# Mild solutions for perturbed evolution equations with finite state-dependent delay

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**Abstract:** In this paper, we give sufficient conditions when the solution is depending on an finite delay to get the existence of mild solutions for two classes of first order perturbed semilinear functional and neutral functional fractional evolution equations by using the nonlinear alternative of Frigon and Granas for contractions maps in Frechet spaces.An example is provide to illustrate the theory.

**Keywords:** Perturbed semilinear functional equations, neutral problem, mild solution, state-dependent delay, fixed point, nonlinear alternative, semigroup theory, Frechet spaces, finite delay.

## 1. Introduction

In this paper, we give the existence of mild solutions defined on a positive real interval $J := [0, b]$ for two classes of first order of semilinear functional and neutral functional perturbed evolution equations with state-dependent delay in a real separable Banach space $(E, |\cdot|)$ when the delay is finite. Firstly, we present some preliminary concepts and results in Section 2 and then in Section 3 we study the following semilinear functional perturbed evolution equations with state-dependent delay

$$y'(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad p.p. \quad t \in J, \tag{1}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{2}$$

where $f, h: J \times C(H; E) \longrightarrow E$, $\rho: J \times C(H; E) \longrightarrow R$ and $\varphi \in C(H; E)$ are given functions and $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from $E$ into $E$ that generate an evolution system of operators $\{U(t,s)\}_{(t,s) \in J \times J}$ for $0 \le s \le t < +\infty$.

For any continuous function $y$ defined on $H$ and any $t \in J$, we denote by $y_t$ the element of $C(H; E)$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in H$. Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time $t$.

Then, in Section 4, we consider the following perturbed neutral functional differential evolution equation with finite delay

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad p.p. \quad t \in J, \tag{3}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{4}$$

where $g: J \times C(H; E) \longrightarrow E$ is a given function.

Finally in Section 5, two examples are given to illustrate the abstract theory.

Our main purpose in this paper is to extend some results from the cite literature devoted to state-dependent delay and those considered on a bounded interval for the evolution problems studied in [15]. We provide sufficient conditions for the existence of mild solutions on a semiinfinite interval $J = [0; +\infty)$ for the two classes of first order semilinear functional and neutral functional perturbed evolution equations with state dependent delay (1)-(2) and with state-dependent delay when the delay is finite using the nonlinear alternative of Frigon-Grannas for contractions maps in Frechet spaces [8] , combined with semigroup theory [4, 35] .

The notion of the phase space B plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was intro-duced by Hale and Kato [20] (see also Kappel and Schappacher [24] and Schumacher [36]. For a detailed discussion on this topic we refer the reader to the book by Hino et al. [23].

Diferential delay equations, or functional diferential equations, have been used in modeling scientific phenomena for many years. see for instance the books [21, 25, 42], and the papers [16, 20]. An extensive theory is developed for evolution equations [3, 4, 18]. Uniqueness and existence results have been established for diferent evolution problems in the papers by Baghli and Benchohra in [2], [9],[11].

However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years. These equations are frequently called equations with state-dependent delay. Existence results and among other things were derived recently for functional diferential equations when the solution is depending on the delay on a bounded interval for impulsive problems. We refer the reader to the papers by Abada et al. [1], Ait Dads and Ezzinbi [5], Anguraj et al. [6], Hernandez et al. [22] and Li et al. [27].

## 2. Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C(H;E)$ be the space of continuous functions from H into E and B(E) be the space of all bounded linear operators from E into E, with the usual supremum norm

$$N \in B(E),\ \| N \|_{B(E)} = \sup\ \{\ |N(y)|\ :\ |y| = 1\ \}.$$

A measurable function $y: [0, +\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [43]).

Let $L^1([0, +\infty), E)$ denotes the Banach space of measurable functions $y: [0, +\infty) \to E$ which are Bochner integrable normed by

$$\| y \|_{L^1} = \int_0^{+\infty} |y(t)|\ dt.$$

In this paper, we will employ an axiomatic definition of the phase space B introduced by Hale and Kato in [20] and follow the terminology used in [23]. Thus, $(B, \|\cdot\|_B)$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

1. If $y: (-\infty, b) \to E, b > 0,$ is continuous on $[0, b]$ and $y_0 \in B$, then for every $t \in [0, b)$ the following conditions hold :

   (i) $y_t \in B$ ;

   (ii) There exists a positive constant H such that $|y(t)| \leq H \| y_t \|_B$ ;

   (iii) There exist two functions $K(\cdot), M(\cdot): R_+ \to R_+$ independent of y with K continuous and M locally bounded such that :

$$\| y_t \|_B \leq K(t) \sup\{\ |y(s)|: 0 \leq s \leq t\} + M(t)\ \| y_0 \|_B.$$

2. For the function y in $(A_1)$, $y_t$ is a $B-$valued continuous function on $[0, b]$.

3. The space B is complete.

Denote $K_b = \sup\{K(t): t \in [0, b]\}$ and $M_b = \sup\{M(t): t \in [0, b]\}$.

**Remark 2.1**

1. (ii) is equivalent to $|\phi(0)| \leq H \| \phi \|_B$ for every $\phi \in B$.

2. Since $\|\cdot\|_B$ is a seminorm, two elements $\phi, \psi \in B$ can verify $\| \phi - \psi \|_B = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.

3. From the equivalence of in the first remark, we can see that for all $\phi, \psi \in B$ such that $\| \phi - \psi \|_B = 0$ : We necessarily have that $\phi(0) = \psi(0)$.

We now indicate some examples of phase spaces. For other details we refer, for instance to the book by Hino et al [23].

**Example 2.2** *Let:*

- the space of bounded continuous functions defined from $(-\infty, 0]$ to E;

- the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;
- $:= \{\phi \in BC: \lim_{\theta\to-\infty}\phi(\theta)$ exist in $E\}$;
- $:= \{\phi \in BC: \lim_{\theta\to-\infty}\phi(\theta) = 0\}$, endowed with the uniform norm

$$\| \phi \| = \sup\{|\phi(\theta)|:\ \theta \le 0\}.$$

We have that the spaces BUC, $C^\infty$ and $C^0$ satisfy conditions $(A_1)-(A_3)$. However, BC satisfies $(A_1),(A_3)$ but $(A_2)$ is not satisfied.

**Example 2.3** *The spaces $C_g$, $UC_g$, $C_g^\infty$ and $C_g^0$.*

Let g be a positive continuous function on $(-\infty, 0]$. We define:

- $:= \left\{\phi \in C((-\infty,0],E): \frac{\phi(\theta)}{g(\theta)}\ \text{ is bounded on } (-\infty,0]\right\}$;
- $:= \left\{\phi \in C_g: \lim_{\theta\to-\infty}\frac{\phi(\theta)}{g(\theta)} = 0\right\}$, endowed with the uniform norm

$$\| \phi \| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)}:\ \theta \le 0\right\}.$$

Then we have that the spaces $C_g$ and $C_g^0$ satisfy conditions $(A_3)$. We consider the following condition on the function g.

- For all $a > 0$, $\sup_{0\le t\le a}\sup\left\{\frac{g(t+\theta)}{g(\theta)}: -\infty < \theta \le -t\right\} < \infty$.

They satisfy conditions $(A_1)$ and $(A_2)$ if $(g_1)$ holds.

**Example 2.4** *The space $C_\gamma$.*

For any real constant $\gamma$, we define the functional space $C_\gamma$ by

$$C_\gamma := \left\{\phi \in C((-\infty,0],E): \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)\ \text{ exists in }\ E\right\}$$

endowed with the following norm

$$\| \phi \| = \sup\{e^{\gamma\theta}|\phi(\theta)|:\ \theta \le 0\}.$$

Then in the space $C_\gamma$ the axioms $(A_1)-(A_3)$ are satisfied.

**Definition 2.5** *A function $f: J \times B \to E$ is said to be an $L^1$ -Carathéodory function if it satisfies :*

- for each $t \in J$ the function $f(t,.): B \to E$ is continuous ;
- for each $y \in B$ the function $f(.,y): J \to E$ is measurable ;
- for every positive integer k there exists $h_k \in L^1(J; R^+)$ such that

$$|f(t,y)| \le h_k(t) \quad \text{for all } \| y \|_B \le k \ \text{ and almost every } t \in J.$$

In what follows, we assume that $\{A(t)\}_{t\ge0}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t .Additionally, we introduce following hypothesis:

1. For $t \in [0,b]$ The domain $D(A(t)) = D$ is independent of t and is dense on $X$.
2. For $t \ge 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all $\lambda$ with $Re(\lambda) \le 0$,and there is a constant $M$ independent of $\lambda$ and $t$ such that

$$\| R(t, A(t) \| \le M(1+|\lambda|)^{-1} \qquad pour\ Re(\lambda) \le 0.$$

3. There exist constant $L > 0$ and $0 < \alpha \le 1$ suth that

$$\| A(t) - A(\theta)A^{-1}(\tau) \| \le L|t-\tau|^\alpha \quad t,\theta,\tau \in J.$$

4. The resolvent $R(t, A(t))(t \ge 0)$ is compact.

**Lemma 2.6** *([3],p.159)Under the assumption $(P1)-(P4)$, the Cauchy problem*

$$y'(t)-A(t)y(t)=0 \quad t\in J: y(0)=y_0,$$

has a unique evolution system $\{U(t,s)/0\leq\theta\leq t\leq b\}$ on $E$ satisfying the following properties:

1. $U(t,t)=I$ where $I$ is the identity operator in $E$,
2. $U(t,s)\ U(s,\tau)=U(t,\tau)$ for $0\leq\tau\leq s\leq t<+\infty$,
3. $U(t,s)\in B(E)$ the space of bounded linear operators on $E$, where for every $(t,s)\in\Delta$ and for each $y\in E$, the mapping $(t,s)\to U(t,s)\ y$ is continuous.
4. For $0\leq\theta\leq t\leq b$, $U(t,s): X\longrightarrow D$ and$t\longrightarrow U(t,s)$ is strongly differentiable on $E$.The derivative $\frac{\partial}{\partial t}U(t,s)\in B(E)$ and it is strongly continuous on $0\leq\theta\leq t\leq b$.Moreover,

$$\frac{\partial}{\partial t}U(t,s)=A(t)U(t,s) \quad \text{for} \quad 0\leq\theta\leq t\leq b,$$

$$\| \frac{\partial}{\partial t}U(t,s)\|_{B(E)}=\| A(t)U(t,s)\|_{B(E)}\leq\frac{C}{t-\theta},$$

$$\| A(t)U(t,s)A^{-1}(s)\|_{B(E)}\leq C \quad \text{for} 0\leq\theta\leq t\leq b.$$

5. For every $v\in D\ and\ t\in(0,b], U(t,s)v \quad isdifferentiablewithrespectto\theta \quad on 0\leq\theta\leq t\leq b$

$$\frac{\partial}{\partial s}U(t,s)v=-U(t,s)A(s)v.$$

6. $U(t,s)$ is compact operator for $0\leq\theta\leq t\leq b$.

And, for each $y_0\in E$, the Cauchy problem has a unique classical solution $y\in C^1(J,E) \quad givenby$

$$y(t)=U(t,0)y_0,\ \ t\in J$$

More details on evolution systems and their properties could be found on the books of Ahmed [3], Engel and Nagel [17] and Pazy [35].

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in N}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\| x\|_1\leq\| x\|_2\leq\ \| x\|_3\leq\ldots \quad forevery\ \ x\in X.$$

Let $Y\subset X$, we say that Y is bounded if for every $n\in N$, there exists $\overline{M}_n>0$ such that

$$\| y\|_n\leq\overline{M}_n \quad for\ \ all\ \ y\in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n,\|\cdot\|_n)\}$ as follows : For every $n\in N$, we consider the equivalence relation $\sim_n$defined by : $x\sim_n y$ if and only if $\| x-y\|_n=0$ for $x,y\in X$. We denote $X^n=(X|_{\sim_n},\|\cdot\|_n)$ the quotient space, the completion of $X^n$ with respect to $\|\cdot\|_n$. To every $Y\subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n\subset X^n$ as follows : For every $x\in X$, we denote $[x]_n$ the equivalence class of x of subset $X^n$ and we defined $Y^n=\{[x]_n: x\in Y\}$. We denote $\overline{Y^n}$, $\mathrm{int}_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of $Y^n$ with respect to $\|\cdot\|_n$ in $X^n$. The corresponding nonlinear alternative result is as follows

**Theorem 2.7** *(Nonlinear Alternative of Granas-Frigon, [19]). Let X be a Fréchet space and $Y\subset X$ a closed subset and let $N: Y\to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements hold :*

$(C1)$ $N$ has a unique fixed point ;

$(C2)$ There exists $\lambda\in[0,1)$, $n\in N$ and $x\in\partial_n Y^n$ such that $\| x-\lambda\ N\ (x)\|_n=0$.

**Lemma 2.8** *([22], Lemma 2.4) If $y: ]-\infty,b]\longrightarrow E$ is a function such that $y_0=\phi$, then :*

$$\| y_s\|\leq(L^{\phi}+M_b)\ \|\phi\| +K_n\sup\{|y(\theta)|;\theta\in[0,max\{0,s\}]\},\ \ s\in R(\rho^-)\cup J,$$

or $L^{\phi}=\sup_{t\in R(\rho^-)}L^{\phi}(t)$.

**Proposition 2.9** *By $(H_{\phi})$, Lemma 2.8 and the property $(A_1)$*

we have for each $t\in[0,n]$ and $\in N$ :

$$\| y_{\rho(t,y_t)} \| \leq K_n |y(t)| + (M_n + L^{\varphi}) \| y_0 \|_B$$

## 3. Equation d'evolution

We give first the definition of a mild solution of the semilinear perturbed evolution problem $(1) - (12)$.

**Definition 3.1** *the function* $y(\cdot): [-r, +\infty[ \longrightarrow E$ is said to be weak solution of

$(1) - (2)$ si $y(t) = \varphi(t)$ for all $t \in H$ et y verifies the following integral equation :

$$y(t) = U(t,0)\ \varphi(0) + \int_0^t U(t,s)[f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\ ds \qquad p.p.\ t \in J. \tag{5}$$

We always assume that $\rho: J \times C(H; E) \to R$ is continuous. Additionally, we introduce following hypothesis:

- The function $t \to \varphi_t$ is continuous from $R(\rho^-)$ into $C(H; E)$ and there exists a continuous and bounded function $L^{\varphi}: R(\rho^-) \to (0, \infty)$ such that

$$\| \varphi_t \| \leq L^{\varphi}(t) \| \varphi \| \qquad for\ every\ \ t \in R(\rho^-).$$

**Remark 3.2** *The condition* $(H_{\phi})$, *is frequently verified by continuous and bounded functions. For more details, see for instance [23].*

We will need to introduce the following hypothesis which are assumed thereafter

- $U(t,s)$ is compact for $t - s > 0$.
- There exists a constant $\widehat{M} \geq 1$ such that

$$\| U(t,s) \|_{B(E)} \leq \widehat{M} \qquad for\ \ every\ (t,s) \in \Delta.$$

There exists a function $p \in L^1_{loc}(J; R_+)$ and a continuous nondecreasing function $\psi: J \longrightarrow ]0, +\infty[$ such that:

$$|f(t,u)| \leq p(t)\ \ \psi(\| u \|) \qquad p.p.\quad t \in J,\ \forall u \in C(H; E).$$

- For all $R > 0$, there exists $l_R \in L^1_{loc}(J; R_+)$ such that:

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \ \| u - v \|$$

$$\forall u, v \in C(H; E) \text{ avec } \| u \| \leq R \text{ and } \| v \| \leq R.$$

- There exists a function $N \in L^1_{loc}(J; R_+)$ and a continuous nondecreasing function $\psi: J \longrightarrow ]0, +\infty[$ such that :

$$|h(t,u)| \leq N(t)\ \ \psi(\| u \|) \qquad p.p.\quad t \in J,\ \forall u \in C(H; E).$$

- There exists a function $\eta \in L^1(J, R_+)$ such that :

$$|h(t,u) - h(t,v)| \leq \eta(t) \quad \| u - v \|_H \qquad p.p.\ t \in J \quad et \quad \forall u, v \in C(H; E).$$

Let us fix $\tau > 1$, we define $C([-r, +\infty[; E)$ the semi-norms by :

$$\| y \|_n = \sup\ \{\ e^{-\tau\ L^*_n(t)}\ \ |y(t)| : t \in [0, n]\ \} \quad \forall n \in N$$

Where $L^*_n(t) = \int_0^t \bar{l}_n(s)\ \ ds$ , $\bar{l}_n(t) = \widehat{M}(l_n(t) + \eta_n(t))$ and $l_n$, $\eta_n$ are the functions from $(H3)$ respectively $(H5)$ .

Then $C([-r, +\infty[; E)$ is a Frechet space with those family of semi-norms $\| \cdot \|_{n \in N}$.

If $y: [-r, b] \to E$ is a function such that $y_0 = \varphi$, then

$$\| y_s \|_B \leq (M_b + L^{\phi}) \| \phi \|_B + K_b \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\},\ \ s \in R(\rho^-) \cup J,$$

where $L^{\phi} = \sup_{t \in R(\rho^-)} L^{\phi}(t)$.

**Theorem 3.3** *[7] Assume that* $(H1) - (H5)$*and* $(H_{\varphi})$ *hold and moreover, we have*

$$\int_c^{+\infty} \frac{ds}{\psi(s)} > \widehat{M} \int_0^n (p(s) + N(s)) ds \qquad \forall n \in N. \tag{6}$$

with $c = (\widehat{M} + L^{\varphi}) \| \varphi \|$. Then the problem $(1) - (12)$ has a unique mild solution in $[-r, +\infty[$.

**Proof.** We transform the problem $(1)-(12)$ into a fixed-point problem. Consider the operator $N: C([-r,+\infty[;E) \longrightarrow C([-r,+\infty[;E)$ defined by

$$N(y)(t)=\begin{cases}\varphi(t), & si\ \ t\in\ \ H;\\ U(t,0)\ \ \varphi(0)+\int_0^t U(t,s)\ \ f(s,y_{\rho(s,y_s)})\ \ ds, & \\ +\int_0^t U(t,s)\ \ h(s,y_{\rho(s,y_s)})\ \ ds, & si\ \ t\in\ \ J.\end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(1)-(12)$.

Let $y$ be a possible solution of the problem $(1)-(12)$. Given $n\in N$, then from $(H1)-(H4)$, $(H_\varphi)$ and Lemma 2.8, we have for each $t\in[0,n]$

$$|y(t)|\leq\parallel U(t,0)\parallel_{B(E)}|\varphi(0)|+\int_0^t\parallel U(t,s)\parallel_{B(E)}|f(s,y_{\rho(s,y_s)})|\ \ ds$$

$$+\int_0^t\parallel U(t,s)\parallel_{B(E)}|h(s,y_{\rho(s,y_s)})|\ \ ds$$

$$\leq\widehat{M}\parallel\varphi\parallel+\widehat{M}\ \ \int_0^t(p(s)+N(s))\psi\big(\parallel y_{\rho(s,y_s)}\parallel\big)\ \ ds$$

$$\leq\widehat{M}\parallel\varphi\parallel+\widehat{M}\ \ \int_0^t(p(s)+N(s))\psi(|y(s)|+L^{\varphi}\parallel\varphi\parallel)\ \ ds.$$

It's follows that

$$|y(t)|+L^{\varphi}\parallel\varphi\parallel\leq\big(\widehat{M}+L^{\varphi}\big)\parallel\varphi\parallel+\widehat{M}\ \ \int_0^t(p(s)+N(s))\psi(|y(s)|+L^{\varphi}\parallel\varphi\parallel)\ \ ds$$

We consider the function $\mu$ defined by

$$\mu(t):=\sup\ \{\ |y(s)|+L^{\varphi}\parallel\varphi\parallel:\ 0\leq s\leq t\ \},\quad 0\leq t<+\infty.$$

Let $t^{\star}\in[-r,t]$ such that $\mu(t)=|y(t^{\star})|+L^{\varphi}\parallel\varphi\parallel$. Si $t^{\star}\in[0,n]$, by the previous inequality, we get :

$$\mu(t)\leq\big(\widehat{M}+L^{\varphi}\big)\parallel\varphi\parallel+\widehat{M}\int_0^t(p(s)+N(s))\psi(\mu(s))\ \ ds\qquad\forall t\in[0,n].$$

Let's take the second member of this inequality as being $v(t)$. Thus we will have :

$$\mu(t)\leq v(t)\ \ \forall t\in[0,n].$$

if,$t^{\star}\in[-r,0]$ then $\mu(t)=\parallel\varphi\parallel$ and the previous inequality holds.

From the definition of $v$, we get :

$$c:=v(0)=\big(\widehat{M}+L^{\varphi}\big)\parallel\varphi\parallel\qquad et\qquad v'(t)=\widehat{M}\ \ (p(t)+N(t))\ \ \psi(\mu(t))\qquad p.p.\ \ t\in[0,n].$$

Let's use the increasing of $\psi$ to get:

$$v'(t)\leq\widehat{M}\ \ (p(t)+N(t))\ \ \psi(v(t))\qquad p.p.\ \ t\in[0,n].$$

Let's use the condition (6) to get :

$$\int_c^{v(t)}\frac{ds}{\psi(s)}\leq\widehat{M}\int_0^t(p(s)+N(s))\ \ ds\leq\widehat{M}\ \ \int_0^n(p(s)+N(s)))ds<\int_c^{+\infty}\frac{ds}{\psi(s)}\qquad\forall t\in[0,n].$$

for all $t\in[0,n]$, there is a constant $\Lambda_n$ such that $v(t)\leq\Lambda_n$ Thus $\mu(t)\leq\Lambda_n$. Since $\parallel y\parallel_n\leq\mu(t)$, so we will have $\parallel y\parallel_n\leq\max\{\parallel\varphi\parallel,\Lambda_n\}:=\Theta_n$.

Consider the set

$$Y=\{\ y\in C([-r,+\infty[;E):\ \sup\{|y(t)|:0\leq t\leq n\}\leq\Theta_n+1,\quad\forall\quad n\in N\}.$$

It's clear that $Y$ is a closed subset of

$C([-r,+\infty[;E)$.

We shall show that $N: Y\to C([-r,+\infty[;E)$ is a contraction operator..

Indeed, consider $y,\overline{y}\in Y$. thus using $(H1)$, $(H3)$ and $(H5)$ for each $t\in[0,n]$ and $n\in N$ :

$$|N(y)(t)-N(\overline{y})(t)|\leq\int_0^t\ \ \parallel U(t,s)\parallel_{B(E)}\Big|f(s,y_{\rho(s,y_s)})-f(s,\overline{y}_{\rho(s,y_s)})\Big|\ \ ds$$

$$\leq\int_0^t\ \ \parallel U(t,s)\parallel_{B(E)}\Big|h(s,y_{\rho(s,y_s)})-h(s,\overline{y}_{\rho(s,y_s)})\Big|\ \ ds$$

$$\leq \int_0^t \widehat{M}\ l_n(s)\ \| y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)} \|\ ds$$

$$+ \int_0^t \widehat{M}\ \eta_n(s)\ \| y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)} \|\ ds.$$

Let's use the Proposition (2.9) we obtain :

$$|N(y)(t) - N(\overline{y})(t)| \leq \int_0^t \widehat{M}(l_n(s) + \eta_n(s))\ |y(s) - \overline{y}(s)|\ ds$$

$$\leq \int_0^t \left[\overline{l}_n(s)\ e^{\tau L_n^*(s)}\right] \left[e^{-\tau\ L_n^*(s)}\ |y(s) - \overline{y}(s)|\right] ds$$

$$\leq \int_0^t \left[\frac{e^{\tau\ L_n^*(s)}}{\tau}\right]' ds\ \ \| y - \overline{y} \|_n$$

$$\leq \frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \ \| y - \overline{y} \|_n.$$

Therefore :

$$\| N(y) - N(\overline{y}) \|_n \leq \frac{1}{\tau}\ \ \| y - \overline{y} \|_n.$$

So for $\tau > 1$, So, the operator $N$ is a contraction for all $n \in N$. From the choice of $Y$, there is no $y \in \partial Y^n$ such that $y = \lambda\ N(y),\ \lambda \in ]0,1[$. So the alternative $(FG2)$ in the Theorem 2.7 is not satisfied. So the Frigon-Granas nonlinear alternative shows that $(FG1)$ is satisfied . Then, we deduce that the operator $N$ admits a unique fixed point which is the unique weak solution of the problem $(1) - (12)$.

## 4. Example:

To illustrate the previous results, consider in this section the following example:

**Example .** Consider the perturbed functional differential equation :

$$\begin{cases} \frac{\partial u}{\partial t}(t,\xi) = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \int_{-\infty}^0 a_1(s-t)u\left[s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right]ds, \quad t \geq 0, \xi \in [0,\pi], \\ + \int_{-\infty}^0 a_3(s-t)u\left[s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right]ds \\ \\ u(t,0) = u(t,\pi) = 0, t \geq 0, \\ \\ u(\theta,\xi) = u_0(\theta,\xi), -r < \theta \leq 0,\ \xi \in [0,\pi]. \end{cases} \tag{7}$$

where $a_0(t,\xi)$ is a continuous function and is uniformly Höolder continuous $t$ ; $a_1, a_3: [-r,0] \to R$ and $a_2: [0,\pi] \to R$, $\rho_i: [0,+\infty[ \to R$ are continuous functions for $i = 1,2$.

To study this system, we consider the space $E = L^2([0,\pi],R)$ and the operator $A: D(A) \subset E \to E$ given by $Aw = w''$.

**Theorem 4.1** *Let $\varphi \in C(H;E)$ . Suppose the condition $(H_\varphi)$ is satisfied and the functions $\rho_i: R_+ \longrightarrow R$ for $i = 1,2$, $a_1, a_3: [-r,0] \to R$ and $a_2: [0,\pi] \to R$ are continue. So there is a unique weak solution of* $(7)$ .

**Proof.** According to the assumptions, we have that :

$$f(t,\psi)(\xi) = \int_{-r}^0 a_1(s)\psi(s,\xi)ds,$$

$$h(t,\psi)(\xi) = \int_{-r}^0 a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined allowing to transform the system (7) to the abstract system $(1) - (2)$. Moreover, the functions $f$ and $h$ are bounded linear. Now, the existence of mild solutions can be deduced from a direct application of Theorem 3.3.

According to the remark 3.2, we obtain the following result:

**Corollary 4.2** *Let $\varphi \in C(H;E)$ a continuous and bounded function. Then there is a unique mild solution to the problem* (7) *on* $[-r,+\infty[$.

## 5. Semilinear Neutral Evolution Equations

We give first the definition of a mild solution of the perturbed semilinear evolution problem $(3)-(34)$.

**Definition 5.1** *The function $y: [-r,+\infty[\longrightarrow E$ is called a weak solution of $(3)-(4)$ if $y(t)=\varphi(t)$ for all $t\in H$ and $y$ satisfies the following integral equation :*

$$\begin{aligned} y(t) &= U(t,0)[\varphi(0)-g(0,\varphi)] + g(t,y_{\rho(t,y_t)}) + \int_0^t U(t,s)A(s)\ g(s,y_{\rho(s,y_s)})\ ds \\ &+ \int_0^t U(t,s)\ [f(s,y_{\rho(s,y_s)}) + h(s,y_{\rho(s,y_s)})]\ ds \quad \forall t\in J. \end{aligned} \tag{8}$$

Suppose the function $\rho: J\times C(H;E)\longrightarrow R$ satisfies the hypothesis $(H_\varphi)$ and Lemma 2.8 . For the sequel, we we will need assumptions the following $(H1)-(H5)$ and we will need the following assumptions:

- There is a constant $\overline{M}_0 > 0$ such that :

$$\| A^{-1}(t) \|_{B(E)} \le \overline{M}_0 \qquad \forall t\in J.$$

- There is a constant $0 < L < \frac{1}{\overline{M}_0}$ such that :

$$|A(t)\ g(t,\varphi)| \le L\ (\| \varphi \| +1) \quad \forall t\in J\ \ et\ \ \forall\ \varphi\in C(H;E).$$

- There is a constant $L_* > 0$ such that :

$$|A(s)\ g(s,\varphi) - A(\overline{s})\ g(\overline{s},\overline{\varphi})| \le L_*\ (|s-\overline{s}| + \| \varphi - \overline{\varphi} \|)$$

$\forall s,\overline{s}\in J$ et $\forall \varphi,\overline{\varphi}\in C(H;E)$.

**Theorem 5.2:** *[7] Assume that the assumptions $(H1)-(H8)$ et $(H_\varphi)$ hold*

$$\int_{\delta_n}^{+\infty} \frac{ds}{s+\psi(s)} > \frac{\widehat{M}}{1-\overline{M}_0 L}\int_0^n \max(L,d(s))ds \qquad \forall n\in N. \tag{9}$$

with $d(s)=p(s)+N(s)$ and

$$\delta_n := L^\varphi \| \varphi \| + \frac{\overline{M}_0 L L^\varphi \| \varphi \| + \overline{M}_0 L(1+\widehat{M}) + \widehat{M}\big(1+\overline{M}_0 L\big) \| \varphi \| + \widehat{M} L n}{1-\overline{M}_0 L}.$$

Then the dependent finite-delay neutral type evolution equation of State $(3)-(4)$ admits a unique weak solution on $[-r,+\infty[$.

**Proof.** We transform the problem $(3)-(34)$ into a fixed-point problem.Consider the operator $\widetilde{N}: C([-r,+\infty[;E)\longrightarrow C([-r,+\infty[;E)$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \varphi(t) & si\ \ t\in\ \ H; \\ U(t,0)\ \ [\varphi(0)-g(0,\varphi)] + g(t,y_{\rho(t,y_t)}) & \\ +\int_0^t U(t,s)A(s)g(s,y_{\rho(s,y_s)})ds & \\ +\int_0^t U(t,s)\big[f(s,y_{\rho(s,y_s)}) + h(s,y_{\rho(s,y_s)})\big]\ ds & si\ \ t\in\ \ J. \end{cases}$$

The fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(3)-(34)$.

Let $y$ a possible solution to the problem

$(3)-(4)$. Given $n\in N$, then according to

$(H1)-(H8)$, and the Proposition (2.9), we have for all $t\in[0,n]$ :

$$|y(t)| \le |g(t,y_{\rho(t,y_t)})| + |U(t,0)[\varphi(0)-g(0,\varphi)]|$$

$$+\int_0^t \left|U(t,s)A(s)g(s,y_{\rho(s,y_s)})\right|ds + \int_0^t \left|U(t,s)f(s,y_{\rho(s,y_s)})\right|ds$$

$$+\int_0^t \left|U(t,s)h(s,y_{\rho(s,y_s)})\right|ds$$

$$\leq \| A^{-1}(t) \|_{B(E)} \| A(t)g(t,y_{\rho(t,y_t)}) \|$$

$$+\| U(t,0) \|_{B(E)} \| \varphi \| + \| U(t,0) \|_{B(E)} \| A^{-1}(0) \| \ \| A(0) \ g(0,\varphi) \|$$

$$+\int_0^t \| U(t,s) \|_{B(E)} \| A(s) \ g(s,y_{\rho(s,y_s)}) \| ds + \int_0^t \| U(t,s) \|_{B(E)} |f(s,y_{\rho(s,y_s)})| ds$$

$$+\int_0^t \| U(t,s) \|_{B(E)} |h(s,y_{\rho(s,y_s)})| ds$$

$$\leq \overline{M}_0 L(\| y_{\rho(t,y_t)} \| + 1) + \widehat{M} \| \varphi \| + \widehat{M}\overline{M}_0 L(\| \varphi \| + 1)$$

$$+\widehat{M}\int_0^t L(\| y_{\rho(s,y_s)} \| + 1)ds + \widehat{M}\int_0^t (p(s)+N(s))\psi(\| y_{\rho(s,y_s)} \|)ds$$

$$\leq \overline{M}_0 L \| y_{\rho(t,y_t)} \| + \overline{M}_0 L(1+\widehat{M}) + \widehat{M}\left(1+\overline{M}_0 L\right) \| \varphi \| + \widehat{M}Ln$$

$$+\widehat{M}L\int_0^t \| y_{\rho(s,y_s)} \| ds + \widehat{M}\int_0^t d(s)\psi(\| y_{\rho(s,y_s)} \|)ds.$$

As $\| y_{\rho(t,y_t)} \| \leq |y(t)| + L^{\varphi} \| \varphi \|$, we obtain :

$$|y(t)| \leq \overline{M}_0 L(|y(t)| + L^{\varphi} \| \varphi \|) + \overline{M}_0 L(1+\widehat{M}) + \widehat{M}(1+\overline{M}_0 L) \| \varphi \| + \widehat{M}Ln$$

$$+\widehat{M}L\int_0^t (|y(s)| + L^{\varphi} \| \varphi \|)ds + \widehat{M}\int_0^t d(s)\psi(|y(s)| + L^{\varphi} \| \varphi \|)ds$$

$$\leq \overline{M}_0 L|y(t)| + \overline{M}_0 L L^{\varphi} \| \varphi \| + \overline{M}_0 L(1+\widehat{M}) + \widehat{M}\left(1+\overline{M}_0 L\right) \| \varphi \| + \widehat{M}Ln$$

$$+\widehat{M}L\int_0^t (|y(s)| + L^{\varphi} \| \varphi \|)ds + \widehat{M}\int_0^t d(s)\psi(|y(s)| + L^{\varphi} \| \varphi \|)ds.$$

So :

$$(1-\overline{M}_0 L)|y(t)| \leq \overline{M}_0 L L^{\varphi} \| \varphi \| + \overline{M}_0 L(1+\widehat{M}) + \widehat{M}\left(1+\overline{M}_0 L\right) \| \varphi \| + \widehat{M}Ln$$

$$+\widehat{M}L\int_0^t (|y(s)| + L^{\varphi} \| \varphi \|)ds + \widehat{M}\int_0^t d(s)\psi(|y(s)| + L^{\varphi} \| \varphi \|)ds.$$

Let's pose $\delta_n := L^{\varphi} \| \varphi \| + \frac{\overline{M}_0 L L^{\varphi}\|\varphi\| + \overline{M}_0 L(1+\widehat{M}) + \widehat{M}\left(1+\overline{M}_0 L\right)\|\varphi\| + \widehat{M}Ln}{1-\overline{M}_0 L}$. So :

$$|y(t)| + L^{\varphi} \| \varphi \| \leq \delta_n + \frac{\widehat{M}L}{1-\overline{M}_0 L}\int_0^t (|y(s)| + L^{\varphi} \| \varphi \|)ds$$

$$+\frac{\widehat{M}}{1-\overline{M}_0 L}\int_0^t d(s)\psi(|y(s)| + L^{\varphi} \| \varphi \|)ds.$$

Consider the function μ defined by :

$$\mu(t) := \sup \{ \ |y(s)| + L^{\varphi} \| \varphi \| : \ 0 \leq s \leq t \ \}, \quad 0 \leq t < +\infty.$$

Soit $t^{\star} \in [-r,t]$ tel que $\mu(t) = |y(t^{\star})| + L^{\varphi} \| \varphi \|$. Si $t^{\star} \in [0,n]$, according to the inequality previous, we will have :

$$\mu(t) \leq \delta_n + \frac{\widehat{M}}{1-\overline{M}_0 L}\left[\int_0^t L \ \mu(s)ds + \int_0^t d(s)\psi(\mu(s))ds\right] \quad \forall t \in [0,n].$$

Take the second member of this inequality to be $v(t)$. Thus we will have :

$$\mu(t) \leq v(t) \ \ \forall t \in [0,n].$$

if $t^{\star} \in [-r,0]$, then $\mu(t) = \| \varphi \|$ and the previous inequality holds.

From the definition of $v$, we obtain : $v(0) = \delta_n$ and

$$v'(t) = \frac{\widehat{M}}{1-\overline{M}_0 L} \ [\ L\mu(t) + d(t) \ \psi(\mu(t)) \ ] \quad p.p. \quad t \in [0,n].$$

Let′s use the increasing of de $\psi$ to get :

$$v'(t) \leq \frac{\widehat{M}L}{1-\overline{M}_0 L} \; [\; L \; v(t) + d(t)\psi(v(t)) \;] \quad p.p. \quad t \in [0,n].$$

let's use it condition (9) to have for all $t \in [0,n]$ :

$$\int_{\delta_n}^{v(t)} \frac{ds}{s+\psi(s)} \leq \frac{\widehat{M}}{1-\overline{M}_0 L} \int_0^t \max(L, d(s))ds \leq \frac{\widehat{M}}{1-\overline{M}_0 L} \int_0^n \max(L, d(s))ds < \int_{\delta_n}^{+\infty} \frac{ds}{s+\psi(s)}.$$

for all $t \in [0,n]$, there exists a constant $\Lambda_n$ such that $v(t) \leq \Lambda_n$ thus $\mu(t) \leq \Lambda_n$. Since $\| y \|_n \leq \mu(t)$, so we will have $\| y \|_n \leq \max\{\| \varphi \|, \Lambda_n\} := \Theta_n$.

Now let show that the operator $\widetilde{N}: Y \longrightarrow C([-r, +\infty[; E)$ is a contraction.

Consider for this $y, \overline{y} \in Y$. According to $(H1)$, $(H3)$, $(H5)$, $(H6)$ and $(H8)$ we will have for all $t \in [0,n]$ et $n \in N$ :

$$|\widetilde{N}(y)(t) - \widetilde{N}(\overline{y})(t)| \leq |g(t, y_{\rho(t,y_t)}) - g(t, \overline{y}_{\rho(t,y_t)})|$$

$$+ \int_0^t \| U(t,s) \|_{B(E)} \; |A(s)[g(s, y_{\rho(s,y_s)}) - g(s, \overline{y}_{\rho(s,y_s)})]| ds$$

$$+ \int_0^t \| U(t,s) \|_{B(E)} \; |f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)})| ds$$

$$+ \int_0^t \| U(t,s) \|_{B(E)} \; |h(s, y_{\rho(s,y_s)}) - h(s, \overline{y}_{\rho(s,y_s)})| ds$$

$$\leq \| A^{-1}(t) \|_{B(E)} \quad |A(t)g(t, y_{\rho(t,y_t)}) - A(t)g(t, \overline{y}_{\rho(t,y_t)})|$$

$$+ \int_0^t \widehat{M} \; |A(s)g(s, y_{\rho(s,y_s)}) - A(s)g(s, \overline{y}_{\rho(s,y_s)})| ds$$

$$+ \int_0^t \widehat{M} \; |f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)})| ds$$

$$+ \int_0^t \widehat{M} \; |h(s, y_{\rho(s,y_s)}) - h(s, \overline{y}_{\rho(s,y_s)})| ds$$

$$\leq \overline{M}_0 L_\star \| y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)} \| + \int_0^t \widehat{M} L_\star \| y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)} \| \, ds$$

$$+ \int_0^t \widehat{M}(l_n(s) + \eta_n(s)) \| y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)} \| \, ds$$

$$\leq \overline{M}_0 L_\star \| y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)} \|$$

$$+ \int_0^t \widehat{M}[L_\star + l_n(s) + \eta_n(s)] \| y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)} \| \, ds.$$

Let's use the Proposition (2.9) to get :

$$|\widetilde{N}(y)(t) - \widetilde{N}(\overline{y})(t)| \leq \overline{M}_0 L_* \; |y(t) - \overline{y}(t)| + \int_0^t \widehat{M}[L_* + l_n(s) + \eta_n(s)] \; |y(s) - \overline{y}(s)| \; ds.$$

Let's take here $\overline{l}_n(t) = \widehat{M}[L_* + l_n(t) + \eta_n(t)]$ for the family of semi-norms $\{\|\cdot\|_n\}_{n\in N}$, then :

$$|\widetilde{N}(y)(t) - \widetilde{N}(\overline{y})(t)| \leq \overline{M}_0 L_* |y(t) - \overline{y}(t)| + \int_0^t \overline{l}_n(s) |y(s) - \overline{y}(s)| ds$$

$$\leq \left[\overline{M}_0 L_* \; e^{\tau \; L_n^*(t)}\right]\left[e^{-\tau L_n^*(t)} \; |y(t) - \overline{y}(t)|\right]$$

$$+ \int_0^t \left[\overline{l}_n(s) \; e^{\tau L_n^*(s)}\right] \; \left[e^{-\tau L_n^*(s)} \; |y(s) - \overline{y}(s)|\right] ds$$

$$\leq \overline{M}_0 L_* \; e^{\tau \; L_n^*(t)} \| y - \overline{y} \|_n + \int_0^t \left[\frac{e^{\tau \; L_n^*(s)}}{\tau}\right]' \; ds \quad \| y - \overline{y} \|_n$$

$$\leq \left[\overline{M}_0 L_* + \frac{1}{\tau}\right] \; e^{\tau L_n^*(t)} \quad \| y - \overline{y} \|_n.$$

Therefore :

$$\| \widetilde{N}(y) - \widetilde{N}(\overline{y}) \|_n \leq \left[\overline{M}_0 L_* + \frac{1}{\tau}\right] \quad \| y - \overline{y} \|_n.$$

So, for an appropriate choice of $L_*$ et $\tau$ such as :

$$\overline{M}_0 L_* + \frac{1}{\tau} < 1,$$

The operator $\widetilde{N}$ is a contraction for all $n \in N$. According to the choice of Y, there is no $y \in \partial Y^n$ such that $y = \lambda\widetilde{N}(y),\ \lambda \in ]0,1[$. Then the alternative (FG2) in Theorem 2.7 is not satisfied. So the Frigon-Granas nonlinear alternative shows that (FG1) is satisfied. Then, we deduce that the operator $\widetilde{N}$ admits only one unique fixed point which is the solution unique mild point of problem $(3)-(34)$.

## 6. Example:

To illustrate the previous results, we give in this section the following example:

**Exemple .** Consider the perturbed neutral functional differential equation of the following type:

$$\begin{cases} \frac{\partial}{\partial t}\left[u(t,\xi) - \int_{-\infty}^{0} a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds\right] \\ = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \int_{-\infty}^{0} a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds, \quad t \geq 0, \xi \in [0,\pi], \\ + \int_{-\infty}^{0} a_4(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds \\ v(t,0) = v(t,\pi) = 0, t \geq 0, \\ v(\theta,\xi) = v_0(\theta,\xi), -r < \theta \leq 0,\ \xi \in [0,\pi]. \end{cases} \tag{10}$$

Where $a_4: R_- \to R$ is a continuous function.

**Theorem 6.1** *Let $\varphi \in C(H;E)$. Suppose the condition $(H_\varphi)$ is satisfied and the functions $d: [0,\pi] \to E$, $\rho_i: R_+ \longrightarrow R$ pour $i = 1,2$, $a_1, a_3, a_4: [-r,0] \to R$ and $a_2: [0,\pi] \to R$ are continuous. So there is a unique mild solution of* $(10)$ .

**Proof.** According to the assumptions, we have that:

$$f(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-r}^{0} a_3(s)\psi(s,\xi)ds,$$

$$h(t,\psi)(\xi) = \int_{-r}^{0} a_4(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined allowing to transform the system $(10)$ to the abstract system $(3)-(4)$. moreover, the functions $f$, $g$ and $h$ are bounded linear. Now, we can ensure the existence of the mild solution by direct application of Theorem 3.3.

According to the remark 3.2, we obtain the following result:

**Corollary 6.2** *Let $\varphi \in C(H;E)$ a continuous and bounded function. Then there exists a unique mild solution of* $(10)$ *on* $[-r,+\infty[$.

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