relationships, that allow covering trigonometry in continuity with the topics of Euclidean geometry, without having to give up on demonstrations. With this work we want also to highlight the importance of geometric reasoning in the didactic field, as its formative value seems even greater, in particular with distance learning. As a matter of fact, in this condition it becomes unrealistic to keep young students' attention focused on long sequences of formal passages, whereas it would be more effective to stimulate them to think about geometric figures.

Keywords: Trigonometric identities, trigonometric formulas, trigonometric relationships, visual proofs

1. Introduction

The development of trigonometric knowledge is a fundamental milestone in mathematical training, especially for its multiple applications; moreover, recent results show its use also in research (see for example [1]). However, the learning outcomes are strongly influenced by the teaching approach, which in some cases points towards the application of trigonometric relations, almost entirely omitting proofs, and in other cases, emphasizes algebraic-formal aspects at the risk of losing students' attention.

Elementary trigonometric relationships are introduced by using simple geometric reasoning, based on congruence and similarity between triangles. It happens for the relation between the sides and the angles of a right triangle, for the reduction formulas, and the trigonometric ratios of particular angles. In the development of the theory, the chord length formula and the law of sines are also obtained geometrically, while the proofs of other relationships use algebraic calculations. However, we can easily prove all the leading trigonometric relationships through Euclidean geometry. The proof of the law of the cosine shown in the following Figure 1 provides the first example. In fact, from the two secants theorem, we have $(a+b) : c = d : (a-b)$, i.e. $a^2-b^2 = cd$. Since $c-d = 2b\cos(\alpha)$, it follows $a^2=b^2+c^2-2bc\cos(\alpha)$.

$$a^2 = b^2 + c^2 - 2bc \cdot cos\alpha$$

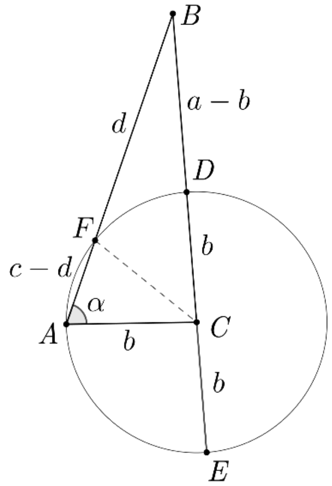

Figure 1: The law of cosines

The aim of this work is to provide teachers with useful tools and demonstrations to develop a complete trigonometric path in continuity with Euclidean geometry. In such a path, already planned and tested in the classroom visual proofs play a fundamental role (see [6] for the methodological references and first results). However, although many useful "visual proof"' can be found in the literature (see [2, p. 270], [3, pp. 75-79], [4], [5, p. 56], [8, p. 145-146], [9 p. 40, 46], [10, p. 62, 63, 66], [11], [12], [13], [14], [15], [16], [17] and [18]), a complete collection of proof of all leading trigonometric relationships is still missing. Therefore, In the following, along the lines of the research started in [6], we present a collection of new geometric proofs.

2. Some visual proofs

In this section some geometric demonstrations of the main trigonometric identities are developed. It should be noted that in order to understand these proofs it is necessary that students know how to apply the trigonometric relationships of right triangles. It is also necessary that they know the main theorems of Euclidean geometry, in particular the similarity criteria and some of their consequences, such as the chord theorem, the law of sines and Euclid's theorems.

Proposition 2.1. - Angle sum identities

$$\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta), \quad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) -\sin(\alpha)\sin(\beta)$$

Proof. Given the angle $\alpha+\beta$ in Figure 2, we draw a perpendicular to the common side at angles α and β. Let R be the circumradius of the triangle obtained, by using the formula for the chord length, we have: $c=2R\sin(\alpha+\beta)$. On the other hand, we have:

$$c = b\sin(\beta)+ a\sin(\alpha) = 2R\sin(\pi/2-\alpha)\sin(\beta)+2R\sin(\pi/2-\beta)\sin(\alpha) = 2R(\cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)\cos(\beta)).$$

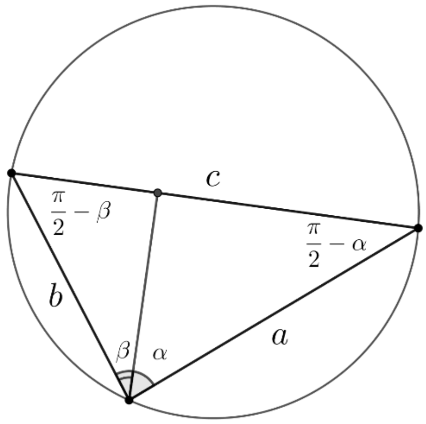

Figure 2: Angle sum identities

Similarly, from $a\cos(\alpha) = b\cos(\beta)$ we have $a^2\cos^2(\alpha) = b^2\cos^2(\beta) = ab\cos(\alpha)\cos(\beta)$. Then $c^2 = (a\sin(\alpha)+b\sin(\beta))^2 = a^2(1-\cos^2(\alpha))+b^2(1-\cos^2(\beta))+2ab\sin(\alpha)\sin(\beta) = a^2+b^2-2ab\cos(\alpha)\cos(\beta)+2ab\sin(\alpha)\sin(\beta)$. The thesis follows by $c^2 = a^2+b^2-2ab\cos(\alpha+\beta)$.

□

Obviously the angle difference identities can be proved substituting β with $-\beta$ in the above formulas. Similarly the double-angle formulae can be proved substituting β with α in the above formulas. However, even for this last formula we propose a simple geometric proof. Other interesting proofs can be found in [3, pp. 75-79].

Proposition 2.2. - Double-angle formulae

$$\cos(2\alpha) = 1-2\sin^2(\alpha), \quad \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

Proof. With reference to Figure 3, let $AO=OB=OC=1$, then $\cos(2\alpha) = OH = 1-HB = 1-BC\sin(\alpha) = 1-2\sin^2(\alpha)$.

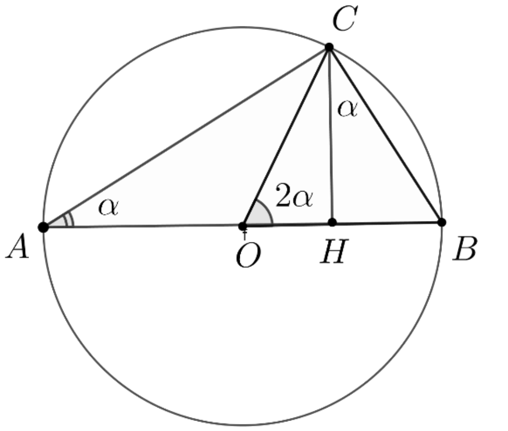

Figure 3: Double-angle formulae

Similarly, $\sin(2\alpha) = CH = AC\sin(\alpha) = AB\cos(\alpha)\sin(\alpha) = 2\sin(\alpha)\cos(\alpha)$. □

Obviously the Half-angle formulae can be proved substituting α with $\alpha/2$ in the above formulas.

Proposition 2.3. - Tangent half-angle formulae

$$\sin(\alpha) = 2\tan(\alpha/2)/(1+\tan^2(\alpha/2)), \qquad \cos(\alpha) = (1-\tan^2(\alpha/2))/(1+\tan^2(\alpha/2))$$

Proof. Let $AO=OB=OC=1$ and $DE \perp OD$ as shown in Figure 4, then $CH: DB = AB: OE$, i.e $\sin(2\alpha) : \tan(\alpha) = 2 : OE$. Moreover, by first Euclid's theorem, we have $OE:OD=OD:OB$, i.e. $OE = 1+\tan^2(\alpha)$. Then $\sin(2\alpha) : \tan(\alpha) = 2 : (1+\tan^2\alpha)$, i.e. $\sin(2\alpha) = 2\tan(\alpha)/(1+\tan^2(\alpha))$.

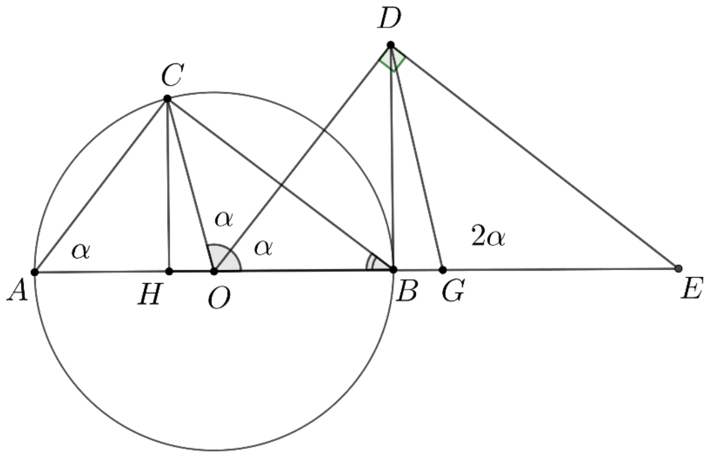

Figure 4: Tangent half-angle formulae for $\sin(2\alpha)$ and $\cos(2\alpha)$

Similarly, $OH : OC = GB : GD$, i.e. $|\cos(2\alpha)| : 1 = (OE/2-1) : (OE/2)$. Then $|\cos(2\alpha)| = 1-2/OE = 1-2/(1+\tan^2(\alpha)) = (\tan^2(\alpha)-1)/(\tan^2(\alpha)+1)$. The thesis follows by analyzing the signs and substituting α with $\alpha/2$ in the above formulas.

□

Proposition 2.4. - Product-to-sum identities or prosthaphaeresis formulae

$$\sin(\alpha)+\sin(\beta) = 2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2), \quad \cos(\alpha)+\cos(\beta) = 2\cos((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$$

$$\sin(\alpha)-\sin(\beta) = 2\cos((\alpha+\beta)/2)\sin((\alpha-\beta)/2), \quad \cos(\alpha)-\cos(\beta) = -2\sin((\alpha+\beta)/2)\sin((\alpha-\beta)/2)$$

Proof. With reference to Figure 5, being $AB = AC = 1$ and $AE \parallel DC$, we have $\theta = \pi-(\pi/2+\alpha)-(\pi-(\alpha+\beta))/2 = (\beta-\alpha)/2$. Moreover, for sines law, we have $AC / \sin(\alpha+\beta) = AB / \sin(\pi/2 - (\alpha+\beta)/2) = 1 / \cos((\alpha+\beta)/2)$, then $AC = \sin(\alpha+\beta) / \cos((\alpha+\beta)/2)$. Thus, using the sine double-angle formula, follows $AC = 2\sin((\alpha+\beta)/2)$. Therefore, from the figure we have: $\sin(\alpha)+\sin(\beta) = AC\cos(\theta) = AC\cos((\alpha-\beta)/2) = 2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$.

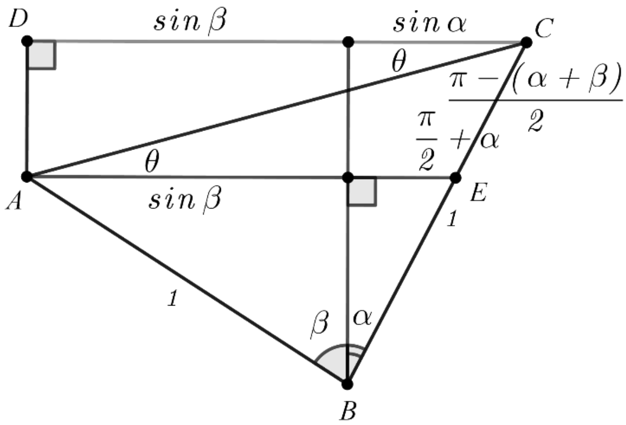

Figure 5: Product-to-sum identities or prosthaphaeresis formulae

The other identities can be proved the same way, replacing α with $\pi/2-\alpha$ or β with $\pi/2-\beta$. □

Obviously the sum-to-product formulae can be proved substituting α with $\gamma+\delta$ and β with $\gamma-\delta$ in the above formulae.

Moreover, applying to a generic triangle of sides a, b, c and opposite angles α, β and γ, the law of sines and the above formulas, we can also deduce the Law of Tangents. In fact, $a : b = \sin(\alpha) : \sin(\beta)$, then

$$(a+b) : (a-b) = (\sin(\alpha) + \sin(\beta)) : (\sin(\alpha) - \sin(\beta)) =$$

$$2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2) / 2\cos((\alpha+\beta)/2)\sin((\alpha-\beta)/2) = \tan(\alpha+\beta)/2 / \tan(\alpha-\beta)/2.$$

In conclusion $(a+b) / (a-b) = \tan(\alpha+\beta)/2 / \tan(\alpha-\beta)/2$.

Proposition 2.5. - The linear combination, or harmonic addition, of sine and cosine $a\sin(x) + b\cos(x) = (a^2+b^2)^{1/2}\sin(x+\theta)$, with $\cos(\theta) = a / (a^2+b^2)^{1/2}$ and $\sin(\theta) = b / (a^2+b^2)^{1/2}$.

Proof. Considering the following Figure 6, being the angle EBA equal to the angle BFC, that equals $\pi-(x+\theta)$, we have $EA=AB\sin(x+\theta)$.

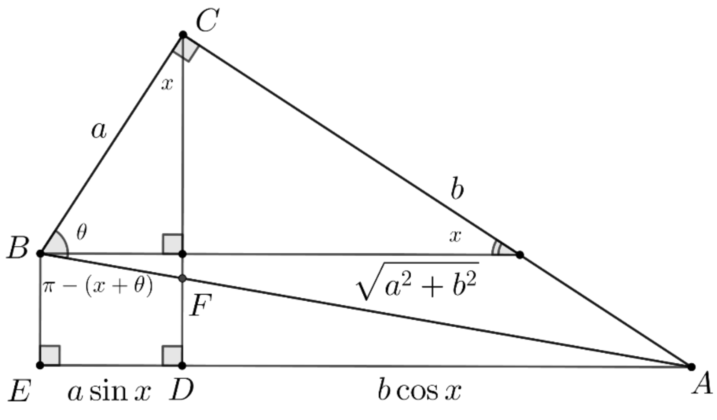

Figure 6: The linear combination of sine and cosine

□

The following formulas adopt their name from the German mathematician and astronomer Karl Brandan Mollweide (1774–1825) [17]. However some authors even attribute them to Sir Isaac Newton (1642–1726) [12, p. 63].

Proposition 2.6. - Mollweide's - Newton's formulas

$$\sin((\alpha-\beta)/2) \,/\, \cos(\gamma/2) = (a-b) \,/\, c, \qquad \cos((\alpha-\beta)/2 \,/\, \sin(\gamma/2) = (a+b) \,/\, c.$$

Proof. By analyzing the following figure we have: $c\cos((\alpha-\beta)/2) = 2b\sin(\gamma/2) + (a-b)\sin(\gamma/2) = (a+b)\sin(\gamma/2)$ and $c\sin((\alpha-\beta)/2) = (a-b)\cos(\gamma/2)$.

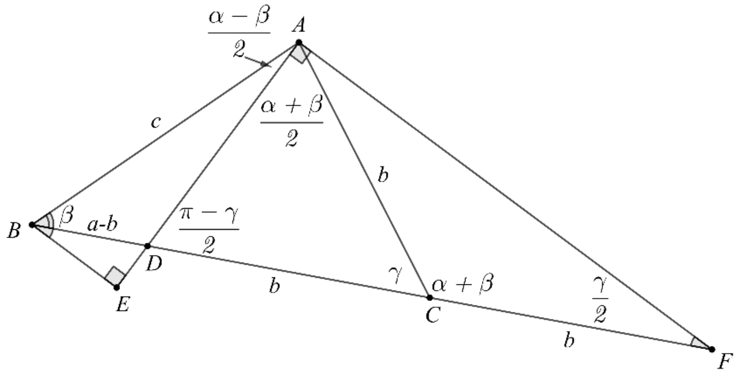

Figure 7: Mollweide's - Newton's formulas

3. Conclusions

With this work we want also to highlight the importance of geometric reasoning in the didactic field, that in the last decade the teaching of Euclidean geometry in secondary schools seems to have lost its vigor. Yet, this part of mathematics is considered fundamental in the development of students' logical abilities by many authors [7]. At present, the formative value of geometry seems even greater, in particular with distance learning. As a matter of fact, in the latter case it becomes unrealistic to keep young students' attention focused on long sequences of formal passages, whereas it would be more effective to stimulate them to think about geometric figures.

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