Local Primary Hollow Modules and its Relation with Some Other Modules

Nada Khalid Abdullah
Department of Mathematics, Collage of Education for Pure Sciences, Tikrit University, Iraq

Abstract
To consider R is an abelian ring with an identical member and Md is non-zero unitary left module over a ring R. Md is said to be a Hollow Module (HM) if each proper sub-mod (sub-mod) of Md is small. A local primary hollow module (LPHM) is defined as a generalization of HM. A module is said to be a LPHM if Md has one primary sub-mod with each small sub-mods. Our paper studies the type of such modules and it introduces some essential concepts that elaborate this issue.
Keywords: Hollow module, Local primary hollow module, Amply supplemented, Indecomposable module, Lifting Module.

1. Introduction

According to the fact that R is represented an abelian identical ring, every module are left unity and a proper sub-mod A of a module Md is said to be small when $A + B \neq Md$ for any proper sub-mod B of Md [1]. A nonzero module Md is called hollow module (HM) if any proper sub-mod of Md is small [2]. A proper sub-mod N of Md is called a primary if an integer number $n > 0$ such that if k , $l \in Md$ satisfy $kl \in N$ then $knMd \subseteq N$ [3]. Md is called a local if Md has only one maximum sub-mod with all the proper sub-mods of Md [4]. In our paper, a generalization of HM will be introduced. The new generalization will be called a Local Primary Hollow Module (LPHM) and it will be defined as a module with a unique primary sub-mod that includes every small sub-mod of Md. The procedure of this work will be done in three sections. In the 1st section , the definition of LPHM's as generalization of HM's will be investigated in terms of the main properties of such types of modules. While in the 2nd section, some conditions under that LPHM's with L-hollow modules (L_HM) and LSP-hollow module (LSP-HM) are equivalent. Then in the 3rd part, the relationship between the LPHM's beside some modules such as indecomposable, lifting and the amply supplemented modules has been introduced .

2. The basic properties of LPHM .

A concepts and some related definitions of LPHM will be introduced in this part then a study for the basic properties of such type of module will be introduced as mention below.

Definition(1): Any R-module Md is called LPHM if Md has only one primary sub-mod which includes every small sub-mod.

3. Examples and Remarks

1- Z_9 is LPH Z-module, while Z_{36} is not LPH Z-module.

2- Every LPHM is Hollow, while the converse is generally does not correct, for example Z_p^{∞} is HM, but it is not LPHM.

3- Every LPH is not is necessary being simple module , for instant z-module Z_{25} is LPHM while it isn't a simple module.

4- A sub-mod of a LPHM is not necessary being LPHM. for instant the Z-module Z is a sub-mod of LPHM (Q) but Z itself is not LPHM.

5- if any $\mathcal{R}$-module Md is finite generating then each LP-H is L-HM.

Proposition(1): If $f: M1 \to M2$ is an epimorphic image of LPHM so that $M1$ is LPHM of $M2$.

Proof: Suppose that the unique primary sub-mod of M_2 is N with $N+K=M_2$ such that K is a proper sub-mod of M_2. $f^{-1}(N)$ has only one primary sub-mod of M_1 otherwise $f^{-1}(N) = M_1$ and hence $f(f^{-1}(N)) = f(M_1) = M_2$ implies that $N = M_2$ but this will be a C! because N is unique primary sub-mod of M_2, so $f^{-1}(N)$ is a unique primary sub-mod of M_1. Now, since M_1 is LPHM then $f^{-1}(N)$ will contain every small sub-mod of M_1 and because $f(f^{-1}(N))$is a small sub-mod of $f(M_1)$. So that M_2 is LPHM.

Proposition(2): Let Md be a module and let $\frac{Md}{K}$ be a LPHM then Md is LPHM, such that K is a small sub-mod of Md .

Proof: Since $\frac{Md}{K}$ is LPHM where K is a small sub-mod of Md, so it has an only one primary sub-mod $\frac{N}{K}$ of $\frac{Md}{K}$ and $A+L=Md$, L is a sub-mod of Md so, $\frac{A+L}{K} = \frac{Md}{K}$ which results that $\frac{A+K}{K} + \frac{L+K}{K} = \frac{Md}{K}$ since $\frac{A+K}{K}$ is a proper sub-mod of $\frac{N}{K}$ and $\frac{Md}{K}$ is LPHM, therefore $\frac{A+K}{K}$ is a small sub-mod of $\frac{Md}{K}$. Thus $\frac{L+K}{K} = \frac{Md}{K}$, so $L+K=Md$, becuase K is a small sub-mod of Md, then $L = Md$. So Md is LPHM .

Corollary(1): If Md is a LPHM with a proper sub-mod N, then $\frac{Md}{N}$ is LPHM.

Proof : $\pi: Md \to \frac{Md}{N}$ be a natural function which is an epimorphism then $\frac{M}{N}$ is LPHM .

Remark (1): Generally, the converse of corollary(1) is not true, for instant $\frac{Z_{12}}{<2>} \cong Z_2$ is LPHM. But Z_{12} is not LPHM .

Recall an projective cover (P,f) of a module Md where P is a project module and $f: P \to Md$ where f is an epimorphism and $kerf$ is a small sub-mod of P (i.e. P is a project cover of Md) [4].

Proposition(3): If $f : M_1 \to M_2$ be a project cover of M_2 and M_2 is LPHM. So M_1 is LPHM .

Proof: Since $f : M_1 \to M_2$ is an epimorphism so $\frac{M_1}{\ker f}$ is an isomorphism to M_2 and then it is LPHM , also $kerf$ is a small submodule of M_1. That means M_1 is LPHM .

Proposition(4): Md is LPHM and $f.g$ module iff Md is a cyclic and has an only one primary sub-mod.

Proof: suppose that Md is a $f.g$ module and LPHM, therefore $Md = R_{x_1} + R_{x_2} + \cdots + R_{x_n}$. When $Md \neq R_{x_1}$ so, R_{x_1} is a proper sub module of Md that give R_{x_1} is a small sub-mod of Md. Since $Md = R_{x_2} + R_{x_3} + \cdots + R_{x_n}$. So, the summand will be canceled one by one till getting $Md = R_{x_i}$ for given i. Thus Md can be a cyclic module, so Md has an only one primary sub-mod .

Conversely, Consider Md is a cyclic module having an only one primary sub- mod N, so Md is *f.g*. Let L is a proper sub-mod of Md with $L + K = Md$ such that K is a sub-mod of Md. Then if L is not small sub-mod of Md gives $K \neq Md$. therefore K is a proper sub-mod of Md and K is sub module of N and since Md is $f.g$, thus K involved in a primary sub-mod. While, Md contains only one primary sub-mod N (by assumption). Thus L is contained in N. So, $L + N = N = M$ C!. Hence $K = Md$, L is a small sub-mod of Md and it is sub-mod of N. Hence Md is LPHM.

Proposition(5): Let Md be LPHM and N be a primary submodule. If $\frac{Md}{N}$ is $f.g$, then Md is $f.g$

Proof: Let $\frac{Md}{N} = R\,(x_1 + N) + R(x_2 + N) + \cdots + R(x_n + N)$ such that $x_i \in Md$ $\forall i = 1, 2, \cdots, n$. the statement $Md = R\,x_1 + R\,x_2 + \cdots + R\,x_n$ will be claimed . Let $m \in Md$, so $m + N \in \frac{Md}{N}$, implies that $m + N = r_1\,(x_1 + N) + r_2\,(x_2 + N) + \cdots + r_n\,(x_n + N) = r_1x_1 + r_2x_2 + \cdots + r_nx_n + N$. Then $m = r_1\,x_1 + r_2\,x_2 + \cdots + r_n\,x_n + n$ for some $n \in N$. So that, $Md = r_1\,x_1 + r_2\,x_2 + \cdots + r_n\,x_n + N$ and since Md is LPHM, then N is an only one prime contains each small sub-mod of Md which implies that $Md = r_1\,x_1 + r_2\,x_2 + \cdots + r_n\,x_n$. Thus Md is $f.g$.

Recall An R-module Md is called a LHM if it has an only one maximum sub-mods that contains each small sub-mods of Md [5].

Corollary(2): Let N be a primary sub-mod of Md. If Md is LPHM and $\frac{Md}{N}$ is $f.g$ then Md is LHM .

Proof: Let $\frac{Md}{N} = R(x_1 + N) + R(x_2 + N) + \cdots + R(x_n + N)$ with $x_i \in Md$ $\forall\, i = 1, 2, \cdots, n$ we claim that $Md = R\,x_1 + R\,x_2 + \cdots + R\,x_n$. Let $m \in Md$ then m+N $\in \frac{Md}{N}$ gives $m + N = r_1(x_1 + N) + r_2(x_2 + N) + \cdots + r_n(x_n + N) = r_1\,x_1 + r_2\,x_2 + \cdots + r_n\,x_n + N$. Implies $m = r_1\,x_1 + r_2\,x_2 + \cdots + r_n\,x_n + n$ for some $n \in N$. Thus $Md = r_1\,x_1 + r_2\,x_2 + \cdots + r_n\,x_n + N$ and since Md is LPHM, then N is a unique prime contains all small sub-mod of Md , implies $Md = r_1\,x_1 + r_2\,x_2 + \cdots + r_n\,x_n$. By remark(1) Md is $f.g$ and it is a LHM .

4. LPHM with LHM and LSPHM :

we shall study in this part of our paper a few conditions when LPHM can be LHM and survey the relation between LPHM and LSPHM .

Proposition(6): Md is a LHM iff Md is a LPHM and cyclic module.

Proof: N is an only one maximum sub-mod that involes all small sub-mod of Md. Suppose $x \in Md$ and $x \notin N$ so R_x will become sub-mod of Md. As similar as $R_x = Md$. If $R_x \neq Md$ so R_x is a small sub-mod of Md hence R_x is a sub-mod of N implies $x \in N$ C!. Thus $R_x = Md$ then Md is a cyclic module. And since Md is LHM then Md is a LPHM .

Conversely, since Md is cyclic and LPHM then it is a $f.g$ module. N is a max. sub module because it is a primary contains all small sub-mod. Given a proper small sub-mod L of Md. If L never containd in N that gives $L + N = Md$, but Md is LPHM so, $N = Md$ and that is a C!. therefore, we conclude that every proper small sub-mod of Md contained by $N$and so, Md will be LHM .

Proposition(7): Md is LHM iff Md is LPHM has an only one primary sub-mod.

Proof: Md has an only one max. sub-mod, hence Md has an only one primary sub-mod.

Conversely, to prove that Md is a cyclic module, suppose that $x \in Md$ with $x \notin N$, so that $R_x + N = Md$ and because Md is LPHM then N is a unique primary contains all small sub module of M and hence $Md = R_x$ .Therefore Md is a cyclic module. Thus Md is LHM .

Proposition(8): Md is LHM iff Rad $Md \neq M$ and Md is LPHM .

Proof: It is clear from the previous propositions .

Conversely, let Md is LPHM and Rad $Md \neq M$, then Rad Md is a small sub-mod of Md. Also by [4] $Rad\ Md$ has an only one max sub-mod of Md so it is a primary sub-mod, and hence $\frac{Md}{Rad\ Md}$ is a simple cyclic module. So $\frac{Md}{Rad\ Md} =< m + Rad\ Md >$ for a given $m \in Md$. We obtain $Md = Rm$. Suppose that $w \in Md$ so $w + RadMd \in \frac{Md}{Rad\ Md}$. There exists $r \in R$ such that $w + RadMd = r(m + RadMd) = rm + RadMd$, implies $w - rm \in RadMd$ so $w - rm = y$ for some $y \in RadMd$. Thus $w = rm + y \in Rm + RadMd$, hence $Md = Rm + RadMd$. But $RadMd$ is a small sub-mod of Md implies $Md = Rm$. Thus Md is a cyclic module. By (proposition 6) Md is LHM.

Recall a sub-mod N of Md is said to be a semi-prime sub-mod if $N \neq Md$ and whenever $r \in R, x \in Md$ such that $r^\wedge nx \in N$ for some n ∈ Z^+, then rx ∈ N [6].

Recall an R-module Md is called LSPHM if Md has an only one semiprime sub-mod contains all small sub-mod of Md [5]. Z_9 is LSPH Z-module , while Z_6 is not LSP-hollow Z-module .

Remark (2): Generally, each LPHM is LSPHM, but the converse is not true. Consider the following example. Let Z_6 is a LSPH Z-module, but Z_6 is not LPH Z-module.

Proposition (9): Md is LPHM iff Md is cyclic and LSPHM .

Proof: Suppose Md is LPHM so it has an only one primary sub-mod N that contains all small sub-mod of Md. Let $x \in Md$ with $x \notin N$ then R_x is a sub-mod of Md. We find that $R_x = Md$. If $R_x \neq Md$ then R_x is a proper small sub-mod of Md and hence R_x is a sub-mod of N, implies $x \in N$ C! . Thus $R_x = Md$ then Md is a cyclic module. Now, since Md is a LPHM, then Md is LSPHM.

Conversely; Let Md be LSPHM and cyclic module then it is $f.g$ module and hence Md has an only one max. so it is primary sub-mod called N contains all proper sub-mod in M. Let L be a proper small sub-mod of Md. If L is not contained in N then $L + N = Md$, but Md is LSPHM thus $N = Md$ C! . Implies every proper small sub- mod of Md is containing in N, thus Md is LPHM .

Corollary(3): Suppose Md is a module, then Md is LPHM iff Md is LSPH and $f.g$.

Proof: Let Md is LPHM, then Md is LSPHM and cyclic module, and since Md is cyclic module, thus Md is $f.g$.

Conversely, Let Md be $f.g$ and LSPHM then $Md = R_{x_1} + R_{x_2} + \cdots + R_{x_n}$. If $Md \neq R_{x_1}$ then R_{x_1} is a proper sub-mod of Md. thus R_{x_1} is a small sub-mod of Md. therefore $Md = R\ _{x_2} + R\ _{x_3} + \cdots + R\ _{x_n}$. Then the summand will be canceled one by one till getting $Md = R\ _{x_i}$ for given i. Thus Md is cyclic module. Therefore Md is LPHM .

Proposition(10): Md is LPHM iff Md is a LSPHM and has an only one semi-prime sub-mod.

Proof: Suppose that Md is LPHM, so Md is LSPHM , then Md has an only one semi-prime sub-mod.

Conversely; Let Md be LSPHM has an only one semi-prime sub-mod, say N, now to prove Md is cyclic module. Let $x \in Md$ such that $x \notin N$, so $R_x + N = Md$ and because Md is LPHM that gives N is a small sub-mod of Md and $Md = R_x$ Thus Md is cyclic and it is LPHM .

Proposition(11): Let Md be a module, Md is LPHM iff Md is $Rad\ Md \neq Md$ and LSPHM .
Proof: Let Md be LPHM, then Md is LSPH and cyclic module, then Md is $f.g$ and $Rad\ Md \neq Md$. In contrast, let Md is LSPHM, then $Rad\ Md$ is a small sub-mod of Md. Also by [proposition 4]. $Rad\ Md$ is the only one semi-primary sub-mod of Md and thus $\frac{Md}{Rad\ Md}$ is simple so it is cyclic. Implies that $\frac{Md}{Rad\ Md} = < m + Rad\ Md >$ for some $m \in Md$. We claim that $Md = Rm$. Let $w \in Md$ then $w + Rad\ Md \in \frac{Md}{Rad\ Md}$. There exists r ∈ R such that $w + Rad\ Md = r(m + Rad\ Md) = rm = Rad\ Md$. Implies that $w - rm \in Rad\ Md$ which implies that $w - rm = y$ for some y ∈ Rad(M). Thus $w = rm + y \in Rm + Rad\ Md$, hence $Md = Rm + RadMd$. But $Rad\ Md$ is a small sub-mod of Md implies $Md = Rm$. Therefore Md is a cyclic module .Therefore, Md is LPHM .

5. LPHM with some other Modules.

In this section we shall study the relation among LPHM and some other modules such as an amply supplemented, lifting and indecomposable .

Definition(2): A module Md is known as an amply supplemented if for each pair of sub-mods U, V of Md such that $Md = U + V$, ∃ a supplement $V_1$of U in Md, where $V_1 \leq V$ [7]. As Z_4 is amply supplemented Z-module . But Z_{12} is not amply supplemented Z-module .

Proposition(12): Each LPHM is an amply supplement .

Proof: Let Md be LPHM and U be a unique primary sub-mod of Md. Since Md is LPHM, then $U + Md = Md$ and $U \cap Md = U$ is a small sub-mod. Thus Md is amply supplemented.

Remark (3): Generally, the opposite of proposition 12 cannot necessary be applied, for instant Z_6 is amply supplemented Z-module, but it is not LPHM.

Definition(3):[1] Any R-module Md is indecomposable if $Md \neq 0$ and the unique direct summands of Md are <$\bar{0}$> and Md. Results that Md has not a direct sum of a couple of nonzero sub-mod.

Proposition(13): Each LPHM is indecomposable .

Proof: Let Md be LPHM, there exists an only one primary sub-mod N involves all small sub-mod of Md, assume that Md is decomposable, so that ∃ proper sub-mods K and L where K and L are sub-mods of N and $Md = K \oplus L$. But Md is LPHM, so L is a small sub-mod of Md where L is sub-mod of N that gives $K = Md$ or K is small sub-mod of Md with K is sub-mod of N that gives $L = Md$ which is C! . Then Md is indecomposable.

Proposition (14): Let Md be a cyclic and LPHM iff each nonzero factor module of Md is indecomposable .

Proof: Let $\frac{Md}{A}$ be a non-zero factor module of Md. Since Md is LPHM, then $\frac{Md}{A}$ is LPHM by (corollary 2). By (proposition13) $\frac{Md}{A}$ is indecomposable.

In contrast, let N be a primary sub-mod of Md and let L be a sub-mod of N. Suppose that $Md = L + K$, where K is a sub-mod of Md we get $\frac{Md}{L\cap K} \cong \frac{Md}{L \oplus Md/K}$ [4]. But $\frac{Md}{L\cap K}$ is indecomposable then either $\frac{Md}{L} = 0$ or $\frac{\mathrm{Md}}{K}$. Since L is a sub-mod of N, and N is a sub-mod of Md. Hence L is a proper sub-mod of Md. Then $\frac{M}{L} \neq 0$ implies that $\frac{\mathrm{Md}}{K} = 0$ and hence $Md = K$. Then L is a small sub-mod of Md. Md is hollow module and since Md is cyclic module. By (proposition 11) Md is LPHM .

Definition(4):[6] Suppose Md is a module, Md is known a lifting module if for each sub-mod N of Md there are sub-mods K and L of Md where $Md = K \oplus L$, K is a sub-mod of N and $N \cap K$ is a small sub-mod of K.

Remark(4): Generally, each lifting module is LPHM , but the converse cannot be necessary correct. We can see this in the following example.

Example (1): Consider Q is LPHM , while it is not lifting module [8].

Proposition(15): Each $f.g$ and LPHM is lifting module.

Proof: Let Md be a Z-module and Z_6 is a lifting module. But, Z-module Z_{12} is not LPHM. There exists a unique primary N of Md has all small sub module, then $Md = Md \oplus \{0\}$ where $\{0\}$ is a sub-mod of N , $N \cap M = N$ and since Md is LPHM, then $N \cap M = N$ is a small sub-mod of Md. Therefore Md is lifting module.

Remark(5): The opposite of proposition15 cannot necessary be applied in general, for instant, the Z-module $Md = Z_3 \oplus Z_4$ is lifting module [8], while it is not LPHM .Since there exists a primary sub-mod $N = Z_3 \oplus (0)$ of Md which not contains every small sub-mods.

Proposition(16): If Md is cyclic indecomposable module and Md is lifting module, then Md is LPHM .

Proof: Suppose N is a proper sub-mod of Md. Now Md is a lifting module then $Md = A + B$, such that A is a sub-mod and $N \cap A$ is small sub-mod of A. In contrast, Md is an indecomposable module, therefore $B = 0$ and hence $A = Md$. that implies that $N \cap M = N$, hence N is a small sub-mod of Md. Hence Md is hollow module and since Md is cyclic module. Then Md is LHM, so it is LPHM .

Conclusion:

The main goal of our study is to represent a new generalized of hollow module that is LPHM, that each LPHM is hollow module, we get that the epimorphic image of LPHM M_1 is LPHM M_2. If Md is LPHM, then $\frac{Md}{N}$ is LPHM for each proper sub-mod N of Md. the convers is true when N is small. Furthermore the relation between LPHM with some other modules as Md is a LHM iff Md is a LPHM and cyclic it has an only one primary , Rad $Md \neq Md$,…etc . And each LPHM is amply supplemented and indecomposable .

Future Studies :

We study and development of concepts generalization locally hollow module over non-commutative rings, locally semi primary lifting modules, locally strong primary hollow modules, some generalization of locally hollow modules, locally primary lifting modules and locally semi primary lifting modules .

References:

[1]. Abbas M. S. and Manhal F. M., 2018 , **d-Small submodules and d-Small Projective Modules** , Int. J. of Algebra, Vol. 12, No. 1, 25-30.

[2]. Mohammed M. and Ahmed A. G., 2013 , **On Some Properties of hollow and hollow dimension modules**, Pure and Applied Math. J. Vol. 2, No. 5, PP. 156-161.

[3]. Ali S. A. , Omar A. A. and Haibat K. M., 2020, Approximaitly Quasi-Primary Submodules, Ibn Al-Haithem Jour. For Pure and Appl. Sci. 33(4) .P.92-101.

[4]. Ali K. M. , 2005 , **Hollow Modules and Semihollow modules**, Thesis College of science , University of Baghdad.

[5].Thaer Z. K., 2019, **Some Generalizations Of L-Hollow Modules, Thesis College of Education for pure sciences ,** University of Tikrit .

[6]. S. H. Mohamed and B. J. Muller, **Continuous and Discrete Modules**, London math. Soc. LNS 147 Cambridge Univ. press, Cambridge, 1990.

[7]. Wisbauer R., 1991 , **Foundations of Module and Ring theory** , Gordon and Brtach Reading. Vol.(3). P. 351.

[8]. Al. Saad A. and S. Nedal Q., **FI-Hollow-Lifting Modules**, Al-Mustansiriyah J. Sci., 24 (5) (2013), PP. 293-306.

[9]. Y. S. Mahmood and H. W. Khalid., 2012, **Pure-Supplemented Modules,** Journal of Science. Vol 53. No 4, PP 882-886.

[10]. Jeathoom R. M. , 2017 , **Some Generalizations of Hollow-lifting Modules** , Thesis College of Sci. Mus. University.