STUDY THE NIL-POTENT SUBGROUPS OF CLASSICAL GROUPS OVERLAPPING WITH COMMUNICATION RINGS

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ABSTRACT

Let there be an infinite division ring, which will be denoted by the letter D, let there be a natural number, which will be denoted by the letter n, and let there be a subnormal subgroup of $GL_n(D)$, which will be denoted by the letter N, such that either n = 1 or the centre of D includes at least five different elements. We give proof that locally nilpotent linear groups are divisible by residually-periodic, and we show that these groups actually benefit from a broad variety of increased characteristics. In addition, we present evidence that locally nilpotent linear groups are divisible by residually-periodic. The use of examples places significant limitations on the scope of possible extensions; however, we do expand the scope of our findings beyond linear groups to groups of automorphisms of both Noetherian modules and Artinian modules over commutative rings. This brings the total number of possible extensions down to a much more manageable level. In conclusion, we have shown that two fascinating theorems concerning nilpotent subgroups may be proven satisfactorily.

Keywords: Subgroup, Nilpotent Group, rings.

1. INTRODUCTION

A ring R is said to be nilpotent if there exists some natural number n greater than one and the equation $R_n = 0$ for that ring. The expression R_n refers to the ideal of R and includes all sums and differences of products that consist of n components from R. The definition of a nilpotent algebra is formulated in a way that is quite analogous to the construction of a nilpotent ring. The degree of nilpotency, often referred to as class, of a ring R is represented by its least significant exponent, which is denoted by the letter n. For instance, the family generates a nilpotent subring inside of $M_2(Q)$, and the degree of nilpotency that this subring exhibits is equivalent to the number 2. It is essential that you keep in mind that if ring R is nilpotent, then $M_i(R)$ is likewise nilpotent for all. $i \in \mathbb{N}$.

$$\left\{\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} : q \in Q\right\}$$

A locally nilpotent algebra has the same definition as a locally nilpotent ring, and a locally nilpotent ring is a ring R with the condition that every finitely created subring of R is nilpotent. Therefore, a locally nilpotent algebra has the same definition as a locally nilpotent ring. Consider, for example, the ring that was crafted by the family. $\{x_i\}_{i\in N}$ whereby $x_i x_i=0$ if j ≥ i for all indices i , j ∈ N.Given the specific circumstances, it has no significance. Each and every ring that is nilpotent also possesses the quality of being nilpotent locally. On the other hand, this is not the case in this particular scenario. Consider, for example, the situation in which we let A_k stand for the commutative algebra over a field k in such a way that the family of symbols satisfies the conditions: $U = \{u_r\}_{r\in(0,1)}$ indexed by the open interval (0, 1) is a basis for A_k, with multiplication defined so that $u_x u_y = u_{x+y}$ if x+y <1 and $u_x u_y = 0$. If this is not the case, then A_k is nilpotent despite being locally nilpotent. If ring R is locally nilpotent, then the condition $M_i(R)$ is true for any and all indices that are smaller than N. An element r in a ring R is said to be nilpotent if and only if the equation r nr = 0 holds true for each and every nr that is part of the ring N. The value of nr N that has the smallest feasible value is the value that is used to establish the nilpotency index of r R. A nil-algebra, often referred to as nil-algebra, has a definition that is comparable to that of a nil-ring and has the same name. A nil-ring is a ring R in which every member is nilpotent. This type of ring is sometimes referred to as a nil-ring. Take, for instance, the matrix ring as an example.

$$\left\{\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} : q \in Q\right\}$$

The structure that was just described is referred to as a nil-ring, and the nilpotency index of each nonzero component that makes up this matrix ring is equal to 2. If there is a definite natural number n such that r n = 0 for each element r in a nil-ring R, then the least such exponent, n, is known as the nil exponent (or nilexponent) of R. This is because r n = 0 for each element r in the nil-ring R. This is due to the fact that r n equals 0 for every element r present in the nil-ring R. The following is true if and only if R is nilpotent, or more especially locally nilpotent:

Each and every subring of R is nilpotent (or locally nilpotent, nil); 2. Each and every homomorphic image of R is nilpotent; and 3. (resp. locally nilpotent, nil),

1. If R is an ideal of the ring R′ and R′ / R is nilpotent (resp. locally nilpotent, nil), then so is R′ .

It is important to take note that the value of each and every locally nilpotent ring is nil. As early as 1945, Jakob Levitzki pondered the difficult problem [12, 22, 25] of determining whether or not there is a nil ring that is not locally nilpotent. Jakob Levitzki's thoughts date back to 1945. This was a challenge that included figuring out whether or not there was a nil ring in existence. Golod tackled this issue in 1964 by developing the first example of a non-locally nilpotent nil ring using the Golod-Shafarevich construction. This was the first step towards solving the problem. This was the very first time that this issue had ever been resolved without any difficulties. The construction of a nil ring that Golod used for a number of years was basically the only known construction of such a ring for a lot of those years, and it was the only one that did not display locally nilpotent behaviour. We are going to discuss about the Golod-Shafarevich 1 construction, which offers a sufficient condition for an algebra to be infinite-dimensional if it is specified by generators and relations. This structure lays the groundwork for an algebra to have an unlimited number of dimensions, which is a necessary but not sufficient requirement.

A ring R is Jacobson radical (or simply radical) if for every $r \in R$ there exists an element $r' \in R$ such that $r + r' + r r' = 0$. To put it another way, R is said to form a group, which is referred to as the adjoint group R of R, when the binary relation is applied to R.

$a \circ b = a + b + ab$ for $a, b \in R$ .The word "Jacobson radical" may also be used to refer to a specific set that is related to a ring; however, we are not going to use this meaning of the term. Instead, we are going to use the first meaning of the term. Jacobson radicals are a kind of radical that may exist in any nil ring. In order to demonstrate this idea in a more succinct fashion, let us assume that R is a nil ring, let r R be random, and let us also assume that The word "Jacobson radical" may also be used to refer to a specific set that is related to a ring; however, we are not going to use this meaning of the term. Instead, we are going to use the first meaning of the term. Each nil ring may be represented by a Jacobson radical in mathematical notation. For the sake of providing a concise illustration of this principle, let R be a nil ring, and let r R be an empty ring. random, and let $n_r \in \mathbb{N}$ be the nilpotency index of r, we have that $r \circ (-r + r^2 - \ldots . \pm r^{n_r - 1}) = 0$.

However it is not the case that every Jacobson radical ring is a nil ring. For example, the subring $R = \{ \frac{x}{y} : x, y \in Z , p \mid x , p|y\}$ of Q is a Jacobson radical ring which is not a nil ring, where p is a fixed prime. We will now briefly show this. Let x and y be integers such that

$p \mid x$ and $p \nmid y$ and consider the expression $\frac{-x}{x+y}$ Since $p \mid x$ and $p \nmid y$. we have that: p divides the numerator of $\frac{-x}{x+y}$ and p does not divide the denominator of $\frac{-x}{x+y}$. So $\frac{-x}{x+y}$ is in the above subring R of Q, and since

$$\frac{x}{y} + \frac{-x}{x+y} + (\frac{x}{y})\,(\frac{-x}{x+y}) = 0$$

We have evidence suggesting that R is the Jacobson radical. Nevertheless, the component $\frac{p}{p+1}$ We are unable to state that R is a nil-ring since R does not satisfy the nilpotent condition. In a Jacobson radical ring denoted by R, nullity is said to have been attained if and only if each and every one of R's subrings is also radical. If ring R is radical, then $M_i(R)$ must likewise be radical for all cases when I is more than N.

The following chain of inferences is intended to serve as a summary of what was discussed before; however, as was said earlier, it is possible that the contrary may not always be true for any particular inference found in this chain.

nilpotent ⇒ locally nilpotent ⇒ nil ⇒ Jacobson radical	.... ………(1)

There are a huge number of highly well-known theorems that demonstrate "partial converses" in respect to the chain of implications that was stated earlier on in this paragraph. For example, there are well-established theorems that state that nil implies nilpotent provided that a particular set of additional requirements are satisfied. This is the case with the theorems cited above.

It is a truth that is generally acknowledged that nil algebras with restricted index are nilpotent locally. This property may be found in nil algebras. Those nil rings are considered to be locally nilpotent if they satisfy either the polynomial identity requirement or the ascending chain condition on left annihilators. The ascending chain requirement can also be satisfied by locally nilpotent nil rings when applied to right annihilators. Every Noetherian nil algebra and every Artinian nil algebra is nilpotent in the sense that applies to them individually.

Let there be an infinite division ring denoted by D, let there be a natural number denoted by n, allow there be a subnormal subgroup of the group denoted by $GL_n(D)$ denoted by N, and let there be a nilpotent maximal subgroup of N denoted by M. This demonstrates that the following theorem is correct: M is said to be abelian if, and only if, either n = 1 or the centre of D has at least five components. This is the sole condition under which M is said to be abelian.

In the third part of this lesson, we are going to discuss polycyclic groups that have a finite skew. We are going to show that it is impossible for $GL_n(D)$ to have a structure that can be described as polycyclic-by-finite (Lemma 3.1). We show that maximal subgroups of $GL_n(D)$ all have the same quality by demonstrating that these maximal subgroups of $GL_n(D)$ are not polycyclic-by-finite. This is how we show that maximal subgroups of $GL_n(D)$ all share this quality. This characteristic shows that maximum subgroups of $GL_n(D)$ are not polycyclic-by-finite in the sense that the term is often understood. In point of fact, we are in possession ofLet there be an endless division ring denoted by D, let there be a natural number denoted by n, and let there be a maximum subgroup denoted by M. $GL_n(D)$. If n = 1 , M cannot be a polycyclic-by-finite structure if the core of D is comprised of at least five distinct components; in this case, D cannot be considered a polycyclic structure. Note that in order for a division ring to be deemed algebraic across its centres and for it to have a characteristic value of zero, there must be at least five elements present in the ring's centre. This is a requirement for all division rings. On the other hand, it would appear that the requirement in Theorems A and B (for n less than 2) that the centre of D must comprise at least five components is not necessary. This is the case since it appears that this condition is not required.

The notation that we make use of is standard practise. For the sake of better precision, the letter F—unless it is clearly specified differently—always indicates the centre of the division ring D. This is the case even in the absence of any other indication. Utilizing the letter F, we will determine where the centre F In of $M_n(D)$ is located. The derived subgroup of GL_n is designated as $SL_n(D)$, where n is an integer more than two, if and only if D is composed of at least four distinct elements (D). Consider that the group G is a subdivision of the GL_n group (D). The symbol F[G] is used to refer to the F-linear hull of G, which is also referred to as the F-algebra that is produced in $M_n(D)$ by elements of G over F. Both of these are referred to as the F-algebra. It is crucial to note that if every element of G is algebraic over F, then F(G) equals F[G], and this is the case if n is equal to one. If n is equal to one, then the division ring generated in D by F and G is symbolised by the symbol F (G). If D_n is understood to be the space of row n-vectors over D, then, in the simplest possible meaning D_n is a D-G bimodule. We refer to G as having the characteristic of either being irreducible, reducible, or totally reducible depending on whether or not it contains the relevant feature as a D-G bimodule when D_n possesses that feature. In addition to this, it is argued that G is completely irreducible if F[G] equals M_n (D). The notation that denotes the derived subgroup of G is denoted by the letter G'. If R is a ring, then the group of its units is represented by the symbol R, and if S is a subset of R, then the centralizer of S in R is represented by the symbol CR. If R is not a ring, then the group of its units is not represented by the symbol R. (S).

2. OBJECTIVES

1. Research into group rings with nilpotent symmetrical components
2. Research into the nilpotent subgroups of classical groups conducted through communication rings

3. NILPOTENT MAXIMAL SUBGROUPS

In this subsection, we present evidence supporting the validity of Theorem A. To get started, we are going to create some useful lemma statements that will also be used in the following portion of the article.

Lemma 1. ([23, Corollary 24]) Corollaries Consider that the Artinian ring A only has one side. Take into consideration the situation in which S possesses a right. The Goldie subring of A and G functions as a locally soluble subgroup of the A unit group that normalizes the structure. Assume that R is a prime number, and assign it the value obtained by subtracting A from S[G]. If that's the case, R and Goldie are both right.

Lemma 2. Let us use the symbol "D" to represent an infinite-dimensional division algebra over its centre. Let's say that a subnormal subgroup of D is denoted by the letter N. And we'll refer to the largest possible subgroup of N as M. It follows that M must be metabelian in order for it to be considered abelian.

Proof : Since M′ is abelian , we can find a maximal normal abelian subgroup A of M containing M′ . Suppose on the contrary that $A \neq M$. If T is a subgroup of M such that

$A < \neq T$. we claim that F(T) = D . In fact , we have $M \subseteq N_N(F(T)^*) \subseteq N$.

If $M = N_N(F(T)^*)$, then $F(T)^*$ is abelian , so T is also abelian , this contradicts the choose of T . Therefore , by the maximality of M in N we may assume $N_N(F(T)^*) = N$. Then

$N \subseteq N_{D^*}(F(T)^*)$, which by [17,13,3.8] we have F(T) = D , as claimed .

Setting K and $T = A\langle a^2 \rangle$. On the other hand , by the fact that $a^2 \notin A$ we conclude that T is a subgroup of M Properly containing A; hence by what we prove before we conclude that F(T)= D. Hence the division ring generated by F[T] , which is exactly its classical ring of quotients , coincides with D. Thus there exist two elements $s_1, s_2 \in F[T]$ such that $a = s_1 s_2^{-1}$.

Write $s_1 = \sum_{i=l}^{m} k_i a^{2i}$ and $s_2 = \sum_{i=l}^{m} k_i' a^{2i}$, where $k_i , k_i' \in K$, for any $l \le i \le m$. Hence

$\sum_{i=l}^{m} a k_i' a^{2i} = \sum_{i=l}^{m} k_i a^{2i}$

If we set $l_i = a k_i' a^{-1}$ for any $l \le i \le m$, then l_i's are of K and we have

$\sum_{i=l}^{m} l_i a^{2i+1} = \sum_{i=l}^{m} k_i a^{2i}$

Which shows that a is algebraic over K, a contradiction

Now let $x \in M \setminus K$ be algebraic over K. Assume that x satisfies an equation of the form $\sum_{i=0}^{n} k_i x^i = 0$, where $k_i \in K$ for any $0 \le i \le n$ and $k_n=1$ Using the fact that x normalizes K and the above equality one can easily show that $R = \sum_{i=0}^{n} K x^i$

is a left vector space that may be thought of as a ring that has a dimension that is less than K. Since this is the case it is a division ring. If we set T = A(x), by what we proved before F(T) = D. Therefore, D is an example of a finite-dimensional division algebra performed over its centre. The existence of this contradiction demonstrates that M is abelian .

In [1] it was proved that $\mathbb{C}^* \cup \mathbb{C}^* j$ is a maximal subgroup of the real quaternion division algebra. Clearly $(\mathbb{C}^* \cup \mathbb{C}^* j)' \subseteq \mathbb{C}^*$ and so $\mathbb{C}^* \cup \mathbb{C}^* j$ is metabelian but not abelian. As a direct consequence of this, Lemma 2's stipulation that D must include an infinite number of dimensions will never be met. Now, with the assistance of Lemma 2, we are able to show that Theorem A is valid in the case when n is equal to 1.

Semi prime group rings

Let us establish the conditions under which $\mathcal{U}^+(KG)$. If KG is a semi prime, it will have no impact whatsoever on the final result. Recall the conclusion reached by Passman, which states that KG is a semi prime if and only if G does not contain any finite normal subgroup of order that is divisible by p. This is the sole condition under which KG may be considered a semi prime. The subsequent discovery, which has been credited to Sehgal and Valenti, will act as our jumping-off point in our investigation.

Lemma 1. Assume for the sake of this theorem that K is an infinite field with a characteristic that differs from 2, that G contains an element of infinite order, and that KG is semiprime. $\mathcal{U}^{+}(KG)$ Satisfies a group identity, then the following assertions hold.

1. If char K = 0, then T is a subgroup of G which is abelian or a Hamiltonian 2-group.

2. If char K = p>2, then T is an abelian p0 -subgroup of G .

3. Every idempotent in KT is central in KG.

4. If KT has a non-symmetric idempotent , then G=T satisfies a group identity.

If, on the other hand, G/T is a u.p. group and G satisfies 1-4, then $\mathcal{U}^{+}(KG)$ satisfies a group identity

Let us make the assumption that the number KG is a nilpotent semiprime and that this property holds true for it. The application of Lemma 1 can be done right away to parts (1)–(3); however, the fourth component will need a large amount of work to be done on it before it can be used. Let's make the assumption that M is a finite subgroup of T and that KM has a primitive central idempotent e that isn't symmetric. This will allow us to go on to the next step. If T is a Hamiltonian 2-group, then T FQ8 E, where Q8 is the quaternion group, is the equivalent elementary abelian 2-group. If this is the case then it will be easy to establish that any central idempotent in KT is symmetric, and as a result, we may make the assumption that T is an abelian group (and a p0 -group if p is greater than 2). If this is the case, then we may come to the conclusion that KM is quite easy to understand, as will be the case with the Wedderburn decomposition of it.

KM = KMe ⊕ KMe* ⊕…,

Since e* is also a primitive central idempotent of KM.

By Lemma 1, every idempotent of KT is central in KG, and so we may write

KG = KGe ⊕ KGe* ⊕…,

Notice that if g ∈ G and ge = e, then g surely lies in the subgroup generated by the support of e, which is contained in M. We have a homomorphism G → $\mathcal{U}$(KGe) given by g ⟼ ge, and as we have just seen, its kernel N is a subgroup of M. Now suppose that $\mathcal{U}^{+}(KG)$ satisfies $(x_{1,\dots\dots,}x_{n)}$ = 1.

We notice that $ge + g^{-1}e^{*} + (1-(e+e^{*})) \in \mathcal{U}^{+}(KG)$ for every g ∈ G,

and hence $(g_1e + g_1^{-1}e^{*} + (1-(e+e^{*})), g_2e + g_2^{-1}e^{*}+1 - (e+e^{*})\dots..)=1$

In particular, looking only at the first component, we have $(g_1e,g_2e,\ldots,g_ne) = e$. Since e is central, $(g_1,g_2,\ldots,g_n)e = e$, so that $(g_1,g_2,\ldots,g_n) \in N \leq T$. That is, G/T satisfies the same nilpotence identity as $\mathcal{U}^+(KG)$

In addition to this, we would want to show how the equation M/N is an important component of the equation G/N. It would be much simpler to deal with everything "if we could just quotient out N and assume that M is a cyclic group instead. This would make the situation much more manageable (being isomorphic to a finite subgroup of a field)". The remark that follows is going to be important for our conversation.

Lemma 2. Suppose that $\mathcal{U}^+(KG)$ satisfies $(x_{1,\ldots\ldots,}x_{n)}$ = 1. Let N be a finite normal subgroup of G, and if p0 0 assume that N is either a p-group or a p0 -group.

Then $\mathcal{U}^+\left(K\left(\frac{G}{N}\right)\right) satisfies$ $(x_{1,\ldots\ldots,}x_{n)}$ = 1 as well.

Proof. As observed in [4, Remark 4], Passman's proof of [11, Lemma 2.1] (which holds for any group identity), works also for the symmetric units, with this restriction upon N.

Clearly $\bar{e}$ is a primitive central idempotent of K$\bar{M}$ = K(M/N), but we must also make sure that $\bar{e}$ is not symmetric . If $\bar{e} = \bar{e}^{\,*}$, then $e - e^* \in \Delta(M,N)$, where $\Delta(M,N)$is the kernel of the natural homomorphism KM $\rightarrow$ K(M/N). By [13, Proposition III.4.18] , this implies that $\widehat{N}(e - e^*) = 0$, $\widehat{N} = \sum_{m \in N} m$. However $\widehat{N}$ is symmetric, and hence $\widehat{N}e^* = (\widehat{N}e)^*$. Since me= e for all m $\in$ N ,we have $\widehat{N}e^*$ = $|N|e^*$, and $\widehat{N}e^*$ = $|N|e^*$. Therefore e = e^*, which is a contradiction Thus we replace G with G/N and assume that M is cyclic and m $\longmapsto$ me is injective on M. We will show that M is central in G.

Note that if $\lambda \in$ KM ,and $\lambda e \neq 0$, then $\lambda e+\lambda^* e^*+(1-(e+e^*)) \in \boldsymbol{\mathcal{U}}^+(\boldsymbol{KG})$,(It is surely symmetric, and since KMe is a field, le is a unit in KMe.) Also, if x is an element of infinite order in G, then $x^{-1}e+xe^*+(1-(e+e^*)) \in \boldsymbol{\mathcal{U}}^+(\boldsymbol{KG})$. Thus, since $\boldsymbol{\mathcal{U}}^+(\boldsymbol{KG})$ is nilpotent, computing only in the KGe component, we get

$(\lambda e, x^{-1}e, x^{-1}e, \ldots, x^{-1}e)=e.$

Let's take a look at the material that's located on [13, pages 188 and 189]. We are in a position to prove not only that x possesses an infinite order, but also that it produces an automorphism of the KMe field and that hxi is linearly independent over that field. Both of these propositions can be shown by our research. (This final assertion is accurate "since Lemma 1 tells us that the idempotent M =|M| is central in KG; hence, M is regarded as a normal subgroup because of this.) This specific automorphism has a restricted order because it transfers elements of M to other elements of M, and M itself is a finite set". As a result, this particular automorphism has a limited order. The following equation demonstrates that this automorphism is the identity; more specifically, x is the centralizer of KMe. This automorphism is shown to be the identity by the following equation, which, in conjunction with [13, Lemma VI.3.17] and the fact that K is infinite, demonstrates that this automorphism is the identity.

Because M is cyclic, we may represent it as M =< ha>. This is because M is cyclic. We have I for some I, and if an is of order n, then in each Wedderburn component it is mapped to an nth root of unity. There is no mapping if an is not of order n. If an is not of order n, then an. Since m $\longmapsto$ Me is injective on M, we are able to reach the conclusion that a maps to a primitive nth root of unity z in the first component. This is possible because of the fact that m $\longmapsto$ I is injective on M. Because m $\longmapsto$ Me is injective on M, this is the result.

Thus

a $\longmapsto (\xi^i, \xi^{i2}, \xi^{i3}, , , ,\ldots\ldots..)$.

Hence also

$x^{-1}ax = a^i \longmapsto (\xi^i, \xi^{i/2}, \xi^{i/3}, , , ,\ldots\ldots..)$.

On the other hand, we are aware that in the first component, x works to centre ae; hence, ξ $=\xi^i$, and any other powers that correspond to it are identical to one another. Therefore, in an order consisting of an indefinitely long series of elements, for every x element. An is guaranteed to commute with components of finite order since T is abelian; as a result, it is fundamental because of this property.

Let it be known that K is an infinite field that possesses the characteristic p0, and then assert the following theorem: 2. Suppose that the group G is one that has a member with an infinitely high order. Take into account the probability that KG is a semiprime, and so is nilpotent as a result of this. If this is the case, the propositions that follow are true.

(1) If p = 0, then T is an abelian subgroup of G, making it a Hamiltonian 2-group. If p > 0, then T is not an abelian subgroup of G.
(2) T is an abelian p'-subgroup of G if and only if p is greater than 2.
(3) Each and every idempotent that exists in KT also exists in KG.
(4) If M is a finite subgroup of T such that KM has a non-symmetric primitive central idempotent e, then G/T is nilpotent, and M/N is central in G/N, where N is the kernel of T. In this case, if M is a finite subgroup of T, then KM has a non-symmetric primitive central idempotent e. g $\mapsto$ge.

Furthermore, if G/T is a u.p. group, and if KG satisfies the above four conditions, then $\boldsymbol{\mathcal{U}}^{+}(\boldsymbol{KG})$ is nilpotent.

Proof. It remains only to check the sufficiency of the conditions. We claim that $\boldsymbol{\mathcal{U}}^{+}(\boldsymbol{KG})$ satisfies $(u_1,\ldots\ldots,u_{v+1}) = 1$, where v is such that G/T satisfies $(g_1,\ldots g_v)=1$ when not all idempotents in KT are symmetric, and that v = 1 is sufficient when all idempotents in KT are symmetric.

So suppose that $u_1,\ldots..,u_{r+1} \in \mathcal{U}^+(\boldsymbol{KG})$. Choosing a transversal X to T in G (with $1 \in X$). We may write $u_i = \sum_j \alpha_{i,j} x_j$, and $u_i^{-1} = \sum_j \beta_{i,j} x_j$ where $\alpha_{i.j}$, $\beta_{i,j} \in$ KT and x_j . Furthermore , for all $x_r,x_s \in X$ appearing in the above finite sums, let $x_r x_s = t_{r,s} y_{r,s}$, with $y_{r,s} \in X$, $t_{r,s} \in T$. Also write each $(x_{d_1}, x_{d_2}, \ldots \ldots, x_{d_w})$ in the form tx, with $t \in T$ and $x \in X$, where the elements x_{d_i} are among the x_j above, and $2 \leq w \leq v+1$. Let E be the subgroup of T generated by the supports of all $\alpha_{i,j}, \beta_{i,j}$,

The tr and s components, in addition to the t components, can be located in each of the individual tx. The conclusion that E is also finite comes logically from the fact that T is either an abelian or Hamiltonian 2-group, both of which are locally finite. As a direct consequence of this, KE may be described as an Artinian of semisimple complexity. Let's take it for granted that e is one of the primary core idempotents of KE. The reason why KEe is a division ring is because of the fact that.

Now $u_i e = (\sum_i \alpha_{i.i}\, e) x_i$ for each i , so that $u_i e = \left(\sum_j k_j\, x_j\right)$,where each kj is in the division ring KEe, and xj A X, and similarly $u_i^{-1} e = \left(\sum_j l_j\, x_j\right)$ *with* $l_j \in$ KEe. Now, in the first component, the image of $u_i u_i^{-1}$ must be the identity, and hence

$$\left(\sum_j k_j\, x_j\right)\left(\sum_j l_j\, x_j\right) = e$$

So that

$$\sum_r \sum_s k_r\,(x_r l_s x_r{}^{-1}) x_r x_s = e,$$

That is

$$\sum_r \sum_s k_r\,(x_r l_s x_r{}^{-1}) t_{r,s} y_{r,s} = e$$

Notices that $k_r(x_r l_s x_r^{-1})t_{r,s} \in$ KEe . By Strojnowski's Theorem (see [15]), the fact that G =T is a u.p. group implies that it is a t.u.p. group. Thus, if either $u_i e$ or $u_i^{-1}e$ is not of the form kx with k A KEe, x A X, then at least two distinct elements yi ; j appear. Since KEe is a division ring, no cancellation occurs, and so this is a contradiction. Therefore we can write $u_i e = kx$ with $k \in KEe$, $x \in X$.

That is $u_i e$ is symmetric , and therefore kx = (kx)* = $x^{-1}k^*$. Thus $x^{-1}k^*x = kx^2$, and

$x^2 = k^{-1}(x^{-1}k^*x)$ KE. Since x is either 1 or an element of infinite order , it is 1.

That is, $u_i e \in (KE)^*$, Since the symmetric elements in KE commute (this is easily seen if T is a Hamiltonian 2-group), it follows that the symmetric units of KG commute, as desired.

Now suppose that e is not symmetric (and therefore T is abelian, as we discussed above, G/T is nilpotent, and E/N is central in G/N, where N is the kernel of $g \mapsto ge$). Then KEe is central in KGe, and therefore central in KG. But we know that $u_i e = k_i x_i$ with $k_i \in KEe$, $x_i \in X$ for each i. It follows that

$(u_1e,.....,u_ve) =(k_1x_1,.....,k_vx_v)e = (x_1,.....,x_v)e$

since each k_i is central . Now G/T satisfies $(g_1,.......,g_v) =1$, and so $(u_1e,.....,u_ve) =e$ For some $t \in E$. However $te \in KEe$ is central and hence

$(u_1e,.....,u_{v+1}e) = (te, u_{v+1}e) = e.$

Since $KG = \oplus_e KGe$, Where e runs over the finite set of primitive central idempotents of KE, the result follows.

We can now resolve the case with the quaternions completely. We need a result due to Lee for the torsion case.

Lemma 3. Suppose that K is a field with characteristic p #2 and H is a torsion group containing Q_8. Then $\boldsymbol{\mathcal{U}}^+(\boldsymbol{KH})$ is nilpotent if and only if

(1) p = 0 and $H \simeq Q_8 \times E$, where E is an elementary abelian 2-group, or
(2) p > 2 and $H \simeq Q_8 \times E \times P$, where E is an elementary abelian 2-group and P is a finite p-group.

4. CONCLUSION

It is not essential to use the limitation 2 R in order to show that the theorem provided in this work can be extended to hyperbolic unitary groups that are defined over commutative form rings. This is because the verification can be done without using the limitation. With reference to an issue of a similar nature involving overgroups of the diagonal subgroup over a semilocal form ring, this was largely achieved in recent works authored by Dybkova (see). It is possible that the proofs of similar results for form rings with noninvertible 2 and a nontrivial involution can be based on the same ideas and carried out along the same lines as the proofs in the present paper. This would be the case if the proofs in the present paper followed the same lines as the proofs in the similar results paper. This is due to the fact that the arguments presented in this work follow similar lines of reasoning. On the other hand, there is a substantial amount of dispute about the question of whether or not they will be noticeably more challenging from a purely technical basis. Because of this, the author came to the conclusion that it would be best to publish the current paper, which examines a variety of computational approaches that were not previously known to exist within the context of the most basic possible situation. In addition, we direct the reader's attention to the survey as well as the recent article, both of which offer a plethora of additional references to works of comparable relevance that were published after the current research was completed.

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