Bifurcation of nonlinear impulsive boundary value problems

Zokha BELATTAR [a], Abdelkader LAKMECHE[b] and Mohamed HELAL[c]

[a] Univ Belhadj Bouchaib, Ain Témouchent, P.B. 101, 46000 , Algeria, e-mail: zokha.saiah@univ-temouchent.edu.dz,
[b] Biomathematics Laboratory, Univ. Sidi Bel-Abbes, P.B. 89, 22000, Algeria, e-mail: lakmeche@yahoo.fr,
[c] Biomathematics Laboratory, Univ. Sidi Bel-Abbes, P.B. 89, 22000, Algeria, e-mail: mhelal_abbes@yahoo.fr,

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Abstract: In this work, we investigate the existence of solutions for a class of second order impulsive differential equations with real parameter using bifurcation techniques by the mean of Krasnosel'ski theorems. Our results generalize some recent works.

Keywords: Branches of solutions, Impulsive differential equation, Bifurcation, Krasnosel'ski theorem.

1. Introduction

The theory of impulsive differential equations is a very active area of research see for instance [9] and [13]. Differential equations involving impulsive effects are found in many applications such as mathematical biology, population dynamics, optimal control and so on (see [6], [10], [11], [22] and [28]). There have been many works on the previous mentioned topics and among of them are interested by the study of the existence of solutions for second order impulsive boundary value problems ([1], [14] − [16], [18], [26], [27], [29]). However, research into bifurcation theory of impulsive differential equations has been modest (see [2] − [5], [12] and [19]). Some papers [17], [18], [21] and [25] introduced Rabinowitz global bifurcation theorems ([23], [24]) to describe the global structure of solutions of second order impulsive boundary value problems.

In this work, we consider the following impulsive boundary value problem (IBVP)

$$v''(t) = g(t, v(t), v'(t), \alpha), \quad t \in (0,1), \quad t \neq t_k, \tag{1.1}$$

$$\Delta v(t_k) = \eta_k(v(t_k), v'(t_k), \alpha), \quad k = 1, \dots, r, \tag{1.2}$$

$$\Delta v'(t_k) = \zeta_k(v(t_k), v'(t_k), \alpha), \tag{1.3}$$

$$v(0) = v(1) = 0. \tag{1.4}$$

Where $\Delta v(t_k) = v(t_k^+) - v(t_k)$, $\Delta v'(t_k) = v'(t_k^+) - v'(t_k)$, $0 = t_0 < t_1 < t_2 < \cdots < t_r < t_{r+1} = 1$, $r \in \mathbb{N}^\star = \mathbb{N} - \{0\}$ and $\alpha \in \mathbb{R}$.

Let $I' := I - \{t_k\}_{k=1}^r$. We assume that $g: I' \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is smooth enough, $\eta_k \in C^1(\mathbb{R}^3, \mathbb{R})$ and $\zeta_k \in C^1(\mathbb{R}^3, \mathbb{R})$, satisfying some assumptions to be specified later.

In [3], we considered the following impulsive boundary value problem

$$(I)\quad \begin{cases} v''(t) = \alpha f(t, v(t), v'(t)), \quad t \in (0,1), \quad t \neq t_k, \\ \Delta v(t_k) = \eta_k(v(t_k), v'(t_k), \alpha), \quad k = 1, \dots, r, \\ \Delta v'(t_k) = \zeta_k(v(t_k), v'(t_k), \alpha), \\ v(0) = v(1) = 0. \end{cases}$$

We studied the existence of multiple solutions for the relative nonlinear second order impulsive differential equations by the mean of Krasnosel'ski bifurcation theory (see [7], [8] and [20]). The key assumption in this theorem is the oddness of the algebraic multiplicity of the eigenvalues of the linearized problem .

The problem (IBVP) were studied in [4], we investigated the existence of solutions using the implicit function theorem and we analyzed the existence of multiple branches of solutions by the mean of Krasnosel'ski theorems where the impulsive effects are any functions depending implicitly on the real parameter α. But whose resolution of this problem we had to assume that the impulses η_k and ζ_k must defend explicitly from real α to apply this theorem.

In this work, we examine the existence of multiple branches of solutions for each case where the functions g, η_k and ζ_k depending implicitly on the real parameter α and are written in the form

$h_i(t, v^p(t), (v')^q(t), \alpha) + \alpha f_i(t, v(t), v'(t))$ with $p \geq 2$ and $q \geq 2$, $p, q \in \mathbb{N}$ for $i = \overline{1,3}$. h_i and f_i are given functions satisfying certain conditions.

Our paper is organized as follows, in section 2, we give some definitions and preliminary results that will be used throughout. Our main results are given in the section 3, which is composed of 3 parts, each one of them contains applications.

2. Preliminaries

For all $i \geq 0$, let

$PC^i(I) := \{v \in C^i(I', \mathbb{R}) / v^{(j)}$ *is left continuous at* t_k, *and* $v^{(j)}(t_k^+)$ *exist for all* k, j; $0 \leq k \leq r, 0 \leq j \leq i\}$.

$(PC^i(I), \|.\|_i)$ is a Banach space with the norm,

$$\| v \|_i = \max(\| v \|_0, \| v' \|_0, \ldots, \| v^{(i)} \|_0),$$

where $\| v \|_0 = \sup\{|w(t)|, t \in I\}$ for $w \in PC^0(I)$.

Let $\mathfrak{L}(PC^i(I))$ be the Banach space of bounded linear operators on $PC^i(I)$ and the standard norm in $\mathfrak{L}(PC^i(I))$, with

$$\| L \|_{\mathfrak{L}(PC^i(I))} = \sup_{\|x\| \leq 1} \| Lx \|_i \qquad ,$$

where $x \in PC^i(I)$ and $L \in \mathfrak{L}(PC^i(I))$.

Definition 2.1 *A pair* $(v, \alpha) \in PC^2(I) \times \mathbb{R}$ *is called a solution of* $(IBVP)$ *if it satisfies* $(1.1) - (1.4)$.

Lemma 2.2 *A pair* $(v, \alpha) \in PC^2(I) \times \mathbb{R}$ *is a solution of* $(IBVP)$ *if and only if* $(v, \alpha) \in PC^1(I) \times \mathbb{R}$ *and it satisfies the following equation*

$$v(t) = \int_0^1 H(t,s) g(s, v(s), v'(s), \alpha) ds + \sum_{0<t_k<t} \{\eta_k(v(t_k), v'(t_k), \alpha) + \zeta_k(v(t_k), v'(t_k), \alpha)(t - t_k)\}$$

$$-t \sum_{k=1}^{r} \{\eta_k(v(t_k), v'(t_k), \alpha) + \zeta_k(v(t_k), v'(t_k), \alpha)(1 - t_k)\}, \qquad \forall t \in I,$$

where H is the Green's function of the linear problem without impulses

$$H(t,s) = \begin{cases} s(t-1) & \text{if } 0 \leq s \leq t \leq 1, \\ t(s-1) & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

$$= ts - \min(t,s), \quad (t,s) \in [0,1]^2.$$

Let $L: D(L) \subset PC^2(I) \to PC^2(I)$ be defined by

$$(Lu)(t) = u''(t), \; t \in I,$$

where $D(L) := \{u \in PC^2(I); u(0) = u(1) = 0\}$.

Proposition 2.3 *The operator L is invertible, and* $L^{-1}: PC^0(I) \to PC^2(I)$ *is given by*

$$(L^{-1}u)(t) = \int_0^1 H(t,s) u(s) ds.$$

Let G be the Nemitskii operator corresponding to g, then

$$G: PC^1(I) \times \mathbb{R} \to PC^0(I),$$
$$G(v,\alpha)(t) \quad := \quad g(t, v(t), v'(t), \alpha).$$

Let $\psi: PC^2(I) \times \mathbb{R} \to PC^2(I)$ be defined by

$$\psi(v,\lambda)(t) \quad = \quad \sum_{0<t_k<t} \{\eta_k(v(t_k), v'(t_k), \alpha) + \zeta_k(v(t_k), v'(t_k), \alpha)(t - t_k)\}$$
$$-t \sum_{k=1}^{r} \{\eta_k(v(t_k), v'(t_k), \alpha) + \zeta_k(v(t_k), v'(t_k), \alpha)(1 - t_k)\}.$$

Consider the map $F: PC^2(I) \times \mathbb{R} \to PC^2(I)$ such that

$$F(v,\alpha) := L^{-1}GJ(v,\alpha) + \psi(v,\alpha),$$

where J is the compact imbedding defined by $J: PC^2(I) \times \mathbb{R} \longrightarrow PC^1(I) \times \mathbb{R}$ with $J(v,\alpha) = (v,\alpha)$.

Then we have

$$L^{-1}[G(J(v,\alpha))](t) \quad = \quad \int_0^1 H(t,s) g(s, v(s), v'(s), \alpha) \mathrm{d}s$$

Lemma 2.4 *The operators ψ and F are compact.*

Lemma 2.5 *$(v,\alpha) \in PC^2(I) \times \mathbb{R}$ is a solution of $(IBVP)$ if and only if $F(v,\alpha) = v$.*

For fixed $\alpha \in \mathbb{R}$, we have $\frac{\partial F}{\partial v}(.,\alpha): PC^2(I) \to \mathfrak{L}(PC^2(I))$ and

$$\frac{\partial F}{\partial v}(v,\alpha).\varphi = \frac{\partial(L^{-1} \circ GJ)}{\partial v}(v,\alpha).\varphi + \frac{\partial \psi}{\partial v}(v,\alpha).\varphi,$$

for $\varphi \in PC^2(I)$.

In the following, we give some theorems to be used in the study of $(IBVP)$.

Let X be a Banach space and consider the following equation

$$M_s(v,\alpha) := v - \alpha \overline{M}_s v + N_s(v,\alpha) = 0, \tag{2.1}$$

where $v \in X$, $\overline{M}_s: X \to X$ is a linear compact operator with $s = \overline{1,3}$, and $N_s: X \times \mathbb{R} \to X$ is a continuous mapping satisfying

$$(H1) \quad N_s(v,\alpha) = \circ\, (\| v \|_X).$$

Remark 2.6 *It is clear that $(v,\alpha) = (0,\alpha)$ is a trivial solution of (2.1). The bifurcation problem of (2.1) is to seek a nontrivial solution $(v_\alpha,\alpha) \neq (0,\alpha^\star)$ of (2.1) from some point $(0,\alpha^\star)$ such that*

$$v_\alpha \to 0 \quad \text{as} \quad \alpha \to \alpha^\star.$$

Theorem 2.7 [24, *Krasnosel'ski Theorem*] *Under the condition $(H1)$, if $\lambda \in \mathbb{R}^\star = \mathbb{R} - \{0\}$ is a real eigenvalue of $\overline{M}_s$ with odd algebraic multiplicity, then $(0,\lambda^{-1})$ is a bifurcation point of (2.1).*

Theorem 2.8 [24, *Theorem* 1.11] *If $\lambda \in \mathbb{R}^\star$ is a simple eigenvalue of $\overline{M}_s$, then the equation (2.1) bifurcates from $(0,\lambda^{-1})$ to exactly two branches Γ_1 and Γ_2.*

3 Bifurcation of branches of solutions for $(IBVP)$

In this section, we investigate the existence of bifurcated solutions for IBVP, we have

$$N_s(v,\alpha) := \alpha \overline{M}_s v - F(v,\alpha), \qquad s = \overline{1,3}.$$

So

Proposition 3.1 *If*

- $g(t,0,0,\alpha) = 0, \quad \forall t \in I$ and $\forall \alpha \in \mathbb{R}$,
- $\eta_k(0,0,\alpha) = 0, \forall \alpha \in \mathbb{R}$
- $\zeta_k(0,0,\alpha) = 0, \forall \alpha \in \mathbb{R}$

Then, for $s = \overline{1,3}$, we have

$$N_s(0,\alpha) = 0.$$

3.1 Case where $g = \Box_1 + \alpha g_1$

In this section, we investigate the existence of bifurcated solutions where

$g(t,v(t),v'(t),\alpha) = h_1(t,v^p(t),(v')^q(t),\alpha) + \alpha g_1(t,v(t),v'(t))$ with $p \geq 2$ and $q \geq 2, p,q \in \mathbb{N}$.

We have

$$\begin{aligned}
\frac{\partial(L^{-1}\circ GJ)}{\partial v}(v,\alpha).\varphi &= \int_0^1 H(t,s)\frac{\partial g}{\partial v}(s,v(s),v'(s),\alpha).\varphi \mathrm{d}s \\
&= \int_0^1 H(t,s)[(pv^{p-1}(s)\frac{\partial h_1}{\partial x}(s,v^p(s),(v')^q(s),\alpha)).\varphi(s) \\
&\quad +(q(v')^{q-1}(s)\frac{\partial h_1}{\partial y}(s,v^p(s),(v')^q(s),\alpha)).\varphi'(s) \\
&\quad +\alpha(\frac{\partial g_1}{\partial x}(s,v(s),v'(s)).\varphi(s) + \frac{\partial g_1}{\partial y}(s,v(s),v'(s)).\varphi'(s))]\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \psi}{\partial v}(v,\alpha).\varphi &= \sum_{0<t_k<t}\Big\{[\frac{\partial \eta_k}{\partial x}(v(t_k),v'(t_k),\alpha).\varphi(t_k) + \frac{\partial \eta_k}{\partial y}(v(t_k),v'(t_k),\alpha).\varphi'(t_k)] \\
&\quad + [\frac{\partial \zeta_k}{\partial x}(v(t_k),v'(t_k),\alpha).\varphi(t_k) + \frac{\partial \zeta_k}{\partial y}(v(t_k),v'(t_k),\alpha).\varphi'(t_k)](t-t_k)\Big\} \\
&\quad -t\sum_{0<t_k<1}\Big\{[\frac{\partial \eta_k}{\partial x}(v(t_k),v'(t_k),\alpha).\varphi(t_k) + \frac{\partial \eta_k}{\partial y}(v(t_k),v'(t_k),\alpha).\varphi'(t_k)] \\
&\quad + [\frac{\partial \zeta_k}{\partial x}(v(t_k),v'(t_k),\alpha).\varphi(t_k) + \frac{\partial \zeta_k}{\partial y}(v(t_k),v'(t_k),\alpha).\varphi'(t_k)](1-t_k)\Big\}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial F}{\partial v}(0,\alpha).\varphi &= \frac{\partial(L^{-1}\circ GJ)}{\partial v}(0,\alpha).\varphi + \frac{\partial \psi}{\partial v}(0,\alpha).\varphi \\
&= \alpha\int_0^1 H(t,s)(\frac{\partial g_1}{\partial x}(s,0,0).\varphi(s) + \frac{\partial g_1}{\partial y}(s,0,0).\varphi'(s))\mathrm{d}s \\
&\quad + \sum_{0<t_k<t}\Big\{[\frac{\partial \eta_k}{\partial x}(0,0,\alpha).\varphi(t_k) + \frac{\partial \eta_k}{\partial y}(0,0,\alpha).\varphi'(t_k)] \\
&\quad + [\frac{\partial \zeta_k}{\partial x}(0,0,\alpha).\varphi(t_k) + \frac{\partial \zeta_k}{\partial y}(0,0,\alpha).\varphi'(t_k)](t-t_k)\Big\} \\
&\quad -t\sum_{0<t_k<1}\Big\{[\frac{\partial \eta_k}{\partial x}(0,0,\alpha).\varphi(t_k) + \frac{\partial \eta_k}{\partial y}(0,0,\alpha).\varphi'(t_k)] \\
&\quad + [\frac{\partial \zeta_k}{\partial x}(0,0,\alpha).\varphi(t_k) + \frac{\partial \zeta_k}{\partial y}(0,0,\alpha).\varphi'(t_k)](1-t_k)\Big\}.
\end{aligned}$$

We put additional hypotheses as follow

- $\frac{\partial \eta_k}{\partial x}(0,0,\alpha) = \frac{\partial \eta_k}{\partial y}(0,0,\alpha) = 0, \ \forall \alpha \in \mathbb{R}$,

- $\frac{\partial \zeta_k}{\partial x}(0,0,\alpha) = \frac{\partial \zeta_k}{\partial y}(0,0,\alpha) = 0, \ \forall \alpha \in \mathbb{R}$.

If $(D_u N_1(0,\alpha))\varphi(t) := \alpha \overline{M}_1 \varphi(t) - \frac{\partial F}{\partial v}(0,\alpha)\varphi(t) = 0$, and from $(H5)-(H6)$, we have

$$\overline{M}_1\varphi(t) \quad = \quad \int_0^1 H(t,s)\left(\frac{\partial g_1}{\partial x}(s,0,0).\varphi(s) + \frac{\partial g_1}{\partial y}(s,0,0).\varphi'(s)\right)\mathrm{d}s.$$

Proposition 3.2 *If the conditions $(H2)-(H6)$ are satisfied, then*

$$N_1(v,\alpha) = \circ\,(\| v \|_2).$$

From [3], we deduce the following results.

Theorem 3.3 *If $(H2)-(H6)$ are satisfied and $\lambda \in \mathbb{R}^\star$ is a real eigenvalue of $\overline{M}_1$, with odd algebraic multiplicity, then $(v,\alpha) = (0,\lambda^{-1})$ is a bifurcation point of $M_1(v,\alpha) = 0$, and $(IBVP)$ has a bifurcation branch of solutions.*

Theorem 3.4 *If $(H2)-(H6)$ are satisfied and $\lambda \in \mathbb{R}^\star$ is a real simple eigenvalue of $\overline{M}_1$, then $(IBVP)$ hasexactly two bifurcated branches of solutions Γ_1 and Γ_2 from $(0,\lambda^{-1})$.*

In the past theorem we assume that the multiplicity of λ is simple, to study the multiplicity of λ,

let $\sigma(s) := \frac{\partial g_1}{\partial x}(s,0,0)$ and $\rho(s) := \frac{\partial g_1}{\partial y}(s,0,0)$, then we have

$$(\overline{M}_1\varphi)(t) = \int_0^1 G(t,s)(\sigma(s).\varphi(s) + \rho(s).\varphi'(s))ds, \qquad \varphi \in PC^2(I).$$

Lemma 3.5 *$\lambda(\neq 0)$ is a real eigenvalue of $\overline{M}_1$, if and only if there exists $\varphi \in PC^2(I) - \{0\}$ such that λ satisfies the boundary value problem (BVP)*

$$\begin{cases} \varphi''(t) - \frac{\rho(t)}{\lambda}\varphi'(t) - \frac{\sigma(t)}{\lambda}\varphi(t) = 0 & \forall t \in I', \\ \varphi(0) = \varphi(1) = 0. \end{cases}$$

Corollary 3.6 *If $(H2)-(H6)$ are satisfied and $\lambda \in \mathbb{R}^\star$ is a real eigenvalue of the boundary value problem (BVP), with odd algebraic multiplicity, then $(IBVP)$ has a bifurcated branches of solutions from $(0,\lambda^{-1})$.*

Corollary 3.7 *If $(H2)-(H6)$ are satisfied and $\lambda \in \mathbb{R}^\star$ is a simple eigenvalue of the boundary value problem (BVP), then $(IBVP)$ has exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $(0,\lambda^{-1})$.*

Application

Consider the following boundary value problem

$$\begin{cases} v''(t) = \ln(1 + \alpha^2 v^4(t) + (v')^2(t)) + \alpha\sin(av(t) + bv'(t)), \quad t \in (0,1), \quad t \neq t_k, \\ \Delta v(t_k) = \eta_k(u(t_k), u'(t_k), \alpha), \quad k = 1,\dots,r, \\ \Delta v'(t_k) = \zeta_k(u(t_k), u'(t_k), \alpha), \quad k = 1,\dots,r, \\ v(0) = v(1) = 0. \end{cases} \tag{3.1}$$

where a and b are constant.

The function $g(t,v,w,\alpha) = \ln(1+\alpha^2 v^4(t) + w^2(t)) + \alpha\sin(av(t)+bw(t))$ is well defined on the neighborhood of $(0,0)$ and it satisfies $(H2)$.

The functions η_k and ζ_k satisfy the conditions $(H3)-(H6)$.

For $g_1(t,v,w) = \sin(av(t)+bw(t))$, we have $\frac{\partial g_1}{\partial x}(t,0,0) = a$ and $\frac{\partial g_1}{\partial y}(t,0,0) = b$. Then

$$\overline{M}_1\varphi(t) = \int_0^1 H(t,s)(a\varphi(s) + b\varphi'(s))ds.$$

We suppose that λ is the eigenvalue of $\overline{M}_1$. Then, if $\lambda < \frac{-b^2}{4a}$ and $a \neq \pm bn\pi, \quad n \geq 1,$

we have $\lambda_n^1 = \frac{-a-\sqrt{a^2-b^2n^2\pi^2}}{2n^2\pi^2}$ and $\lambda_n^2 = \frac{-a+\sqrt{a^2-b^2n^2\pi^2}}{2n^2\pi^2}$ are the real simple eigenvalues of $\overline{M}_1$.

Corollary 3.7 implies that if $\alpha = \lambda_n^1$ or $\alpha = \lambda_n^2$ for $n \in \mathbb{N}^\star$, then the problem (3.1) has exactly two branches Γ_1^j and Γ_2^j bifurcating from $(0,(\lambda_n^j)^{-1})$ with $j = 1,2$ (see [3] for more details).

3.2 Case where $\boldsymbol{\eta_k = \square_2 + \alpha\xi_k}$

In this section, we investigate the existence of bifurcated solutions where

$\eta_k(v(t_k),v'(t_k),\alpha) = h_2(v^p(t_k),(v')^q(t_k),\alpha) + \alpha\xi_k(v(t_k),v'(t_k))$ with $p \geq 2$ and $q \geq 2,\ p,q \in \mathbb{N}$.

We have

$$\frac{\partial(L^{-1}\circ GJ)}{\partial v}(v,\alpha).\varphi = \int_0^1 H(t,s)[\frac{\partial g}{\partial x}(s,v(s),v'(s),\alpha).\varphi(s) + \frac{\partial g}{\partial y}(s,v(s),v'(s),\alpha).\varphi'(s)]ds$$

and

$$\begin{aligned}\frac{\partial\psi}{\partial v}(v,\alpha).\varphi = {} & \sum_{0<t_k<t}\Big\{[(pv^{p-1}(t_k)\frac{\partial h_2}{\partial x}(v^p(t_k),(v')^q(t_k),\alpha)).\varphi(t_k)\\
&+(q(v')^{q-1}(t_k)\frac{\partial h_2}{\partial y}(v^p(t_k),(v')^q(t_k),\alpha)).\varphi'(t_k)]\\
&+\alpha[\frac{\partial\xi_k}{\partial x}(v(t_k),v'(t_k)).\varphi(t_k) + \frac{\partial\xi_k}{\partial y}(v(t_k),v'(t_k)).\varphi'(t_k)]\\
&+[\frac{\partial\zeta_k}{\partial x}(v(t_k),v'(t_k),\alpha).\varphi(t_k) + \frac{\partial\zeta_k}{\partial y}(v(t_k),v'(t_k),\alpha).\varphi'(t_k)](t-t_k)\Big\}\\
&-t\sum_{0<t_k<1}\Big\{[(pv^{p-1}(t_k)\frac{\partial h_2}{\partial x}(v^p(t_k),(v')^q(t_k),\alpha)).\varphi(t_k)\\
&+(q(v')^{q-1}(t_k)\frac{\partial h_2}{\partial y}(v^p(t_k),(v')^q(t_k),\alpha)).\varphi'(t_k)]\\
&+\alpha[\frac{\partial\xi_k}{\partial x}(v(t_k),v'(t_k)).\varphi(t_k) + \frac{\partial\xi_k}{\partial y}(v(t_k),v'(t_k)).\varphi'(t_k)]\\
&+[\frac{\partial\zeta_k}{\partial x}(v(t_k),v'(t_k),\alpha)\varphi(t_k) + \frac{\partial\zeta_k}{\partial y}(v(t_k),v'(t_k),\alpha).\varphi'(t_k)](1-t_k)\Big\}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial F}{\partial v}(0,\alpha).\varphi &= \frac{\partial(L^{-1}\circ GJ)}{\partial v}(0,\alpha).\varphi + \frac{\partial\psi}{\partial v}(0,\alpha).\varphi\\
&= \int_0^1 H(t,s)(\frac{\partial g}{\partial x}(s,0,0,\alpha).\varphi(s) + \frac{\partial g}{\partial y}(s,0,0,\alpha).\varphi'(s))\\
&\quad+\sum_{0<t_k<t}[\alpha(\frac{\partial\xi_k}{\partial x}(0,0).\varphi(t_k) + \frac{\partial\xi_k}{\partial y}(0,0).\varphi'(t_k))\\
&\quad+(\frac{\partial\zeta_k}{\partial x}(0,0,\alpha).\varphi(t_k) + \frac{\partial\zeta_k}{\partial y}(0,0,\alpha).\varphi'(t_k))(t-t_k)]\\
&\quad-t\sum_{0<t_k<1}[\alpha(\frac{\partial\xi_k}{\partial x}(0,0).\varphi(t_k) + \frac{\partial\xi_k}{\partial y}(0,0).\varphi'(t_k))\\
&\quad+(\frac{\partial\zeta_k}{\partial x}(0,0,\alpha).\varphi(t_k) + \frac{\partial\zeta_k}{\partial y}(0,0,\alpha).\varphi'(t_k))(1-t_k)]\end{aligned}$$

We put additional hypotheses as follow

- $\frac{\partial g}{\partial x}(t,0,0,\alpha) = \frac{\partial g}{\partial y}(t,0,0,\alpha) = 0,\ \forall \alpha \in \mathbb{R}$,
- $\frac{\partial \zeta_k}{\partial x}(0,0,\alpha) = \frac{\partial \zeta_k}{\partial y}(0,0,\alpha) = 0,\ \forall \alpha \in \mathbb{R}$.

If $(D_u N_2(0,\alpha))\varphi(t) := \alpha \overline{M}_2 . \varphi(t) - \frac{\partial F}{\partial v}(0,\alpha).\varphi(t) = 0$, and from $(H7)$ and $(H8)$, we have

$\overline{M}_2\varphi(t) = \sum_{0<t_k<t} (\frac{\partial \xi_k}{\partial x}(0,0).\varphi(t_k) + \frac{\partial \xi_k}{\partial y}(0,0).\varphi'(t_k)) - t\sum_{0<t_k<1} (\frac{\partial \xi_k}{\partial x}(0,0).\varphi(t_k) + \frac{\partial \xi_k}{\partial y}(0,0).\varphi'(t_k))$.

Proposition 3.8 *If the conditions $(H2)-(H4),(H7)$ and $(H8)$ are satisfied, then*

$$N_2(v,\alpha) = \circ\,(\| v \|_2).$$

Then, from theorem 2.7 we have

Theorem 3.9 *If the hypotheses $(H2)-(H4),(H7)$ and $(H8)$ are satisfied and $\lambda \in \mathbb{R}^\star$ is a real eigenvalue of M_2, with odd algebraic multiplicity, then $(v,\alpha) = (0,\lambda^{-1})$ is a bifurcation point of $M_2(v,\alpha) = 0$ and $(1.1)-(1.4)$ has a bifurcated branches of solutions.*

And from theorem 2.8 we have

Theorem 3.10 *If the hypotheses $(H2)-(H4)$ and $(H7)-(H8)$ are satisfied and $\lambda \in \mathbb{R}^\star$ is a simple eigenvalue of $\overline{M}_2$, then $(1.1)-(1.4)$ has exactly two bifurcated branches of solutions from $(0,\lambda^{-1})$.*

In the following we study the multiplicity of the eigenvalues of $\overline{M}_2$ to determine the number of branches of solutions.

To do that let $b_k := \frac{\partial \xi_k}{\partial x}(0,0)$, $c_k := \frac{\partial \xi_k}{\partial y}(0,0)$ and put $A_k := b_k t_k + c_k$.

Let $f_k(t) = h_k(t).t$ with

$$h_k(t) = \begin{cases} 1 & if \quad t \in]t_k, t_{k+1}[, \\ 0 & otherwise\ , \end{cases}$$

$k = 0,1,2,\ldots,r$.

Proposition 3.11 *For $s = 2,3$.*

Let $\mathbb{E} := \{\varphi \in PC^2(I) / \varphi(t) = \sum_{k=0}^{r} \gamma_k f_k(t) + \beta_k h_k(t),\ t \neq t_k\}$.

Then $\mathbb{E}$ be a Banach space with $\dim\mathbb{E} = 2r+2$, moreover $\forall \varphi \in PC^2(I)$, $\overline{M}_s\varphi \in \mathbb{E}$.

Remark 3.12 *For $s = 2,3$. Let λ be an eigenvalue of $\overline{M}_s$ and φ_λ an eigenvector of $\overline{M}_s$ associated to λ. Then*

$$\varphi_\lambda(t) = \begin{cases} 0 & if \quad t = 0, \\ \sum_{k=1}^{r} \gamma_k(\varphi_\lambda) f_k(t) + \beta_k(\varphi_\lambda) h_k(t) & if \quad t \neq t_k, \\ \gamma_{k-1}(\varphi_\lambda) t_k + \beta_{k-1}(\varphi_\lambda) & if \quad t = t_k, \\ 0 & if \quad t = 1. \end{cases}$$

We denote $\gamma_k(\varphi_\lambda) := \gamma_k$ and $\beta_k(\varphi_\lambda) := \beta_k$.

Proposition 3.13 *Let $\lambda \in \mathbb{R}^*$. Then λ is an eigenvalue of $\overline{M}_2$ if and only if there exist $\gamma_0,\ldots,\gamma_r,\beta_0,\ldots,\beta_r \in \mathbb{R}$ such that λ satisfies the following system with $(2r+2)$ equations*

$$
(II)\begin{cases}
(\lambda + A_1)\gamma_0 + \sum_{i=2}^{r} A_i\gamma_{i-1} + \sum_{i=1}^{r} b_i\beta_{i-1} = 0,\\
\lambda\beta_0 = 0,\\
A_1\gamma_0 + (\lambda + A_2)\gamma_1 + \sum_{i=3}^{r} A_i\gamma_{i-1} + \sum_{i=1}^{r} b_i\beta_{i-1} = 0,\\
-A_1\gamma_0 - b_1\beta_0 + \lambda\beta_1 = 0,\\
\vdots\\
\sum_{i=1}^{k} A_i\gamma_{i-1} + (\lambda + A_{k+1})\gamma_k + \sum_{i=k+2}^{r} A_i\gamma_{i-1} + \sum_{i=1}^{r} b_i\beta_{i-1} = 0,\\
\sum_{i=1}^{k} - A_i\gamma_{i-1} + \sum_{i=1}^{k} - b_i\beta_{i-1} + \lambda\beta_k = 0,\\
\vdots\\
\sum_{i=1}^{r-1} A_i\gamma_{i-1} + \gamma_{r-1}(\lambda + A_r) + \sum_{i=1}^{r} b_i\beta_{i-1} = 0,\\
\sum_{i=1}^{r-1} - A_i\gamma_{i-1} + \sum_{i=1}^{r-1} - b_i\beta_{i-1} + \lambda\beta_{r-1} = 0,\\
\sum_{i=1}^{r} A_i\gamma_{i-1} + \lambda\gamma_r + \sum_{i=1}^{r} b_i\beta_{i-1} = 0,\\
\sum_{i=1}^{r} - A_i\gamma_{i-1} - \sum_{i=1}^{r} b_i\beta_{i-1} + \lambda\beta_r = 0.
\end{cases}
$$

with $k = 1, \ldots, r-1$.

Moreover the eigenvector associated to λ is given by

$$
\begin{aligned}
\varphi_\lambda(t) &= \sum_{k=1}^{r} \gamma_k f_k(t) + \beta_k h_k(t)\\
&= \sum_{k=1}^{r} h_k(t)(\gamma_k t + \beta_k),\ t \neq t_k,\ t \in [0,1].
\end{aligned}
$$

Proof. If $t \in]0, t_1[$, $\overline{M}_2\varphi(t) = \lambda\varphi(t)$ is equivalent to

$$\sum_{0<t_k<t} (b_k\varphi(t_k) + c_k\varphi'(t_k)) - t\sum_{k=1}^{r} (b_k\varphi(t_k) + c_k\varphi'(t_k)) = \lambda(\gamma_0 t + \beta_0).$$

Then,

$$-t\sum_{k=1}^{r} (b_k\varphi(t_k) + c_k\varphi'(t_k)) = \lambda(\gamma_0 t + \beta_0),\ \forall t \in]0, t_1[.$$

We obtain

$$t[\lambda\gamma_0 + b_1(\gamma_0 t_1 + \beta_0) + c_1\gamma_0 + \sum_{i=2}^{r} b_i(\gamma_{i-1}t_i + \beta_{i-1}) + c_i\gamma_{i-1}] + \lambda\beta_0 = 0,\ \forall t \in]0, t_1[.$$

Then,

$$
\begin{cases}
\lambda\beta_0 = 0,\\
\gamma_0(\lambda + A_1) + \sum_{i=2}^{r} A_i\gamma_{i-1} + \sum_{i=1}^{r} b_i\beta_{i-1} = 0.
\end{cases}
$$

and

$$
\begin{cases}
\sum_{i=1}^{k} - A_i\gamma_{i-1} + \sum_{i=1}^{k} - b_i\beta_{i-1} + \lambda\beta_k = 0,\\
\sum_{i=1}^{k} A_i\gamma_{i-1} + (\lambda + A_{k+1})\gamma_k + \sum_{i=k+2}^{r} A_i\gamma_{i-1} + \sum_{i=1}^{r} b_i\beta_{i-1} = 0.
\end{cases}
$$

For $t \in]t_r, 1[$, $\overline{M}_2\varphi(t) = \lambda\varphi(t)$ is equivalent to

$$\lambda(\gamma_r t + \beta_r) = \sum_{k=1}^{r} b_k\varphi(t_k) + c_k\varphi'(t_k) - t\sum_{k=1}^{r} b_k\varphi(t_k) + c_k\varphi'(t_k)$$

Then,

$$
\begin{cases}
-\sum_{i=1}^{r} A_i\gamma_{i-1} - \sum_{i=1}^{r} b_i\beta_{i-1} + \lambda\beta_r = 0,\\
\sum_{i=1}^{r} A_i\gamma_{i-1} + \lambda\alpha_r + \sum_{i=1}^{r} b_i\beta_{i-1} = 0.
\end{cases}
$$

Lemma 3.14 *Let $\lambda \in \mathbb{R}^\star$. Then λ is an eigenvalue of $\overline{M}_2$ if and only if there exist $\gamma_0, \ldots, \gamma_r, \beta_0, \ldots, \beta_r \in \mathbb{R}$ such that*

$$(III) \qquad T_2(\lambda)\begin{pmatrix}\gamma_0\\ \gamma_1\\ \vdots\\ \gamma_r\\ \beta_0\\ \beta_1\\ \vdots\\ \beta_r\end{pmatrix} = 0,$$

where $T_2(\lambda)$ is the $(2r+2)$ square matrix such that

$$T_2(\lambda) = \begin{pmatrix}\tilde{A} & \tilde{B}\end{pmatrix}$$

where $\tilde{A}$ is a $(2r+2)\times(r+1)$ matrix and $\tilde{B}$ is a $(2r+2)\times(r+1)$ matrix satisfying

- $\tilde{A} = (a_{ij})$ with

$$\begin{cases} a_{(2i)j} = \lambda + A_j \ \ with \ \ i = \overline{1,r} \quad for \quad j = i, \\ a_{(2i)j} = A_j \ \ with \ \ i = \overline{2,r+1} \quad for \quad 1 \le j < i, \\ a_{(2i)j} = A_j \ \ with \ \ i = \overline{1,r-1} \quad and \quad j = \overline{2,r} \quad for \quad j > i, \\ a_{(2i+1)j} = -A_j \ \ with \ \ i = \overline{1,r} \quad for \quad 1 \le j \le i, \\ a_{(2i+1)j} = 0 \quad with \ \ i = \overline{0,r} \quad and \quad j = \overline{1,r+1} \quad for \quad j > i. \end{cases}$$

Then,

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \mu + A_1 & A_2 & A_3 & \dots & A_{r-1} & A_r & 0 \\ -A_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ A_1 & \lambda + A_2 & A_3 & \dots & A_{r-1} & A_r & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ -A_1 & -A_2 & -A_3 & \dots & -A_{r-1} & -A_r & 0 \\ A_1 & A_2 & A_3 & \dots & A_{r-1} & \lambda + A_r & 0 \\ -A_1 & -A_2 & -A_3 & \dots & -A_{r-1} & -A_r & 0 \\ A_1 & A_2 & A_3 & \dots & A_{r-1} & A_r & \lambda \end{pmatrix}.$$

- $\tilde{B} = (a_{ij})$ with

$$\begin{cases} a_{(2i)j} = b_{j-(r+1)}, \ \ a_{(2i)(2r+2)} = 0 \ \ with \quad i = \overline{1,r+1} \ \ and \ \ j = \overline{r+2,2r+1} \quad for \quad j-(r+1) \ge 1, \\ a_{(2i+1)j} = \lambda \quad with \quad i = \overline{0,r} \quad for \quad j-(r+2) = i, \\ a_{(2i+1)j} = -b_{j-(r+1)} \ \ with \ \ i = \overline{1,r} \quad for \quad 1 \le j-(r+1) \le i, \\ a_{(2i+1)j} = 0 \ \ with \ \ i = \overline{0,r-1} \ \ and \ \ j = \overline{r+3,2r+2} \quad for \quad j-(r+2) > i. \end{cases}$$

Then,

$$\tilde{B}=\begin{pmatrix}\lambda & 0 & 0 & \dots & 0 & 0 & 0\\ b_1 & b_2 & b_3 & \dots & b_{r-1} & b_r & 0\\ -b_1 & \lambda & 0 & \dots & 0 & 0 & 0\\ b_1 & b_2 & b_3 & \dots & b_{r-1} & b_r & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ -b_1 & -b_2 & -b_3 & \dots & -b_{r-1} & \lambda & 0\\ b_1 & b_2 & b_3 & \dots & b_{r-1} & b_r & 0\\ -b_1 & -b_2 & -b_3 & \dots & -b_{r-1} & -b_r & \lambda\\ b_1 & b_2 & b_3 & \dots & b_{r-1} & b_r & 0\end{pmatrix}.$$

Proof. From the proposition 3.13, the system (II) is equivalent to (III).

Put $P_s(\lambda)=detT_s(\lambda)$ with $s=2,3$, then $\lambda\in\mathbb{R}^*$ is an eigenvalue of $\overline{M}_s$ if and only if $T_s(\lambda)$ is not invertible, i.e. $P_s(\lambda)=0$.

Remark 3.15 *Let λ be a real eigenvalue of $\overline{M}_s$ with $s=2,3$. If λ satisfies*

$(H9)$ $\quad P_s(\lambda)=P_s'(\lambda)=P_s''(\lambda)=\ldots=P_s^{2\kappa}(\lambda)=0$ and $P_s^{2\kappa+1}(\lambda)\neq 0,\ \kappa\in\mathbb{N}$,

then it is an eigenvalue with odd algebraic multiplicity $2\kappa+1$.

If λ is a simple eigenvalue of $\overline{M}_s$, i.e $\kappa=0$, then

$$(H10)\qquad P_s(\lambda)=0 \text{ and } P_s'(\lambda)\neq 0.$$

From theorem 2.7, we have

Corollary 3.16 *If $(H2)-(H4)$ and $(H7)-(H9)$ are satisfied with $\lambda\in\mathbb{R}^*$, then $(1.1)-(1.4)$ has a bifurcated branches of solutions from $(0,\lambda^{-1})$.*

From theorem 2.8, we have

Corollary 3.17 *If $(H2)-(H4)$, $(H7)-(H8)$ and $(H10)$ are satisfied with $\lambda\in\mathbb{R}^*$, then $(1.1)-(1.4)$ has exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $(0,\lambda^{-1})$.*

Application

Case1: For $r=1$, we obtain

$$T_2(\lambda)=\begin{pmatrix}0 & 0 & \lambda & 0\\ \lambda+A_1 & 0 & b_1 & 0\\ A_1 & \lambda & b_1 & 0\\ -A_1 & 0 & -b_1 & \lambda\end{pmatrix}.$$

Then $P_2(\lambda)=\lambda^3(\lambda+A_1)$, moreover the eigenvalues of $\overline{M}_2$ are 0 and $-A_1$. We suppose that $t_1\neq-\frac{c_1}{b_1}$, then $(-b_1t_1-c_1)$ is a simple eigenvalue of $\overline{M}_2$. So, from corollary 3.17 the problem $(IBVP)$ has exactly two branches of solutions Γ_1 and Γ_2 bifurcating from $(0,(-b_1t_1-c_1)^{-1})$.

Case2: For $r=2$, we obtain

$$T_2(\lambda)=\begin{pmatrix}0 & 0 & 0 & \lambda & 0 & 0\\ \lambda+A_1 & A_2 & 0 & b_1 & b_2 & 0\\ A_1 & \lambda+A_2 & 0 & b_1 & b_2 & 0\\ -A_1 & 0 & 0 & -b_1 & \lambda & 0\\ A_1 & A_2 & \lambda & b_1 & b_2 & 0\\ -A_1 & -A_2 & 0 & -b_1 & -b_2 & \lambda\end{pmatrix}.$$

Then, $P_2(\lambda)=\lambda^4[\lambda^2+(A_1+A_2)\lambda+A_1b_2]$.

We suppose that $b_1\neq 0$ and $b_2\neq 0$. If $t_1=-\frac{c_1}{b_1}$ then $P_2(\lambda)=\lambda^5[\lambda+(A_1+A_2)]$. So the eigenvalues of $\overline{M}_2$ are 0 and $-(A_1+A_2)$.

We suppose that $t_2\neq-\frac{b_1t_1+(c_1+c_2)}{b_2}$, then $-(b_1t_1+b_2t_2+(c_1+c_2))$ is a simple eigenvalue of $\overline{M}_2$. So, from corollary 3.17 the problem $(IBVP)$ has exactly two branches of solutions Γ_1 and Γ_2 bifurcating from

$$(0,(-b_1t_1-b_2t_2-(c_1+c_2))^{-1}).$$

If $t_1 \neq -\frac{c_1}{b_1}$ and $(b_1t_1+c_1)^2+2(b_1t_1+c_1)(b_2t_2+c_2)+(b_2t_2+c_2)^2>4b_2(b_1t_1+c_1)$

then the characteristic equation $\lambda^2+(A_1+A_2)\lambda+A_1b_2=0$ admits two solutions

$\lambda_1=-\frac{1}{2}[b_1t_1+b_2t_2+c_1+c_2+\sqrt{(b_1t_1+b_2t_2+c_1+c_2)^2-4b_2(b_1t_1+c_1)}]$ and

$$\lambda_2=-\frac{1}{2}[b_1t_1+b_2t_2+c_1+c_2-\sqrt{(b_1t_1+b_2t_2+c_1+c_2)^2-4b_2(b_1t_1+c_1)}].$$

Moreover, the eigenvalues of $\overline{M}_2$ are 0 and the simple eigenvalues λ_1 and λ_2.

Theorem 3.10 implies that if either $\alpha=\lambda_1$ or $\alpha=\lambda_2$, then the problem $(IBVP)$ has exactly two branches of solutions Γ_1^j and Γ_2^j bifurcating from $(0,(\lambda_j)^{-1})$, with $j=1,2$.

Remarks 3.18

1. If $t_2=\frac{-(b_1t_1+c_1)(b_2+1)-c_2-1}{b_2}$, then the eigenvalues of

$P_2(\lambda)=0$ are $\lambda_1=1$ and $\lambda_2=(b_1t_1+c_1)b_2$. From Theorem 3.10, if either $\alpha=1$ or $\alpha=(b_1t_1+c_1)b_2$, then the problem $(IBVP)$ has exactly two branches of solutions Γ_1^1 and Γ_2^1 bifurcating from $(0,1)$ and two branches of solutions Γ_1^2 and Γ_2^2 bifurcating from $(0,((b_1t_1+c_1)b_2)^{-1})$.

2. If $t_2=\frac{(b_1t_1+c_1)(b_2-1)-c_2+1}{b_2}$, then the eigenvalues of $P_2(\lambda)=0$ are $\lambda_1=-1$ and

$\lambda_2=-(b_1t_1+c_1)b_2$. From Theorem 3.10, if either $\alpha=-1$ or $\alpha=-(b_1t_1+c_1)b_2$, then the problem $(IBVP)$ has exactly two branches of solutions Γ_1^1 and Γ_2^1 bifurcating from $(0,-1)$ and two branches of solutions Γ_1^2 and Γ_2^2 bifurcating from $(0,(-(b_1t_1+c_1)b_2)^{-1})$.

Remark 3.19 *Let $A_k=0$ for $k=1,\dots,r$. We have $P_2(\lambda)=\lambda^{2r+1}(-b_r)$, moreover the eigenvalues of $\overline{M}_2$ is only* 0.

3.3 Case where $\boldsymbol{\zeta_k = \square_3 + \alpha\varsigma_k}$

In this section, we investigate the existence of bifurcated solutions where

$\zeta_k(v(t_k),v'(t_k),\alpha)=h_3(v^p(t_k),(v')^q(t_k),\alpha)+\alpha\varsigma_k(v(t_k),v'(t_k))$ with $p\geq 2$ and $q\geq 2$, $p,q\in\mathbb{N}$.

We have

$$\begin{aligned}
\frac{\partial\psi}{\partial v}(v,\alpha).\varphi \quad = \quad & \sum_{0<t_k<t}\Big\{[(pv^{p-1}(t_k)\frac{\partial h_3}{\partial x}(v^p(t_k),(v')^q(t_k),\alpha)).\varphi(t_k)\\
&+(q\left(v'\right)^{q-1}(t_k)\frac{\partial h_3}{\partial y}\left(v^p(t_k),(v')^q(t_k),\alpha)\right).\varphi'(t_k)\Big](t-t_k)\\
&+\alpha[\frac{\partial\varsigma_k}{\partial x}(v(t_k),v'(t_k)).\varphi(t_k)+\frac{\partial\varsigma_k}{\partial y}(v(t_k),v'(t_k)).\varphi'(t_k)](t-t_k)\\
&+[\frac{\partial\eta_k}{\partial x}(v(t_k),v'(t_k),\alpha).\varphi(t_k)+\frac{\partial\eta_k}{\partial y}(v(t_k),v'(t_k),\alpha).\varphi'(t_k)]\Big\}\\
&-t\sum_{0<t_k<1}\Big\{[(pv^{p-1}(t_k)\frac{\partial h_3}{\partial x}(v^p(t_k),(v')^q(t_k),\alpha)).\varphi(t_k)\\
&+(q(v')^{q-1}(t_k)\frac{\partial h_3}{\partial y}(v^p(t_k),(v')^q(t_k),\alpha)).\varphi'(t_k)](1-t_k)\\
&+\alpha[\frac{\partial\varsigma_k}{\partial x}(v(t_k),v'(t_k)).\varphi(t_k)+\frac{\partial\varsigma_k}{\partial y}(v(t_k),v'(t_k)).\varphi'(t_k)](1-t_k)\\
&+[\frac{\partial\eta_k}{\partial x}(v(t_k),v'(t_k),\alpha).\varphi(t_k)+\frac{\partial\eta_k}{\partial y}(v(t_k),v'(t_k),\alpha).\varphi'(t_k)]\Big\}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial F}{\partial v}(0,\alpha).\varphi &= \frac{\partial (L^{-1}\circ GJ)}{\partial v}(0,\alpha).\varphi + \frac{\partial \psi}{\partial v}(0,\alpha).\varphi \\
&= \int_0^1 H(t,s)(\frac{\partial g}{\partial x}(s,0,0,\alpha).\varphi(s) + \frac{\partial g}{\partial y}(s,0,0,\alpha).\varphi'(s)) \\
&\quad + \sum_{0<t_k<t} [\alpha(\frac{\partial \varsigma_k}{\partial x}(0,0).\varphi(t_k) + \frac{\partial \varsigma_k}{\partial y}(0,0).\varphi'(t_k))(t-t_k) \\
&\quad + (\frac{\partial \eta_k}{\partial x}(0,0,\alpha).\varphi(t_k) + \frac{\partial \eta_k}{\partial y}(0,0,\alpha).\varphi'(t_k))] \\
&\quad - t\sum_{0<t_k<1} [\alpha(\frac{\partial \varsigma_k}{\partial x}(0,0).\varphi(t_k) + \frac{\partial \varsigma_k}{\partial y}(0,0).\varphi'(t_k))(1-t_k) \\
&\quad + (\frac{\partial \eta_k}{\partial x}(0,0,\alpha).\varphi(t_k) + \frac{\partial \eta_k}{\partial y}(0,0,\alpha).\varphi'(t_k))]
\end{aligned}$$

We put additional hypotheses as follow

- $\frac{\partial \eta_k}{\partial x}(0,0,\alpha) = \frac{\partial \eta_k}{\partial y}(0,0,\alpha) = 0, \ \forall \alpha \in \mathbb{R}.$

If $(D_u N_3(0,\alpha))\varphi(t) := \alpha \overline{M}_3.\varphi(t) - \frac{\partial F}{\partial v}(0,\alpha).\varphi(t) = 0$, and From $(H7)$ and $(H11)$, we have

$$\begin{aligned}
\overline{M}_3\varphi(t) &= \sum_{0<t_k<t} (\frac{\partial \varsigma_k}{\partial x}(0,0).\varphi(t_k) + \frac{\partial \varsigma_k}{\partial y}(0,0).\varphi'(t_k))(t-t_k) \\
&\quad - t\sum_{0<t_k<1} (\frac{\partial \varsigma_k}{\partial x}(0,0).\varphi(t_k) + \frac{\partial \varsigma_k}{\partial y}(0,0).\varphi'(t_k))(1-t_k).
\end{aligned}$$

Proposition 3.20 *If the conditions $(H2)-(H4),(H7)$ and $(H11)$ are satisfied, then*

$$N_3(v,\alpha) = \circ(\| v \|_2).$$

Then, from theorem 2.7, we have

Theorem 3.21 *If the hypotheses $(H2)-(H4),(H7)$ and $(H11)$ are satisfied and $\lambda \in \mathbb{R}^\star$ is a real eigenvalue of $\overline{M}_3$, with odd algebraic multiplicity, then $(v,\alpha) = (0,\lambda^{-1})$ is a bifurcation point of $M_3(v,\alpha) = 0$ and $(1.1)-(1.4)$ has a bifurcated branches of solutions.*

And from theorem 2.8, we have

Theorem 3.22 *If the hypotheses $(H2)-(H4),(H7)$ and $(H11)$ are satisfied and $\lambda \in \mathbb{R}^\star$ is a simple eigenvalue of $\overline{M}_3$, then $(1.1)-(1.4)$ has exactly two bifurcated branches of solutions from $(0,\lambda^{-1})$.*

In the following we study the multiplicity of the eigenvalues of $\overline{M}_3$ to determine the number of branches of solutions.

To do that let $d_k := \frac{\partial^2 \varsigma_k}{\partial \alpha \partial x}(0,0,0)$, $e_k := \frac{\partial^2 \varsigma_k}{\partial \alpha \partial y}(0,0,0)$ and put $B_k := d_k t_k$,

$C_k := (1-t_k)d_k$, $D_k := (1-t_k)(d_k t_k + e_k)$ and $E_k := t_k(d_k t_k + e_k)$.

Proposition 3.23 *Let $\lambda \in \mathbb{R}^*$. Then λ is an eigenvalue of $\overline{M}_3$ if and only if there exist $\gamma_0,\ldots,\gamma_r,\beta_0,\ldots,\beta_r \in \mathbb{R}$ such that λ satisfies the following system with $(2r+2)$ equations*

$$(IV)\begin{cases}(\lambda+D_1)\gamma_0+\sum_{i=2}^{r} D_i\gamma_{i-1}+\sum_{i=1}^{r} C_i\beta_{i-1}=0,\\ \lambda\beta_0=0,\\ -E_1\gamma_0+(\lambda+D_2)\gamma_1+\sum_{i=3}^{r} D_i\gamma_{i-1}-B_1\beta_0+\sum_{i=2}^{r} C_i\beta_{i-1}=0,\\ E_1\gamma_0+B_1\beta_0+\lambda\beta_1=0,\\ \vdots\\ \sum_{i=1}^{k}-E_i\gamma_{i-1}+(\lambda+D_{k+1})\gamma_k+\sum_{i=k+2}^{r} D_i\gamma_{i-1}+\sum_{i=1}^{k}-B_i\beta_{i-1}+\sum_{i=k+1}^{r} C_i\beta_{i-1}=0,\\ \sum_{i=1}^{k} E_i\gamma_{i-1}+\sum_{i=1}^{k} B_i\beta_{i-1}+\lambda\beta_k=0,\\ \vdots\\ \sum_{i=1}^{r-1}-E_i\gamma_{i-1}+(\lambda+D_r)\gamma_{r-1}+\sum_{i=1}^{r-1}-B_i\beta_{i-1}+C_r\beta_{r-1}=0,\\ \sum_{i=1}^{r-1} E_i\gamma_{i-1}+\sum_{i=1}^{r-1} B_i\beta_{i-1}+\lambda\beta_{r-1}=0,\\ \sum_{i=1}^{r}-E_i\gamma_{i-1}+\lambda\gamma_r+\sum_{i=1}^{r}-B_i\beta_{i-1}=0,\\ \sum_{i=1}^{r} E_i\gamma_{i-1}+\sum_{i=1}^{r} B_i\beta_{i-1}+\lambda\beta_r=0.\end{cases}$$

with $k=1,\ldots,r-1$.

Moreover the eigenvector associated to λ is given by

$$\begin{aligned}\varphi_\lambda(t) &= \sum_{k=1}^{r}\gamma_k f_k(t)+\beta_k h_k(t)\\ &= \sum_{k=1}^{r} h_k(t)(\gamma_k t+\beta_k),\ t\neq t_k,\ t\in[0,1].\end{aligned}$$

Proof. If $t\in]0,t_1[$, $\overline{M}_3\varphi(t)=\lambda\varphi(t)$ is equivalent to

$$\sum_{0<t_k<t}(d_k\varphi(t_k)+e_k\varphi'(t_k))(t-t_k)-t\sum_{k=1}^{r}(d_k\varphi(t_k)+e_k\varphi'(t_k))(1-t_k)=\lambda(\gamma_0 t+\beta_0).$$

Then,

$$-t\sum_{k=1}^{r}(d_k\varphi(t_k)+e_k\varphi'(t_k))(1-t_k)=\lambda(\gamma_0 t+\beta_0),\qquad \forall t\in]0,t_1[,$$

We obtain,

$$t[\lambda\gamma_0+d_1(\gamma_0t_1+\beta_0)(1-t_1)+e_1\gamma_0(1-t_1)+\sum_{i=2}^{r} d_i(\gamma_{i-1}t_i+\beta_{i-1})(1-t_i)+e_i\gamma_{i-1}(1-t_i)]+\lambda\beta_0=0,$$
$\forall t\in]0,t_1[$.

Then,

$$\begin{cases}\lambda\beta_0=0,\\ \gamma_0(\lambda+D_1)+\sum_{i=2}^{r} D_i\gamma_{i-1}+\sum_{i=1}^{r} C_i\beta_{i-1}=0.\end{cases}$$

Similarly, for $t\in]t_k,t_{k+1}[$, we obtain the following result

$$\begin{cases}\sum_{i=1}^{k} E_i\gamma_{i-1}+\sum_{i=1}^{k} B_i\beta_{i-1}+\lambda\beta_k=0,\\ \sum_{i=1}^{k}-E_i\gamma_{i-1}+(\lambda+D_{k+1})\gamma_k+\sum_{i=k+2}^{r} D_i\gamma_{i-1}+\sum_{i=1}^{k}-B_i\beta_{i-1}+\sum_{i=k+1}^{r} C_i\beta_{i-1}=0.\end{cases}$$

For $t\in]t_r,1[$, $\overline{M}_3\varphi(t)=\lambda\varphi(t)$ is equivalent to

$$\lambda(\gamma_r t+\beta_r) = \sum_{k=1}^{r}(d_k\varphi(t_k)+e_k\varphi'(t_k))(t-t_k)-t\sum_{k=1}^{r}(d_k\varphi(t_k)+e_k\varphi'(t_k))(1-t_k)$$

Then,

$$\begin{cases}\sum_{i=1}^{r} E_i\gamma_{i-1}+\sum_{i=1}^{r} B_i\beta_{i-1}+\lambda\beta_r=0,\\ \sum_{i=1}^{r}-E_i\gamma_{i-1}+\lambda\gamma_r+\sum_{i=1}^{r}-B_i\beta_{i-1}=0.\end{cases}$$

Lemma 3.24 *Let $\lambda \in \mathbb{R}^\star$. Then λ is an eigenvalue of $\overline{M}_3$ if and only if there exist $\gamma_0, \dots, \gamma_r, \beta_0, \dots, \beta_r \in \mathbb{R}$ such that*

$$(V) \qquad T_3(\lambda)\begin{pmatrix}\gamma_0\\ \gamma_1\\ \vdots\\ \gamma_r\\ \beta_0\\ \beta_1\\ \vdots\\ \beta_r\end{pmatrix} = 0,$$

where $T_3(\lambda)$ is the $(2r+2)$ square matrix given by

$$T_3(\lambda) = \begin{pmatrix}\tilde{A}_2 & \tilde{B}_2\\ \tilde{C}_2 & \tilde{D}_2\end{pmatrix}$$

where $\tilde{A}_2$ is a $(r+1)\times(r+1)$ matrix, $\tilde{B}_2$ is a $(r+1)\times(r+1)$ matrix, $\tilde{C}_2$ is a $(r+1)\times(r+1)$ matrix and $\tilde{D}_2$ is a $(r+1)\times(r+1)$ matrix such that:

• $\tilde{A}_2 = (a_{ij})$ with

$$\begin{cases} a_{ij} = \lambda + D_j \ \ with \ \ i = \overline{1,r} \quad for \quad j = i,\\ a_{ij} = D_j \ \ with \ \ i = \overline{1,r-1} \quad and \quad j = \overline{2,r} \quad for \quad j \geq i,\\ a_{ij} = -E_j \ \ with \ \ i = \overline{2,r+1} \quad for \quad 1 \leq j < i,\\ a_{i(r+1)} = 0 \quad for \quad i = \overline{1,r} \quad and \quad a_{(r+1)(r+1)} = \lambda. \end{cases}$$

Then,

$$\tilde{A}_2 = \begin{pmatrix} \lambda + D_1 & D_2 & D_3 & D_4 & \dots & D_r & 0\\ -E_1 & \lambda + D_2 & D_3 & D_4 & \dots & D_r & 0\\ -E_1 & -E_2 & \lambda + D_3 & D_4 & \dots & D_r & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \\ -E_1 & -E_2 & -E_3 & -E_4 & \dots & \lambda + D_r & 0\\ -E_1 & -E_2 & -E_3 & -E_4 & \dots & -E_r & \lambda \end{pmatrix}.$$

• $\tilde{B}_2 = (a_{ij})$ with

$$\begin{cases} a_{ij} = c_{j-(r+1)}, \ \ with \quad i = \overline{1,r} \ \ and \ \ j = \overline{r+2, 2r+1} \quad for \quad j-(r+1) \geq i,\\ a_{ij} = -B_{j-(r+1)} \quad with \quad i = \overline{2,r+1} \quad for \quad 1 \leq j-(r+1) < i,\\ a_{i(2r+2)} = 0 \quad with \ \ i = \overline{1,r+1}. \end{cases}$$

Then,

$$\tilde{B}_2 = \begin{pmatrix} C_1 & C_2 & C_3 & \dots & C_{r-1} & C_r & 0\\ -B_1 & C_2 & C_3 & \dots & C_{r-1} & C_r & 0\\ -B_1 & -B_2 & C_3 & \dots & C_{r-1} & C_r & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \\ -B_1 & -B_2 & -B_3 & \dots & -B_{r-1} & C_r & 0\\ -B_1 & -B_2 & -B_3 & \dots & -B_{r-1} & -B_r & 0 \end{pmatrix}.$$

• $\tilde{C}_2 = (a_{ij})$ with

$$\begin{cases} a_{ij} = 0, \ with \quad i = \overline{r+2,2r+2} \ \ and \ \ j = \overline{1,r+1} \quad for \quad j \geq i-(r+1), \\ a_{ij} = E_j \quad with \quad i = \overline{r+3,2r+2} \quad for \quad 1 \leq j < i-(r+1). \end{cases}$$

Then,

$$\tilde{C}_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ E_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ E_1 & E_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \\ E_1 & E_2 & E_3 & \dots & E_{r-1} & 0 & 0 \\ E_1 & E_2 & E_3 & \dots & E_{r-1} & E_r & 0 \end{pmatrix}.$$

• $\widetilde{D}_2 = (a_{ij})$ with

$$\begin{cases} a_{ij} = \lambda, \ with \quad i = \overline{r+2,2r+2} \quad for \quad i = j, \\ a_{ij} = B_{j-(r+1)} \quad with \quad i = \overline{r+3,2r+2} \quad for \quad r+2 \leq j < i, \\ a_{i(2r+2)} = 0 \ \ with \ \ i = \overline{r+2,2r+1} \quad and \quad j = \overline{r+3,2r+2} \quad for \quad j > i. \end{cases}$$

Then,

$$\widetilde{D}_2 = \begin{pmatrix} \lambda & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ B_1 & \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ B_1 & B_2 & \lambda & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & & \\ B_1 & B_2 & B_3 & B_4 & \dots & B_{r-1} & \lambda & 0 \\ B_1 & B_2 & B_3 & B_4 & \dots & B_{r-1} & B_r & \lambda \end{pmatrix}.$$

From theorem 3.21, we have

Corollary 3.25 *If $(H2)-(H4)$, $(H7),(H9)$ and $(H11)$ are satisfied with $\lambda \in \mathbb{R}^*$, then $(1.1)-(1.4)$ has a bifurcated branches of solutions from $(0,\lambda^{-1})$.*

From theorem 3.22, we have

Corollary 3.26 *If $(H2)-(H4)$, $(H7)$ and $(H10)-(H11)$ are satisfied with $\lambda \in \mathbb{R}^*$, then $(1.1)-(1.4)$ has exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $(0,\lambda^{-1})$.*

Application

Case1: For $r = 1$, we obtain

$$T_3(\lambda) = \begin{pmatrix} \lambda + D_1 & 0 & C_1 & 0 \\ -E_1 & \lambda & -B_1 & 0 \\ 0 & 0 & \lambda & 0 \\ E_1 & 0 & B_1 & \lambda \end{pmatrix}.$$

Then $P_3(\lambda) = \lambda^3(\lambda + D_1)$, moreover the eigenvalues of $\overline{M}_3$ are 0 and $-D_1$. We suppose that $d_1 \neq 0, e_1 \neq 0$ and $t_1 \neq -\frac{d_1}{e_1}$, then $(t_1-1)(d_1t_1+e_1)$ is a simple eigenvalue of $\overline{M}_3$. So, from corollary 3.26, the problem $(IBVP)$ has exactly two branches of solutions Γ_1 and Γ_2 bifurcating from $(0,((t_1-1)(d_1t_1+e_1))^{-1})$.

Case2: For $r = 2$, we obtain

$$T_3(\lambda) = \begin{pmatrix} \lambda + D_1 & D_2 & 0 & C_1 & C_2 & 0 \\ -E_1 & \lambda + D_2 & 0 & -B_1 & C_2 & 0 \\ -E_1 & -E_2 & \lambda & -B_1 & -B_2 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ E_1 & 0 & 0 & B_1 & \lambda & 0 \\ E_1 & E_2 & 0 & B_1 & B_2 & \lambda \end{pmatrix}.$$

Then $P_3(\lambda) = \lambda^4[\lambda^2 + (D_1 + D_2)\lambda + D_1 D_2 + E_1(D_2 - C_2)]$.

We suppose that $d_1 \neq 0$, $d_2 \neq 0$ and $e_1 \neq 0$. If either $t_1 = -\frac{e_1}{d_1}$ or $t_1 = \frac{d_2 t_2 + e_2}{d_2}$, then $P_3(\lambda) = \lambda^5[\lambda + (D_1 + D_2)]$. The eigenvalues of $\overline{M}_3$ are 0 and $-(D_1 + D_2)$.

We suppose $(1 - t_1)(d_1 t_1 + e_1) + (1 - t_2)(d_2 t_2 + e_2) \neq 0$, then $(t_1 - 1)(d_1 t_1 + e_1) + (t_2 - 1)(d_2 t_2 + e_2)$ is a simple eigenvalue of $\overline{M}_3$. So, from corollary 3.26, the problem $(IBVP)$ has exactly two branches of solutions Γ_1 and Γ_2 bifurcating from $(0, ((t_1 - 1)(d_1 t_1 + e_1) + (t_2 - 1)(d_2 t_2 + e_2))^{-1})$.

If $t_1 \neq -\frac{e_1}{d_1}$ and $t_1 \neq \frac{d_2 t_2 + e_2}{d_2}$, for $\Delta := (1 - t_1)^2(d_1 t_1 + e_1)^2 + (1 - t_2)^2(d_2 t_2 + e_2)^2 - 2(1 - t_1)(1 - t_2)(d_1 t_1 + e_1)(d_2 t_2 + e_2) - 4t_1(1 - t_2)(d_1 t_1 + e_1)(d_2 t_2 + e_2 - d_2) > 0$, then the characteristic equation $\lambda^2 + (D_1 + D_2)\lambda + D_1 D_2 + E_1(D_2 - C_2) = 0$ admits two solutions

$\lambda_1 = \frac{1}{2}[(t_1 - 1)(d_1 t_1 + e_1) + (t_2 - 1)(d_2 t_2 + e_2) - \sqrt{\Delta}]$ and

$$\lambda_2 = \frac{1}{2}[(t_1 - 1)(d_1 t_1 + e_1) + (t_2 - 1)(d_2 t_2 + e_2) + \sqrt{\Delta}].$$

Corollary 3.26 implies that if either $\alpha = \lambda_1$ or $\alpha = \lambda_2$, then the problem $(IBVP)$ has exactly two branches of solutions Γ_1^j and Γ_2^j bifurcating from $(0, (\lambda_j)^{-1})$ with $j = 1,2$.

Remarks 3.27

1. If $1 + (1 - t_1)(d_1 t_1 + e_1) + (1 - t_2)(d_2 t_2 + e_2) + (1 - t_1)(1 - t_2)(d_1 t_1 + e_1)(d_2 t_2 + e_2) + t_1(1 - t_2)(d_1 t_1 + e_1)(d_2 t_2 + e_2 - d_2) = 0$, then the eigenvalues of $P_3(\lambda) = 0$ are $\lambda_1 = 1$

and $\lambda_2 = -1 - (1 - t_1)(d_1 t_1 + e_1) - (1 - t_2)(d_2 t_2 + e_2)$. From corollary 3.26 if either $\alpha = 1$ or

$\alpha = -1 - (1 - t_1)(d_1 t_1 + e_1) - (1 - t_2)(d_2 t_2 + e_2)$, then the problem $(IBVP)$ has exactly two branches of solutions Γ_1^1 and Γ_2^1 bifurcating from (0,1) and two branches of solutions Γ_1^2 and Γ_2^2 bifurcating from

$(0, (-1 - (1 - t_1)(d_1 t_1 + e_1) - (1 - t_2)(d_2 t_2 + e_2))^{-1})$.

2. If $1 + (1 - t_1)(1 - t_2)(d_1 t_1 + e_1)(d_2 t_2 + e_2) + t_1(1 - t_2)(d_1 t_1 + e_1)(d_2 t_2 + e_2 - d_2) = (1 - t_1)(d_1 t_1 + e_1) + (1 - t_2)(d_2 t_2 + e_2)$, then the eigenvalues of $P_3(\lambda) = 0$ are $\lambda_1 = -1$ and

$\lambda_2 = 1 - (1 - t_1)(d_1 t_1 + e_1) - (1 - t_2)(d_2 t_2 + e_2)$. From corollary 3.26, if either $\alpha = -1$ or

$\alpha = 1 - (1 - t_1)(d_1 t_1 + e_1) - (1 - t_2)(d_2 t_2 + e_2)$, then the problem $(IBVP)$ has exactly two branches of solutions Γ_1^1 and Γ_2^1 bifurcating from $(0, -1)$ and two branches of solutions Γ_1^2 and Γ_2^2 bifurcating from

$(0, (1 - (1 - t_1)(d_1 t_1 + e_1) - (1 - t_2)(d_2 t_2 + e_2))^{-1})$.

Remark 3.28 *Let* $t_k = \frac{-e_k}{d_k}$ *for* $k = 1, \dots, r$. *We have* $E_k = 0$ *and* $D_k = 0$, *then* $P_3(\lambda) = \lambda^{2r+2}$. *So the eigenvalues of* $\overline{M}_3$ *is only* 0.

4 Concluding remarks

In this work, we have studied the existence of multiple branches of solutions of second order impulsive differential equation with real parameter α. We have obtained many results according to the hypothesis on the nonlinearity functions and the functions in the impulsive part of the problem $(IBVP)$. It will be very interesting to consider the case with both nonlinear term and impulse functions depending implicitly on the real parameter. That's why we're going to find some form of functions that ensures this approach using Krasnosel'ski bifurcation theorem. However, the two approachs used in [3] and [4] examined only the case with explicit dependence on the parameter in the differential equation and impulse equations.

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