Positive Solution of Random Differential Equation in Banach Spaces

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Abstract : Here, we investigate two point boundary value problem for the second order functional random differential equation

$$\begin{aligned} &p''(t,\omega)+\lambda\, p'(t,\omega)+u\left(t, p(h_1(t,\omega)), p(h_2(t,\omega), p(h_3(t,\omega)),\omega\right)=0,\\ &ap(-1,\omega)-bp'(-1,\omega)=0,\\ &cp(1,\omega)+dp'(1,\omega)=0 \end{aligned}$$

where the function u takes values in a cone K of a Banach space E. For $h_1(t,\omega)=(t,\omega)$ and $h_2(t,\omega)=(-t,\omega)\, and\, h_3(t,\omega)=(t,\omega)$, $\omega\in\Omega$ we obtain the boundary value problem with reflection of the argument. we prove the existence of positive solution in the space $C\left([-1,1],E,\Omega\right)$ using random fixed point theorem.

Keywords: Random differential equation, Boundary value problem, Banach space, positive solution, fixed point theorem.

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1.Introduction

Let K be a cone in a real Banach space E. We will assume that the norm $\|\cdot\|_E$ in E is monotonic with respect to K, that is, if $\theta<u<v$ then $\|u\|_E\le\|v\|_E$, where < denotes the partial ordering defined by K and θ stands for the zero element of E. Further, denote by $C(I,E,\Omega)$ the space of all continuous functions defined on the interval $[-1,1]$ and taking values in E, equipped with the norm

$$\|u\|=\max_{\substack{t\in I\\ \omega\in\Omega}}\|u(t,\omega)\|_E .$$

Clearly, $C(I,E,\Omega)$ is a Banach space.

Let $Q(\omega)=\{x\in C(I,E,\Omega):\theta<x(t,\omega)\text{ for } t\in I,\omega\in\Omega\}$

It is easy to prove that Q is a cone in $C(I,E,\Omega)$.

we will study two point boundary value problem for the second order functional random differential equation

$$\begin{aligned} &p''(t,\omega)+\lambda\, p'(t,\omega)+u\left(t,\, p(h_1(t,\omega)),\, p(h_2(t,\omega),\, p(h_3(t,\omega)),\omega\right)=0,\\ &\qquad ap(-1,\omega)-bp'(-1,\omega)=0,\\ &\qquad cp(1,\omega)+dp'(1,\omega)=0 \end{aligned} \tag{1.1}$$

Where $t\in I, k\in R,\ a,b,c,d\geq 0$ and $ad+bc+ac>0$, $\omega\in\Omega$.

we will assume that

1. $u: I\times K\times K\times K\times\Omega\to K$ is a continuous function.
2. $h_1,h_2,h_3: I\times\Omega\to I$ are continuous functions mapping the interval I onto itself.

Notice that for $h_1(t,\omega)=(t,\omega)$ and $h_2(t,\omega)=(-t,\omega)\, and\, h_3(t,\omega)=(t,\omega)$ we obtain the boundary value problem involving reflection of the argument

$$\begin{aligned} &p''(t,\omega)+\lambda\, p'(t,\omega)+u\left(t,\, p(h_1(t,\omega)),\, p(h_2(t,\omega),\, p(h_3(t,\omega)),\omega\right)=0,\\ &\qquad ap(-1,\omega)-bp'(-1,\omega)=0,\\ &\qquad cp(1,\omega)+dp'(1,\omega)=0 \end{aligned}$$

Such problems have been considered in[5,9] for $\lambda=0$ and $u: I\times K\times K\times K\times\Omega\to K$ and for f taking values in a real Hilbert space.

Our purpose is to discuss the existence of positive solution of (1.1). We will use the following fixed point theorem from [7] which is a modification of well-known Krasnoselskii theorem on operators compressing and expanding a cone.

Proposition1.1. Let P be a cone of a real Banach space X , and let the norm $\|\cdot\|$ in X be monotonic with respect to P . Let $B_r=\{p\in X:\|p\|<r\},\ B_R=\{p\in X:\|p\|<R\},\ )<r<R$.

Suppose that $F: P\bigcap \overline{B}_R \to P$ is a strict set-contraction which satisfies one of the following conditions:

(i). $p \in P\bigcap \partial B_r \Rightarrow \|Fp\| \le \|p\|$ and $p \in P\bigcap \partial B_R \Rightarrow \|Fp\| \ge \|p\|$ or

(ii). $p \in P\bigcap \partial B_R \Rightarrow \|Fp\| \le \|p\|$ and $p \in P\bigcap \partial B_r \Rightarrow \|Fp\| \ge \|p\|$.

Then F has a fixed point in $P\bigcap(\overline{B_R} \setminus B_r)$.

Recall that $F: D \to X, D \subset X,$ is said to be a strict set-contraction if F is continuous and bounded and there exists $0 \le L < 1$ such that $\alpha(F(S)) \le L\alpha(S)$ for all bounded subsets S of D, where α denotes the Kuratowski measure of noncompactness

2.Auxialary Results

First we will study some properties of the functions

$$G(t,s) = \begin{cases} \dfrac{1}{\rho} e^{-kt}\left[e^{k(s-1)}\mu_1 + c\right]\left[e^{k(t+1)}\mu_2 - a\right], & -1 \le t \le s \le 1 \\ \dfrac{1}{\rho} e^{-kt}\left[e^{k(s+1)}\mu_2 - a\right]\left[e^{k(t-1)}\mu_1 + c\right], & -1 \le s \le t \le 1 \end{cases} \tag{2.1}$$

where

$$k \ne 0,\ \mu_1 = dk - c,\ \mu_2 = bk + a \text{ and } \rho = \frac{1}{k}[ae^{-k}\mu_1 + ce^{k}\mu_2]$$

and

$$G^*(t,s) = \begin{cases} \dfrac{1}{\rho^*}(c+d-cs)(a+b+at), & -1 \le t \le s \le 1 \\ \dfrac{1}{\rho^*}(a+b+as)(c+d-ct), & -1 \le s \le t \le 1 \end{cases} \tag{2.2}$$

where $\rho^* = 2ac + bc + ad$. It is easy to show that the function (2.1) fulfils the following inequalities:

$$\bigwedge_{t,s\in I} G(t,s) \geq 0 \tag{2.3}$$

and

$$\bigwedge_{t,s\in I} G(t,s) \leq G(s,s) \tag{2.4}$$

Moreover, for any $-1 \leq \gamma < \delta \leq 1$ and $t \in [\gamma, \delta]$ we have

$$G(t,s) \geq mG(s,s) \tag{2.5}$$

where $s \in I$ and

$$m = \min\left\{ \frac{e^k \mu_2 - ae^{-k\gamma}}{e^k \mu_2 - ae^{-k}}, \frac{e^{-k}\mu_1 + ce^{-k\delta}}{e^{-k}\mu_1 + ce^k} \right\} \tag{2.6}$$

It is easily seen that $m < 1$. The function G^* also satisfies the inequalities (2.3), (2.4) and (2.5) with m replaced by

$$m^* = \min\left\{ \frac{a+b+a\gamma}{2a+b}, \frac{c+d-c\delta}{2c+d} \right\} \tag{2.7}$$

Clearly, $m^* < 1$.

Next, consider the integral-functional operator

$$(Fp)(t,\omega) = \int_{-1}^{1} G(t,s) u\left(s, p(h_1(s,\omega)), p(h_2(s,\omega),\omega),\right) p(h_3(s,\omega),\omega) ds \tag{2.8}$$

Where $t \in I$, $p \in Q$, the function G is defined by (2.1) and u, h_1 , h_2 *and* h_3 satisfy 1and 2.

Let

$$M = \max_{t,s\in I} G(t,s)$$
$$\overline{T_r} = \{ p \in E : \|p\|_E \leq r \},$$

and

$$\overline{B_r} = \{ p \in C(I, E, \Omega) : \|p\| \le r \}$$

Lemma 2.1[6]. Assume that for any $r > 0$:

(3) the function f is uniformly continuous on $I \times (K \cap \overline{T_r}) \times (K \cap \overline{T_r})$

(4) there exists a non-negative constant L_r, such that $4ML_r < 1$

$$\alpha(f(t, \Phi, \Phi)) \le L_r \alpha(\Phi) \quad (2.9)$$

for all $t \in I$ and $\Phi \subset K \cap \overline{T_r}$.

Then, for any $r > 0$ the operator (9) is a strict set-contraction on $\Phi \cap \overline{B_r}$.

3.Main Result

Now we state and prove our results on positive solutions of (1.1). Consider the case $k \neq 0$.

Theorem1.Let G be given by (2.1) and let $-1 \le \gamma < \delta < \omega \le 1$ be such that $h_i : [\gamma, \delta, \omega] \to [\gamma, \delta, \omega]$, $i = 1, 2, 3$. Suppose that the assumptions 1-4 are satisfied and

(5) there exists $k \in K, k \neq \theta$, such that

$$u(t, p, q, \omega) < \left[\int_{-1}^{1} G(s, s) ds \right]^{-1} k$$

for all $t \in I$ and $p, q \in K, \omega \in \Omega$ such that $\|p\|_E, \|q\|_E \in \left[0, \|k\|_E \right]$,

(6) there exist $\eta \in K, \eta \neq \theta, \|\eta\|_E \neq \|k\|_E$ and $t_0 \in I$ such that

$$\left[\int_{\gamma}^{\delta} G(t_0, s) ds \right]^{-1} \eta < u(t, p, q, \omega)$$

for all $t \in I$ and $p, q \in K$ such that $\|p\|_E, \|q\|_E \in \left[m\|\eta\|_E, \|\eta\|_E \right]$, where m is given by (2.6). Then the problem (1.1) has at least one positive solution.

Proof. Notice that each positive solution of the problem (1.1) (with $\neq 0$) is a fixed point of the integral-functional operator (2.8), that is

$$(Fp)(t,\omega)=\int_{-1}^{1}G(t,s)u\big(s,\,p(h_1(s,\omega)),\,p(h_2(s,\omega),\omega),\,p(h_3(s,\omega),\omega)\big)\,ds$$

Where $t\in I,\, p\in C(I,E,\Omega)$ and the function G is given by (2.1). On the other hand, if u belonging to Q is a fixed point of F , then p is a solution of (1.1) . Thus, to prove our theorem it is enough to show that F has a fixed point in Q . In the space $C(I,E,\Omega)$ consider the set

$$P=\left\{p\in C(I,E,\Omega):\theta<p(t,\omega)\text{ on } I \text{ and } \mathop{\wedge}_{t\in[\gamma,\delta]}\mathop{\wedge}_{s\in I} mp(s,\omega)<p(t,\omega)\right\}.$$

Clearly, P is a cone in $C(I,E,\Omega)$ and the norm $\|\cdot\|$ in $C(I,E,\Omega)$ is monotonic with respect to P . Consider the operator (2.8) for $t\in I$ and $p\in P$. We will show that F satisfies the assumptions of Proposition 1.1. First, we will prove that $F(P)\subset P$. To this end observe that by 1 and (4) $\theta<(Fp)(t,\omega)$ (3.1)

for every $t\in I$ and $p\in P$. Moreover, it follows from (2.5) that for any $t\in[\gamma,\delta]$ and $s\in I$

$$\begin{aligned}m(Fp)(s,\omega)&=m\int_{-1}^{1}G(s,s)u\big(s,\,p(h_1(s,\omega)),\,p(h_2(s,\omega),\,p(h_3(s,\omega)),\omega\big)\,ds\\&<\int_{-1}^{1}G(t,s)u\big(s,\,p(h_1(s,\omega)),\,p(h_2(s,\omega),\,p(h_3(s,\omega)),\omega\big)\,ds\\&=(Fp)(t,\omega)\end{aligned}$$

Combining it with (3.1) we conclude that $F(P)\subset P$. Without loss of generality we may assume that $\|k\|_E<\|\eta\|_E$. Fix $r=\|k\|_E$ and $R=\|\eta\|_E$.

By Lemma 2.1, F is a strict set-contraction on $P\cap\overline{B_R}$. Moreover, for $p\in P\cap\partial B_r$ we have $\theta<p(h_1(t,\omega))$ on I and $\|p\|=\|k\|_E$, hence

$$\mathop{\wedge}_{t\in I}\big\|p\big(h_1(t,\omega)\big)\big\|_E\leq\|k\|_E$$

Analogously

$$\underset{t\in I}{\wedge}\left\|p\left(h_2(t,\omega)\right)\right\|_E \le \|k\|_E \,,\ \underset{t\in I}{\wedge}\left\|p\left(h_3(t,\omega)\right)\right\|_E \le \|k\|_E$$

Thus, by 5, for any $t \in I$ we obtain

$$(Fp)(t,\omega) < \int_{-1}^{1} G(s,s)u\left(s,\, p(h_1(s,\omega)),\, p(h_2(s,\omega)),\, p(h_2(s,\omega)),\, \omega\right)ds$$
$$< \int_{-1}^{1} G(s,s)\left(\int_{-1}^{1} G(\tau,\tau)\right)^{-1} kds = k$$

Hence, in view of monotonicity of $\|\cdot\|_E$ we get

$$\underset{t\in I}{\wedge}\left\|\left((Fp)(t,\omega)\right)\right\|_E \le \|k\|_E$$

and in consequence $\|Fp\| \le \|p\|$ on $P \cap \partial B_r$. Furthermore, for $p \in P \cap \partial B_r$ we have

$$\underset{t\in[\gamma,\delta]}{\wedge}\ \underset{s\in I}{\wedge}\ \theta < mp\left(h_1(s,\omega)\right) < p\left(h_1(t,\omega)\right)$$

Since the norm $\|\cdot\|_E$ is monotonic we obtain

$$\underset{t\in[\gamma,\delta]}{\wedge}\ \underset{s\in I}{\wedge}\left\|mp\left(h_1(s,\omega)\right)\right\|_E \le \left\|p\left(h_1(t,\omega)\right)\right\|_E$$

which gives

$$\underset{t\in[\gamma,\delta]}{\wedge} m\max_{s\in I}\left\|p\left(h_1(s,\omega)\right)\right\|_E \le \left\|p\left(h_1(t,\omega)\right)\right\|_E$$

But h_1 maps I onto itself, hence for $\|p\| = \|\eta\|_E$

$$\underset{t\in[\gamma,\delta]}{\wedge} m\|\eta\|_E \le \left\|p\left(h_1(t,\omega)\right)\right\|_E \le \|\eta\|_E$$

In the same manner we get

$$\underset{t\in[\gamma,\delta]}{\wedge} m\|\eta\|_E \le \left\|p\left(h_2(t,\omega)\right)\right\|_E \le \|\eta\|_E \,,\ \underset{t\in[\gamma,\delta]}{\wedge} m\|\eta\|_E \le \left\|p\left(h_3(t,\omega)\right)\right\|_E \le \|\eta\|_E$$

Thus in view of 6

$$\begin{aligned}\eta &= \int_{\gamma}^{\delta} G(t_0,s)\left(\int_{\gamma}^{\delta} G(t_0,\tau)d\tau\right)^{-1} \eta ds \\ &\prec \int_{\gamma}^{\delta} G(t_0,s)u\left(s, p(h_1(s,\omega)), p(h_2(s,\omega)), p(h_3(s,\omega)),\omega\right)ds \text{ so} \\ &\prec \int_{-1}^{1} G(t_0,s)\, f\left(s, x(h_1(s,\omega)), x(h_2(s,\omega)), p(h_3(s,\omega)),\omega\right)ds \\ &= (Fp)(t_0)\end{aligned}$$

$$\|(Fp)(t_0)\|_E \geq \|\eta\|_E ,$$

which implies $\|Fp\| \geq \|p\|$ on $P \cap \partial B_r$. By Proposition 1.1 the operator F has a fixed point in the set $P \cap \left(\overline{B}_R \setminus B_r\right)$. This means that the problem (1.1) has at least one positive solution $p \in P$ such that

$\|\lambda\|_E \leq \|p\| \leq \|\eta\|_E$. This ends the proof of Theorem (1.1).

References

[1]. J. Banasand , K. Goebel, Measures of noncompactness in Banach spaces, Marcel Dekker, New York, Basel, 1980.

[2]. N.G. Các, J.A. Gatica, Fixed point theorems for mappings in ordered Banach spaces, J. Math. Anal.Appl.71(1971), 547-557.

[3].D. Guo,V. Lakshmikantham, Multiple solutions of two-point boundary value problems of ordinary differential equations in Banach spaces, J. Math. Anal. Appl. 129(1988),211-222.

[4].D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.

[5].C.P. Gupta, Existence and uniqueness theorems for boundary value problems involving reflection of the argument, Nonlinear Anal.11(1987),1075-1083.

[6].M.A. Krasnoselskii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.

[7].G. Li, A new fixed point theorem on demi-compact 1-set-contraction mappings, Proc.Amer. Math. Soc. 97(1986), 277-280.

[8].R.H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, Wiley, New York, 1976.

[9].J. Wiener, A.R. Aftabizadeh, Boundary value problems for differential equations with reflection of the argument, Internat. J. Math. Math. Sci. 8(1985), 151-163.