NAVIER-STOKES USE THE SOLUBLE DESIGN METHOD TO EVALUATE FINITE ELEMENTS

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ABSTRACT.
We propose a data assimilation algorithm for the Navier-Stokes equations, based on the Azouani, Olson, and Titi (AOT) algorithm, but applied to the Navier-Stokes-Voigt equations. Adapting the AOT algorithm to regularized versions of Navier-Stokes has been done before, but the innovation of this work is to drive the assimilation equation with observational data, rather than data from a regularized system. We first prove that this new system is globally well-posed. Moreover, we prove that for any admissible initial data, the L2 and H1 norms of error are bounded by a constant times a power of the Voigt-regularization parameter $\alpha > 0$, plus a term which decays exponentially fast in time. In particular, the large-time error goes to zero algebraically as α goes to zero. Assuming more smoothness on the initial data and forcing, we also prove similar results for the H2 norm.

I INTRODUCTION

The finite number of degrees of freedom such as determining modes, determining nodes, determining volume elements has been proved by Korn for some αmodels in fluid mechanics, including Navier-Stokes-α, Leray-α and Navier-Stokes-ω equations. Azouani and Titi proposed a new feedback control for controlling general dissipative evolution equations using any of the determining systems of parameters (modes, nodes, volume elements, etc...) without requiring the presence of separation in spatial scales, i.e. without assuming the existence of an inertial manifold. Then it is applied to stabilize the nonlinear reaction-diffusion equations by using finite parameters feedback controls; see also a more recent result for damped nonlinear wave equations. The computational study of a simple finite dimensional feedback control algorithm for stabilizing solutions to some nonlinear dissipative systems was presented. In this paper, following the general lines of the approach we propose a simple finite-dimensional feedback control scheme for stabilizing stationary solutions of Navier-Stokes-Voigt equations with periodic boundary conditions. Here the feedback control scheme only uses finitely many of observables and controllers, such as finite number of determining Fourier modes, determining nodes, and determining finite volumes for a unified theory of such determining functional. The paper is organized as follows. For convenience of the reader, we recall the functional setting of the Navier-Stokes-Voigt equations. we first stabilize an unstable stationary solution to the Navier-Stokes-Voigt equations (in both cases of dimension two and dimension three) by using finite-dimensional feedback control scheme employing finite volume elements or projection onto Fourier modes. Then in the case of two dimensions, we show that an unstable stationary solution can be stabilized by using a finite-dimensional feedback control employing finitely many nodal values.

The NS-Voigt equations were first proposed by Oskolkov model for Kelvin–Voigt fluids, but were later viewed as a regularization for the NS equations where also the Euler-Voigt equations were first introduced and studied. The Voigt-regularization is related to the wider class of α-models, including the NS-α (NS-α) model and the Leray-α model. The Voigt model enjoys two features that the other α-models are not known to have in the 3D case. First, it is known to be globally well-posed in the inviscid case. Second, in the viscous case, it is well-posed in the physical case of "noslip" homogeneous Dirichlet boundary conditions, with no need to impose artificial boundary conditions. Although we only work in 2D, we focus on the NS-Voigt model due in part to these attractive features, and also due to its simplicity. The study of the AOT-algorithm applied to other models but still driven by observable data, and the errors resulting from the mismatch, will be the subject of a forthcoming work.

In this paper, following the general lines of the approach introduced, we propose a simple finite-dimensional feedback control scheme for stabilizing stationary solutions of Navier-Stokes-Voigt equations with periodic boundary conditions. Here the feedback control scheme only uses finitely many of observables and controllers, such as finite number of determining Fourier modes, determining nodes, and determining finite volumes for a unified theory of such determining functionals.

2. MODEL FORMULATION

We follow the same notation as introduced parametrize the surface $S \subset R^3$ by the local coordinates θ, ϕ, i.e.,

$x : R^2 \supset U \to R^3 ; (\theta, \phi) \to x(\theta, \phi)$.

Thus, the embedded R^3 representation of the surface is given by S = x(U). The unit outer normal of Sat point x is denoted by ν(x). We denote by (∂θ x, ∂ϕx) the canonical basis to describe the (tangential) velocity $v(x) \in T_xS$, i.e., $v = v^\theta \partial_\theta x + v^\phi \partial_\phi x$ at a point x ∈ S. In a (tubular) neighborhood Ωδ of S, defined a signed-distance function are defined by

$$\tilde{p}(\widetilde{\mathbf{x}}) := p(\mathbf{x}) \quad \text{and} \quad \tilde{\mathbf{v}}(\widetilde{\mathbf{x}}) := \mathbf{v}(\mathbf{x}),$$

respectively, for $x \in \Omega_\delta$ and x the corresponding coordinate projection. To embed the R_3 vector space structure to the tangential bundle of the surface, we use the pointwise defined normal projection

$$\begin{aligned}\pi_{\mathcal{S}}(\mathbf{x}) : \mathsf{T}_\mathbf{x}\mathbb{R}^3 \cong \mathbb{R}^3 \to \mathsf{T}_\mathbf{x}\mathcal{S};\\ \widehat{\mathbf{v}}(\mathbf{x}) \mapsto \widehat{\mathbf{v}}(\mathbf{x}) - \boldsymbol{\nu}(\mathbf{x})(\boldsymbol{\nu}(\mathbf{x}) \cdot \widehat{\mathbf{v}}(\mathbf{x})) = \mathbf{v}(\mathbf{x}),\end{aligned}$$

However, the $rot_S\, rot_S$ v term leads to a heavy workload in terms of implementation and assembly time, as 36 second order operators, 72 first order operators, and 36 zero order operators have to be considered. This effort can drastically be reduced by rotating the velocity field in the tangent plane. Instead ofDv we consider w = ν ×v as unknown. Applying ν×

$$\begin{aligned}\partial_t \widehat{\mathbf{w}} - \mathrm{div}_{\mathcal{S}}\, \widehat{\mathbf{w}} \boldsymbol{\nu} \times \widehat{\mathbf{w}} = -\mathrm{rot}_{\mathcal{S}}\, \tilde{p} + \frac{1}{\mathrm{Re}}\,(\mathrm{grad}_{\mathcal{S}}\, \mathrm{div}_{\mathcal{S}}\, \widehat{\mathbf{w}} + 2\kappa \widehat{\mathbf{w}}\\ - \alpha(\widehat{\mathbf{w}} \cdot \boldsymbol{\nu})\boldsymbol{\nu},\\ \mathrm{rot}_{\mathcal{S}}\, \widehat{\mathbf{w}} = 0\end{aligned}$$

2.1 ENERGY BUDGET FOR THE NSV MODEL

The global regularity result for solutions of the Navier-Stokes-Voigt equations established in [30] implies that the following energy equality holds for every t ∈ [0, ∞):

$$\frac{d}{dt}(\frac{1}{2}|\mathbf{v}(\cdot,t)|^2 + \frac{\alpha^2}{2}\|\mathbf{v}(\cdot,t)\|^2) = (\mathbf{f},\mathbf{v}(\cdot,t)) - \nu\|\mathbf{v}(\cdot,t)\|^2,$$

Similarly, the global regularity results established in [2] imply that the solutions of the NSV equations in the inviscid (i.e. Euler-Voigt model, ν = 0) and unforced setting, f = 0, satisfy for every t ∈ R:

$$\frac{d}{dt}(\frac{1}{2}\left|\mathbf{v}^0(\cdot,t)\right|^2 + \frac{\alpha^2}{2}\left\|\mathbf{v}^0(\cdot,t)\right\|^2) = 0.$$

Therefore, the conserved quantity in the inviscid and unforced setting of the NSV (i.e. Euler-Voigt) model is
Then, we can write the projected Navier-Stokes-Voigt equations on the shell [κ ′ , κ′′)

$$\mathcal{E}_\alpha = \frac{1}{2}|\mathbf{v}|^2 + \frac{\alpha^2}{2}\|\mathbf{v}\|^2,$$

$$\frac{d}{dt}(\mathbf{v}_{\kappa',\kappa''} + \alpha^2 A\mathbf{v}_{\kappa',\kappa''}) + \nu A\mathbf{v}_{\kappa',\kappa''} + B(\mathbf{v},\mathbf{v})_{\kappa',\kappa''} = \mathbf{f}_{\kappa',\kappa''}.$$

2.2 Two-phase Navier-Stokes equations with Boussinesq-Scriven interface stresses
The two Navier-Stokes equations in Ωi , i = 1, 2, in (2.1) together with the interfacial conditions can be reformulated in one Navier-Stokes equation on the whole domain **Ω** with a surface tension force term localized at the interface. Combining this with the level set method leads to the following model for the two-phase problem in **Ω** × [0, T], with unknowns u(x, t), p(x, t) and the level set function φ(x,

$$\rho(\phi)\Big(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) = -\nabla p + \rho(\phi)\mathbf{g} + \operatorname{div}(\mu(\phi)\mathbf{D}(\mathbf{u})) + \delta_\Gamma \operatorname{div}_\Gamma \boldsymbol{\sigma}_\Gamma$$
$$\operatorname{div}\mathbf{u} = 0$$
$$\frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = 0.$$

t)

We now specify the domain, boundary and initial conditions that are used in the numerical experiments in section 4. For **Ω** we take a rectangular box with lengths L_x, L_y, L_z in the three coordinate directions. The interface Γ(0) is defined as a sphere at the centreline of the box with radius r. The subdomain $\Omega_1(0)$ is the interior of this sphere. The boundary conditions for u are as follows. On the z-boundaries ($z = \pm L_z$) we use periodic boundary conditions. On the y-boundaries we take Dirichlet no slip conditions (u = 0). On the x-inflow boundary we prescribe a Poiseuille profile that is constant in z-direction and has the form

$$\mathbf{u}^{\mathrm{P}}(y) = U_{center}\big(1 - (\frac{2y}{L_y})^2\big)\vec{\mathbf{e}}_x.$$

3. RESULTS AND DISCUTION

A general form of a level-set function for a n-torus can be written as L(x) = Qn i=1 T(x − mi) − (n − 1) δ with a constant δ > 0 and the midpoints of the tori mi ∈ R 3 for i = 1, . . ., n. In the following examples, we consider the fully discrete scheme for Problem 2 and use Re = 10, τ = 0.1, α = 3000, R = 1, and r = 0.5. For the Gaussian curvature κ, we use the analytic formula. The initial condition is considered to be v0 = rotS ψ0 = ν × gradS ψ0 with ψ0 = 1 2 (x + y + z) which ensures the incompressibility constraint. Figure 4 (Multimedia view) (top) shows the time evolution on the 1-torus with m1 = 0. The initial state has four defects, two vortices with Indxv = +1, indicated as red dots, and two saddles with Indxv = 1, indicated as blue dots (one vortex and one saddle are not visible). These defects annihilate during the evolution. The final state is again a Killing vector field without any defects. For n > 1, the rotational symmetry is broken and Killing vector fields are no longer possible. We thus expect dissipation of the kinetic energy and convergence to v = 0 for any initial condition. Figure 4 (Multimedia view) (middle) shows the time evolution on a 2-torus where we have used the midpoints m1 = (1.2, 0, 0) and m2 = m1 as well as δ = 1. The initial state has two vortices and four saddles and thus P x∈v −1 (0) Indxv = −2. Two vortex-saddle pairs annihilate each other and the final defect configuration consists of two saddles located at the center of the 2-torus (one is not visible). The velocity field decays towards v = 0. Figure 4 (Multimedia view) (bottom) shows the time evolution on a 3-torus with midpoints m1 = (1.2,0.75, 0), m2 = (1.2, 0.75, 0), and m3 = (0, 1.33, 0) as well as δ = 10. Initially we have three vortices and seven saddles and thus P x∈v −1 (0) Indxv = −4, which is also fulfilled for the final defect configuration with two vortices and six saddles at the center of the 3-torus (one vortex and three saddles are not visible). Again the velocity field decays towards v = 0. To show the differences in the evolution on the n-tori before and after the final defect configuration is reached, we consider the H 1 semi-norm of the rescaled velocity field v¯ = Dv/kDvkL 2 . If the defects do not move, this quantity is constant. Figure 5 shows the evolution over time together with the decay of the kinetic energy E = 1 2 ∫S kDvk 2 dS. These results clearly show the strong interplay between topology, geometric properties, and defect positions.

4. CONCLUSIONS

We have proposed a discretization approach for the incompressible surface Navier-Stokes equation on general surfaces independent of the genus g(S). The approach only requires standard ingredients which most finite element implementations can offer. It is based on a reformulation of the equation in Cartesian coordinates of the embedding R 3 , penalization of the normal component, a Chorin projection method, and discretization in space by globally continuous, piecewise linear Lagrange surface finite elements for each component. A further rotation of the velocity field leads to a drastic reduction of the complexity of the equation and the required computing time. The fully discrete scheme is described in detail and its accuracy is validated against a DEC solution on a 1-torus. The interesting interplay between the topology of the surface, its geometric properties, and defects in the flow field are shown on n-tori for n = 1, 2, 3. Even if the formulation of the incompressible surface Navier-Stokes equation is relatively old,15,22,34 numerical treatments on general surfaces are very rare. We are only 012107-6 S. Reuther and A. Voigt Phys. Fluids 30, 012107 (2018) and therefore expect the proposed approach to initiate a broader use and advances in the mentioned applications in Sec. I. We further expect it to be the basis for further developments, e.g., coupling of the surface flow with bulk flow in two-phase flow problems, as, e.g., using a vorticity-stream function approach or in 4 within an alternative formulation based on the bulk velocity and projection operators. Another extension considers evolving surfaces. With a prescribed normal velocity, this has already been again using a vorticity-stream function approach. The corresponding equations are using a global variational approach as a thin-film limit. A mathematical derivation of the evolution equation for the normal component is still controversial. The derivation is based on local conservation of mass and linear momentum in tangential and normal directions, while the derivation is based on local conservation of mass and total linear momentum. The resulting equations differ. However, in the special case of a stationary surface, all these models coincide with the incompressible surface Navier-Stokes equation In all considered examples, the Gaussian curvature was analytically given. However, this is not necessary. For appropriate algorithms to compute κ from a given surface triangulation

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