A NOVEL PROBLEM FOR SOLVING CORDIAL LABELING OF CORONA PRODUCT BETWEEN PATH AND THIRD ORDER OF CONE GRAPHS

A. Elrokh [a], M. M. Ali Al-shamiri [b], Atef Abd El-hay [c]

[a] Dept. of Math., Faculty of Science, Menoufia University, Shebeen Elkom, Egypt.
[b] Dept.of Math, Faculty of Science and Art, Muharyl Asser ,King khalid University, Abha, Saudi Arabia.
[b] Department of Mathematics and Computer, Faculty of Science, Ibb University, Ibb, Yemen
[c] Computer science dep. Higher Institute of Computers and Information Technology, Shorouk Academy.

E mail address: [a]ashraf.hefnawy68@yahoo.com, [b]mal-shamiri@kku.edu.sa and [c]atef_1992@yahoo.com
Article History: *Do not touch during review process(xxxx)*

Abstract: A simple graph is said to be cordial if it admits 0-1 labeling that satisfies certain conditions. In this paper we investigate necessary and sufficient conditions for cordial labeling of the corona Product between path and third order of cone graphs.
Keywords: Path, Cone, Third power of graph, Corona Product, Cordial labeling.

1. Introduction

Let G be a graph with p vertices and q edges. All graphs considered here are simple, finite, connected and undirected. A labeling of a graph G is a process of allocating numbers or labels to the nodes of G or lines of G or both through mathematical functions [2]. There are several types of labeling and a complete survey of graph labeling is available in [6]. Graph theory has a good development in the graph labeling and has a broad range of applications which reported in the work of Yegnanaryanan and Vaidhyanathan [9]. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by Cahit in [3]. Based on this labeling, more papers published in cordial labeling such as mean cordial labeling, H_1- and H_2-cordial labeling of some graphs [7]. In 1990, Chait [4], proved the following: each tree is cordial; an Euerlian graph is not cordial if its size is congruent to $2(mod\ 4)$; a complete graph K_n is cordial if and only if $n \leq 3$ and a complete bipartite graph $K_{n,m}$ is cordial for all positive integers n and m. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona $G_1 \odot G_2$ of two graphs G_1 (with n_1 vertices , m_1 edges) and G_2 (with n_2 vertices , m_2 edges) is defined as the graph obtained by taking one copy of G_1 and copies of G_2 , and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2 . It is easy to see that the corona $G_1 \odot G_2$ that has $n_1 + n_1 n_2$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges. We will give brief summary of definitions which are useful for the present investigations.

Definition 1. A mapping $f: V \to \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called *the label of the vertex v of G under f*. If for an edge $e = uv$, the induced edge labeling $f^*: E(G) \to \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$, where $u, v \in V$. Let $v_f(i)$ be the numbers of vertices of G labeled i under f , and $e_f(i)$ be the numbers of edges of G labeled i under f^* where $i \in \{0,1\}$.

Definition 2. Binary vertex labeling of a graph G is called *cordial* if $\left|(v_f)_0 - (v_f)_1\right| \leq 1$ and $|(e_f)_0 - (e_f)_1| \leq 1$. A graph G is called *Cordial* if it admits cordial labeling.

Definition 3. The cone graph is the join between Null graph N_n and a cycles C_m denoted by $C_{n,m}$

Definition 4. The third power of a cone denoted by $C_{n,m}^3$, is $C_{n,m} \cup J$, where J is the set of all edges of the form edges $v_i v_j$ such that $2 \leq d(v_i v_j) \leq 3$ and $i < j$ where $d(v_i v_j)$ is the shortest path from v_i to v_j .

2. Terminologies and Notations

we can use these symbols of labeling as follows

L'_{8s}	11000011…(s$-time$)11000011
L_{8s}	00111100…(s$-time$)00111100

S'_{8s}	01101001…(s−$time$)01101001
S_{8s}	10010110…(s−$time$)10010110
M'_{8s}	01011010…(s−$time$)01011010
M_{8s}	10100101…(s−$time$)10100101
N'_{4r}	1100…..(s−$time$)1100
N_{8s}	0011…(s−$time$)0011
F'_{4s}	0101…(s−$time$)0101
F_{4s}	1010…(s−$time$)1010
Q'_{4s}	1001…(s−$time$)1001
Q_{4s}	0110…(s−$time$)0110

Table 1. The symbols of labeling.

Suppose that A_a, A'_a, A''_a andA'''_a is a collection of labeling of a cycle c_k where $k = a(mod4)$ and for the special p_k we choose the labeling C_k, C'_k, C''_k and C'''_k, where $k = 1,2,3$.

Suppose that $j = 0,1,2,3$. let B_0^j meaning the labeling of $C^3_{n,4r+j}$ where r is odd and B_e^j meaning the labeling of $C^3_{n,4r+j}$ where r is even.

If L is a labeling for a path P_k and M is a labeling for third power of cone $C^3_{n,m}$, then we use the notation $[L; M]$ to represent the labeling of the corona $P_k \odot C^3_{n,m}$. Additional notation that we use is the following: for a given labeling of the corona $P_k \odot C^3_{n,m}$, we let v_i and e_i (for $i = 0,1$) be the numbers of vertices and edges, respectively, that are labeled by i of the corona $P_k \odot C^3_{n,m}$, and let x_i and a_i be the corresponding quantities for P_k, and we let y_i and b_i be those for $C^3_{n,m}$. It is easy to verify that $v_0 = x_0 + ky_0$, $v_1 = x_1 + ky_1$, $e_0 = a_0 + kb_0 + x_0y_0 + x_1y_1$ and $e_1 = a_1 + kb_1 + x_0y_1 + x_1y_0$. Thus, $v_0 - v_1 = (x_0 - x_1) + k(y_0 - y_1) +$ and $e_0 - e_1 = (a_0 - a_1) + k(b_0 - b_1) + +(x_0 - x_1)(y_0 - y_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.

3. Main result

In this section we study the necessary and sufficient condition of the cordial labeling of a corona between paths and a third power of Cone graphs denoted by $P_k \odot C^3_{n,m}$ for all k, m, n.

the next table illustrate the vertex and edges of the path P_k where $k = 1,2,3$

P_k $k = 1,2,3$	x_0	x_1	a_0	a_1
$P_1 = 0$ $P'_1 = 1$	1 0	0 1	0 0	0 0
$P_2 = 01$ $P'_2 = 0_2$ $P''_2 = 1_2$	1 2 0	1 0 2	0 1 1	1 0 0
$P_3 = 010$ $P'_3 = 0_3$ $P''_3 = 1_3$ $P'''_3 = 0_21$ $P''''_3 = 1_20$	2 3 0 2 1	1 0 3 1 2	0 2 2 1 1	2 0 0 1 1

Table 2. Vertex labeling and edges of a path. P_k

the next table illustrate the vertex and edges of a path P_k where $k \equiv i(mod4)$i.e. $k = 4s + i$, $\forall i = 0,1,2,3$.

P_k $k \equiv i(mod4)$ $i = 0,1,2,3$	x_0	x_1	a_0	a_1
$A_0 = 0_{4s}$ $A_0' = F_{4s}'$ $A_0'' = 1_{4s}$ $A_0''' = N_{4s}$	$4s$ $2s$ 0 $2s$	0 $2s$ $4s$ $2s$	$4s - 1$ 0 $4s - 1$ $2s$	0 $4s - 1$ 0 $2s - 1$
$A_1 = 0_{4s}0$ $A_1' = F_{4s}'0$ $A_1'' = 1_{4s}1$ $A_1''' =_{4s} 0$	$4s + 1$ $2s + 1$ 0 $2s + 1$	0 $2s$ $4s + 1$ $2s$	$4s$ 0 $4s$ $2s$	0 $4s$ 0 $2s$
$A_2 = 0_{4s}0_2$ $A_2' = F_{4s}'01$ $A_2'' = 1_{4s}1_2$ $A_2''' = N_{4s}10$	$4s + 2$ $2s + 1$ 0 $2s + 1$	0 $2s + 1$ $4s + 2$ $2s + 1$	$4s + 1$ 0 $4s + 1$ $2s$	0 $4s + 1$ 0 $4s$
$A_3 = 0_{4s}0_3$ $A_3' = F_{4s}'010$ $A_3'' = 1_{4s}1_3$ $A_3''' = N_{4s}0_21$	$4s + 3$ $2s + 2$ 0 $2s + 2$	0 $2s + 1$ $4s + 3$ $2s + 1$	$4s + 2$ 0 $4s + 2$ $2s + 1$	0 $4s + 2$ 0 $2s + 1$

Taple 3. Vertex labeling and edge of a path. P_k **where** $k \equiv i(mod4)$ $\forall i = 0,1,2,3$

Lemma 3.1 $P_k \odot C_{n,4r}^3$, $m \equiv 0(mod4)$ is cordial , for all $r > 1$.

The next table (4) illustrate the labeling of the Cone $C_{n,4r}^3$, $n = 1,2,3$.

n	labeling of cone $C_{n,4r}^3$	y_0	y_1	b_0	b_1
1	$B_e = 0; L_{8r}'$ $B_e' = 0; 1L_{8r-8}'N_4'0_21$ $B_o = 0; S_{8r}Q_4'$ $B_O' = 0; 101_2L_{8r-8}'N_4'0_210$	$4r$ $4r + 1$ $4r + 2$ $4r + 3$	$4r + 1$ $4r$ $4r + 3$ $4r + 2$	$16r - 3$ $16r - 2$ $16r + 5$ $16r + 6$	$16r - 2$ $16r - 3$ $16r + 6$ $16r + 5$
2	$B_e = 01; L_{8r}'$ $B_e' = 01; 1L_{8r-8}'N_4'0_21$ $B_o = 01; S_{8r}Q_4'$ $B_o' = 01; 101_2L_{8r-8}'N_4'0_210$	$4r + 1$ $4r + 1$ $4r + 3$ $4r + 3$	$4r + 1$ $4r + 1$ $4r + 3$ $4r + 3$	$20r - 3$ $20r - 2$ $20r + 7$ $20r + 8$	$20r - 2$ $20r - 3$ $20r + 8$ $20r + 7$
3	$B_e = 010; L_{8r}'$ $B_e' = 010; 1L_{8r-8}'N_4'0_21$ $B_o = 010; S_{8r}Q_4'$ $B_o' = 010; 101_2L_{8r-8}'N_4'0_210$	$4r + 2$ $4r + 1$ $4r + 3$ $4r + 4$	$4r + 1$ $4r + 2$ $4r + 4$ $4r + 3$	$24r - 3$ $24r - 2$ $24r + 9$ $24r + 10$	$24r - 2$ $24r - 3$ $24r + 10$ $24r + 9$

Table 4. Vertex labeling and edge of a cone $\boldsymbol{C_{n,4r}^3}$**,** $\boldsymbol{n = 1, 2, 3}$.

By using table (2), we study the cordiality of $P_k \odot C_{n,4r}^3$, where $k = 1,2,3$.

n	p_k	$C_{n,4r}^3$	$v_0 - v_1$	$e_0 - e_1$
1	P_1'	B_o' B_e'	0	0
2	P_1	B_o' B_e'	1	1
3	P_1'	B_o' B_e'	0	0
1	P''_2	B_o', B_o' B_e', B_e'	0	-1
2	P_2	B_o', B_o' B_e', B_e'	0	1
3	P_2''	B_o', B_o' B_e', B_e'	0	-1
1	P_3''''	B_o', B_o', B_o' B_e', B_e', B_e'	0	0
2	P_3	B_o', B_o', B_o' B_e', B_e', B_e'	1	1
3	P_3''''	B_o', B_o', B_o' B_e', B_e', B_e'	0	0

Table 5. Vertex labeling and edge of $P_k \odot C_{n,4r}^3$, $n = 1,2,3$.

By using table (3), we study the cordiality of $P_k \odot C_{n,4r}^3$ when $k = i(mode)4\ \forall i = 0,1,2,3$.

n	P_k	$C_{n,4s}^3$	$v_0 - v_1$	$e_0 - e_1$
1	A_0''	$B_o', B_o', B_o, B_o \dots$ $B_e', B_e', B_e, B_e \dots$	0	1
2	A_0'	$B_o', B_o', B_o', B_o' \dots$ $B_e', B_e', B_e', B_e' \dots$	0	1
3	A_0''	$B_o', B_o', B_o, B_o \dots$ $B_e', B_e', B_e, B_e \dots$	0	1
1	A_1''	$B_o', B_o', B_o, B_o \dots, B_o$ $B_e', B_e', B_e, B_e \dots, B_e$	0	0
2	A_1'	$B_o', B_o', B_o', B_o' \dots, B_o'$ $B_e', B_e', B_e', B_e' \dots, B_e'$	1	1
3	A_1''	$B_o', B_o', B_o, B_o \dots, B_o$ $B_e', B_e', B_e, B_e \dots, B_e$	0	0
1	A_2''	$B_o', B_o', B_o, B_o \dots, B_o', B_o$ $B_e', B_e', B_e, B_e \dots, B_e', B_e$	0	-1
2	A_2'	$B_o', B_o', B_o', B_o' \dots, B_o', B_o'$ $B_e', B_e', B_e', B_e' \dots, B_e', B_e'$	0	-1
3	A_2''	$B_o', B_o', B_o, B_o \dots, B_o', B_o$ $B_e', B_e', B_e, B_e \dots, B_e', B_e$	0	-1
1	A_3''	$B_o', B_o', B_o, B_o \dots, B_o', B_o, B_o'$ $B_e', B_e', B_e, B_e \dots, B_e', B_e, B_e'$	0	-1
2	A_3'	$B_o', B_o, B_o', B_o \dots, B_o', B_o, B_o'$ $B_e', B_e, B_e', B_e \dots, B_e', B_e, B_e'$	1	1
3	A_3''	$B_o', B_o, B_o', B_o \dots, B_o', B_o, B_o'$ $B_e', B_e, B_e', B_e \dots, B_e', B_e, B_e'$	0	1

Table 6. Vertex labeling and edge of $\boldsymbol{P_k \odot C_{n,4r}^3}$

subcase (3-1): if $n = i(mod4)$, where $i = 0,1,2,3$.

The next table (7) illustrate the labeling of the Cone $C^3_{n,4r}$.

i	labeling of cone $C^3_{n,4r}$	y_0	y_1	b_0	b_1
0	$B_e = F'_{4t}; L'_{8r}$	$4r + 2t$	$4r + 2t$	$12r + 16rt - 3$	$12r + 16rt - 2$
	$B'_e = F'_{4t}; 1L'_{8r-8}N'_4 0_2 1$	$4r + 2t$	$4r + 2t$	$12 + 16t - 2$	$12r + 16t - 3$
	$B_o = F'_{4t}; S_{8r}Q'_4$	$4r + 2t + 2$	$4r + 2t + 2$	$12r + 16rt + 8t + 3$	$12r + 16rt + 8t + 4$
	$B'_O = F'_{4t}; 101_2 L'_{8r-8}N'_4 0_2 10$	$4r + 2t + 2$	$4r + 2t + 2$	$12r + 16rt + 8t + 4$	$12r + 16rt + 8t + 3$
1	$B_e = F'_{4t}0; L'_{8r}$	$4r + 2t + 1$	$4r + 2t$	$16r + 16rt - 3$	$16r + 16rt - 2$
	$B'_e = F'_{4t}0; 1L'_{8r-8}N'_4 0_2 1$	$4r + 2t + 1$	$4r + 2t$	$16r + 16rt - 2$	$16r + 16rt - 3$
	$B_o = F'_{4t}0; S_{8r}Q'_4$	$4r + 2t + 3$	$4r + 2t + 2$	$16r + 16rt + 8t + 5$	$16r + 16rt + 8t + 6$
	$B'_o = F'_{4t}0; 101_2 L'_{8r-8}N'_4 0_2 10$	$4r + 2t + 3$	$4r + 2t + 2$	$16r + 16rt + 8t + 6$	$16r + 16rt + 8t + 5$
2	$B_e = F'_{4t}01; L'_{8r}$	$4r + 2t + 1$	$4r + 2t + 1$	$20r + 16rt - 3$	$20r + 16rt - 2$
	$B'_e = F'_{4t}01; 1L'_{8r-8}N'_4 0_2 1$	$4r + 2t + 1$	$4r + 2t + 1$	$20r + 16rt - 2$	$20r + 16rt - 3$
	$B_o = F'_{4t}01; S_{8r}Q'_4$	$4r + 2t + 3$	$4r + 2t + 3$	$20r + 16rt + 8t + 7$	$20r + 16rt + 8t + 8$
	$B'_o = F'_{4t}01; 101_2 L'_{8r-8}N'_4 0_2 10$	$4r + 2t + 3$	$4r + 2t + 3$	$20r + 16rt + 8t + 8$	$20r + 16rt + 8t + 7$
3	$B_e = F'_{4t}010; L'_{8r}$	$4r + 2t + 2$	$4r + 2t + 1$	$24r + 16rt - 3$	$24r + 16rt - 2$
	$B'_e = F'_{4t}010; 1L'_{8r-8}N'_4 0_2 1$	$4r + 2t + 2$	$4r + 2t + 1$	$24r + 16rt - 2$	$24r + 16rt - 3$
	$B_o = F'_{4t}010; S_{8r}Q'_4$	$4r + 2t + 4$	$4r + 2t + 3$	$24r + 16rt + 8t + 9$	$24r + 16rt + 8t + 10$
	$B'_o = F'_{4t}010; 101_2 L_{8r-8}N'_4 0_2 10$	$4r + 2t + 4$	$4r + 2t + 3$	$24r + 16rt + 8t + 10$	$24r + 16rt + 8t + 9$

Table 7. Vertex labeling and edge of a cone $C^3_{n,4r}$.

By using table (2), we study the cordiality of $P_3 \odot C^3_{n,4r}$.

i	P_k	$C^3_{n,4s}$	$v_0 - v_1$	$e_0 - e_1$
0	P_1	B'_o B'_e	1	1
1	P'_1	B'_o B'_e	0	0
2	P_1	B'_o B'_e	1	1
3	P'_1	B'_o B'_e	0	0
0	P_2	B'_o, B'_o B'_e, B'_e	0	1
1	P''_2	B'_o, B'_o B'_e, B'_e	0	1
2	P_2	B'_o, B'_o B'_e, B'_e	0	1
3	P''_2	B'_o, B'_o B'_e, B'_e	0	1
0	P_3	B'_o, B'_o, B'_o B'_e, B'_e, B'_e	1	1
1	P'''_3	B'_o, B'_o, B'_o B'_e, B'_e, B'_e	0	0
2	P_3		1	1

		B'_o, B'_o, B'_o B'_e, B'_e, B'_e		
3	P'''_3	B'_o, B'_o, B'_o B'_e, B'_e, B'_e	0	0

Table 8. Vertex labeling and edge of $P_k \odot C^3_{n,4r}$.

By using table (3), we study the cordiality of $P_k \odot C^3_{n,4r+j}$ when $k = i(mode)4$ where $i = 0,1,2,3$.

i	P_k	$C^3_{n,4s}$	$v_0 - v_1$	$e_0 - e_1$
0	A'_0	$B'_o, B'_o, B'_o, B'_o\ldots$ $B'_e, B'_e, B'_e, B'_e\ldots$	0	1
1	A''_0	$B'_o, B'_o, B_o, B_o\ldots$ $B'_e, B'_e, B_e, B_e\ldots$	0	1
2	A'_0	$B'_o, B'_o, B'_o, B'_o\ldots$ $B'_e, B'_e, B'_e, B'_e\ldots$	0	1
3	A''_0	$B'_o, B'_o, B_o, B_o\ldots$ $B'_e, B'_e, B_e, B_e\ldots$	0	1
0	A'_1	$B'_o, B'_o, B'_o, B'_o\ldots, B'_o$ $B'_e, B'_e, B'_e, B'_e\ldots, B'_e$	1	1
1	A''_1	$B_o, B'_o, B_o, B'_o\ldots, B_o$ $B_e, B'_e, B_e, B'_e\ldots, B_e$	0	0
2	A'_1	$B'_o, B'_o, B'_o, B'_o\ldots, B'_o$ $B'_e, B'_e, B'_e, B'_e\ldots, B'_e$	1	1
3	A''_1	$B'_o, B'_o, B_o, B_o\ldots, B_o$ $B'_e, B'_e, B_e, B_e\ldots, B_e$	0	0
0	A'_2	$B'_o, B'_o, B'_o, B'_o\ldots, B'_o, B'_o$ $B'_e, B'_e, B'_e, B'_e\ldots, B'_e, B'_e$	0	1
1	A''_2	$B'_o, B'_o, B_o, B_o\ldots, B_o, B'_o$ $B'_e, B'_e, B_e, B_e\ldots, B_e, B'_e$	0	-1
2	A'_2	$B'_o, B'_o, B'_o, B'_o\ldots, B'_o, B'_o$ $B'_e, B'_e, B'_e, B'_e\ldots, B'_e, B'_e$	0	1
3	A''_2	$B'_o, B'_o, B_o, B_o\ldots, B'_o, B_o$ $B'_e, B'_e, B_e, B_e\ldots, B'_e, B_e$	0	-1
0	A'_3	$B'_o, B'_o, B'_o, B'_o\ldots, B'_o, B'_o, B'_o$ $B'_e, B'_e, B'_e, B'_e\ldots, B'_e, B'_e, B'_e$	1	1
1	A''_3	$B'_o, B_o, B'_o, B_o\ldots, B'_o, B_o, B'_o$ $B'_e, B_e, B'_e, B_e\ldots, B'_e, B_e, B'_e$	0	1
2	A'_3	$B'_o, B'_o, B'_o, B'_o\ldots, B'_o, B'_o, B'_o$ $B'_e, B'_e, B'_e, B'_e\ldots, B'_e, B'_e, B'_e$	1	1
3	A''_3	$B'_o, B_o, B'_o, B_o\ldots, B'_o, B_o, B'_o$ $B'_e, B_e, B'_e, B_e\ldots, B'_e, B_e, B'_e$	0	1

Table 9. Vertex labeling and edge of $P_k \odot C^3_{n,4r}$

Lemma 3.2 $P_k \odot C^3_{n,m}$ is cordial , $m \equiv 1(mod4)$, i.e. $m = 4r + 1$, except at $r = 1$.

subcase (3-2-1): if $n = 1,2,3$.

The next table (10) illustrate the labeling of the Cone $C^3_{n,4r+1}$.

n	labeling of cone $C^3_{n,4r+1}$	y_0	y_1	b_0	b_1
1	$B^1_e = 0; L'_{8r}1$ $B^{1\prime}_e = 0; L_{8r-8}N_4 1 N'_4$ $B^1_o = 0; 1_2 L'_{8r}0_2 1$ $B^{1\prime}_O = 0; 1_2 L'_{8r-8}N'_4 0 N_4 10$	$4r+1$ $4r+1$ $4r+3$ $4r+3$	$4r+1$ $4r+1$ $4r+3$ $4r+3$	$16r-1$ $16r$ $16r+7$ $16r+8$	$16r$ $16r-1$ $16r+8$ $16r+7$
2	$B^1_e = 01; L_{8r}1$ $B^1_o = 01; 1_2 L_{8r}0_2 1$	$4r+1$ $4r+3$	$4r+2$ $4r+4$	$20r$ $20r+10$	$20r$ $20r+10$
3	$B^1_e = 010; L'_{8r}1$ $B^{1\prime}_e = 010; L_{8r-8}N_4 1 N'_4$ $B^1_o = 010; 1_2 L'_{8r}0_2 1$ $B^{1\prime}_o = 010; 1_2 L'_{8r-8}N'_4 0 N_4 10$	$4r+2$ $4r+2$ $4r+4$ $4r+4$	$4s+2$ $4s+2$ $4s+4$ $4s+4$	$24r$ $24r+1$ $24r+12$ $24r+13$	$24r+1$ $24r$ $24r+13$ $24r+12$

Table 10. Vertex labeling and edge of a cone $C^3_{n,4r+1}$.

By using table (2), we study the cordiality of $P_k \odot C^3_{n,4r+1}$, where $k = 1,2,3$.

n	p_k	$C^3_{n,4r+1}$	$v_0 - v_1$	$e_0 - e_1$
1	P_1	$B^{1\prime}_o$ $B^{1\prime}_e$	1	1
2	P_1	$B^{1\prime}_o$ $B^{1\prime}_e$	0	1
3	P_1	$B^{1\prime}_o$ $B^{1\prime}_e$	1	1
1	P_2	$B^{1\prime}_o, B^{1\prime}_o$ $B^{1\prime}_e, B^{1\prime}_e$	0	1
2	P'_2	$B^{1\prime}_o, B^{1\prime}_o$ $B^{1\prime}_e, B^{1\prime}_e$	0	-1
3	P_2	$B^{1\prime}_o, B^{1\prime}_o$ $B^{1\prime}_e, B^{1\prime}_e$	0	1
1	P_3	$B^{1\prime}_o, B^{1\prime}_o, B^{1\prime}_o$ $B^{1\prime}_e, B^{1\prime}_e, B^{1\prime}_e$	1	1
2	P'_3	$B^{1\prime}_o, B^{1\prime}_o, B^{1\prime}_o$ $B^{1\prime}_e, B^{1\prime}_e, B^{1\prime}_e$	0	1
3	P_3	$B^{1\prime}_o, B^{1\prime}_o, B^{1\prime}_o$ $B^{1\prime}_e, B^{1\prime}_e, B^{1\prime}_e$	1	1

Table 11. Vertex labeling and edge of $P_k \odot C^3_{n,4r+1}$.

By using table (3), we study the cordiality of $P_k \odot C^3_{n,4r+1}$ when $k = i(mode)4\ \forall i = 0,1,2,3$.

n	P_k	$C^3_{n,4s+1}$	$v_0 - v_1$	$e_0 - e_1$
1	A'_0	$B^{1\prime}_o, B^{1\prime}_o, B^{1\prime}_o, B^{1\prime}_o \ldots$ $B^{1\prime}_e, B^{1\prime}_e, B^{1\prime}_e, B^{1\prime}_e \ldots$	0	1
2	A_0	$B^1_o, B^1_o, B^1_o, B^1_o \ldots$ $B^1_e, B^1_e, B^1_e, B^1_e \ldots$	0	-1
3	A'_0	$B^{1\prime}_o, B^{1\prime}_o, B^{1\prime}_o, B^{1\prime}_o \ldots$ $B^{1\prime}_e, B^{1\prime}_e, B^{1\prime}_e, B^{1\prime}_e \ldots$	0	1

1	A'_1	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}$	1	1
2	A_1	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1$	0	-1
3	A'_1	$B_o^{1'}, B_o^1, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}$ $B_e^{1'}, B_e^1, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}$	1	1
1	A'_2	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}, B_e^{1'}$	0	1
2	A_2	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1, B_e^1$	0	-1
3	A'_2	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}, B_e^{1'}$	0	1
1	A'_3	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}, B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}, B_e^{1'}, B_e^{1'}$	1	1
2	A_3	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1, B_o^1, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1, B_e^1, B_e^1$	0	-1
3	A'_3	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}, B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}, B_e^{1'}, B_e^{1'}$	1	1

Table 12. Vertex labeling and edge of $P_k \odot C_{n,4r+1}^3$

<u>subcase (3-2-2):</u> if $n = i(mod4)$, where $i = 0,1,2,3$.

The next table (2.4) illustrate the labeling of the Cone $C_{n,4r+1}^3$.

n	*labeling of cone* $C_{n,4r+1}^3$	y_0	y_1	b_0	b_1
0	$B_e^1 = F'_{4t}; L'_{8r}1$ $B_o^1 = F'_{4t}; 1_2 L'_{8r} 0_2 1$	$4r + 2t$ $4r + 2t + 2$	$4r + 2t + 1$ $4r + 2t + 3$	$12r + 16rt + 2t + 1$ $12r + 16rt + 10t + 5$	$12r + 16rt + 2t + 1$ $12r + 16rt + 10t + 5$
1	$B_e^1 = F'_{4t}0; L'_{8r}1$ $B_e^{1'} = F'_{4t}0; L_{8r-8} N_4 1 N'_4$ $B_o^1 = F'_{4t}0; 1_2 L'_{8r} 0_2 1$ $B_o^{1'} = F'_{4t}0; 1_2 L'_{8r-8} N'_4 0 N_4 10$	$4r + 2t + 1$ $4r + 2t + 1$ $4r + 2t + 3$ $4r + 2t + 3$	$4r + 2t + 1$ $4r + 2t + 1$ $4r + 2t + 3$ $4r + 2t + 3$	$16r + 16rt + 2t - 1$ $16r + 16rt + 2t$ $16r + 16rt + 10t + 7$ $16r + 16rt + 10t + 8$	$16r + 16rt + 2t$ $16r + 16rt + 2t - 1$ $16r + 16rt + 10t + 8$ $16r + 16rt + 10t + 7$
2	$B_e^1 = F'_{4t}10; L'_{8r}1$ $B_o^1 = F'_{4t}10; 1_2 L'_{8r} 0_2 1$	$4r + 2t + 1$ $4r + 2t + 3$	$4r + 2t + 2$ $4r + 2t + 4$	$20r + 16rt + 2t$ $20r + 16rt + 10t + 10$	$20r + 16rt + 2t$ $20r + 16rt + +10t + 10$
3	$B_e^1 = F'_{4t}010; L'_{8r}1$ $B_e^{1'} = F'_{4t}010; L_{8r-8} N_4 1 N'_4$ $B_o^1 = F'_{4t}010; 1_2 L'_{8r} 0_2 1$ $B_o^{1'} = F'_{4t}010; 1_2 L'_{8r-8} N'_4 0 N_4 10$	$4r + 2t + 2$ $4r + 2t + 2$ $4r + 2t + 4$ $4r + 2t + 4$	$4r + 2t + 2$ $4r + 2t + 2$ $4r + 2t + 4$ $4r + 2t + 4$	$24r + 16rt + 2t$ $24r + 16rt + 2t + 1$ $24r + 16rt + 10t - 13$ $24r + 16rt + 10t - 12$	$24r + 16rt + 2t + 1$ $24r + 16rt + 2t$ $24r + 16rt + 10t - 12$ $24r + 16rt + 10t - 13$

Table 13. Vertex labeling and edge of a cone $C_{n,4r+1}^3$.

By using table (2), we study the cordiality of $P_k \odot C_{n,4r+1}^3$, where $k = 1,2,3$.

i	P_k	$C_{n,4r+1}^3$	$v_0 - v_1$	$e_0 - e_1$
0	p_1	$B_o^{1'}$ $B_e^{1'}$	0	1
1	P_1	$B_o^{1'}$ $B_e^{1'}$	1	1

2	P_1	$B_o^{1'}$ $B_e^{1'}$	0	1
3	P_1	$B_o^{1'}$ $B_e^{1'}$	1	1
0	P_2'	B_o^1, B_o^1 B_e^1, B_e^1	0	-1
1	P_2	$B_o^{1'}, B_o^{1'}$ $B_e^1, B_e^{1'}$	0	1
2	P_2'	B_o^1, B_o^1 B_e^1, B_e^1	0	-1
3	P_2	$B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}$	0	-1
0	P_3'	B_o^1, B_o^1, B_o^1 B_e^1, B_e^1, B_e^1	0	1
1	P_3	$B_o^{1'}, B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}$	1	1
2	P_3'	B_o^1, B_o^1, B_o^1 B_e^1, B_e^1, B_e^1	0	1
3	P_3	$B_o^{1'}, B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}$	1	1

Table 14. Vertex labeling and edge of $P_k \odot C_{n,4r+1}^3$.

By using table (3), we study the cordiality of $P_k \odot C_{n,4r+1}^3$ when $k = i(mode)4$ where $i = 0,1,2,3$.

i	P_k	$C_{n,4s+1}^3$	$v_0 - v_1$	$e_0 - e_1$
0	A_0	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots$	0	-1
1	A_0'	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots$	0	1
2	A_0	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots$	0	-1
3	A_0'	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots$	0	1
0	A_1	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1$	0	1
1	A_1'	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}$	1	1
2	A_1	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1$	0	-1
3	A_1'	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}$	1	1
0	A_2	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1, B_e^1$	0	-1
1	A_2'	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}, B_e^{1'}$	0	1
2	A_2	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1, B_e^1$	0	-1
3	A_2'	$B_o^{1'}, B_o^{1'}, B_o^{1'}, B_o^{1'} \ldots, B_o^{1'}, B_o^{1'}$ $B_e^{1'}, B_e^{1'}, B_e^{1'}, B_e^{1'} \ldots, B_e^{1'}, B_e^{1'}$	0	1

0	A_3	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1, B_o^1, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1, B_e^1, B_e^1$	0	−1
1	A_3'	$B_o^{1\prime}, B_o^1, B_o^{1\prime}, B_o^{1\prime} \ldots, B_o^{1\prime}, B_o^{1\prime}, B_o^{1\prime}$ $B_e^{1\prime}, B_e^1, B_e^{1\prime}, B_e^{1\prime} \ldots, B_e^{1\prime}, B_e^{1\prime}, B_e^{1\prime}$	1	1
2	A_3	$B_o^1, B_o^1, B_o^1, B_o^1 \ldots, B_o^1, B_o^1, B_o^1$ $B_e^1, B_e^1, B_e^1, B_e^1 \ldots, B_e^1, B_e^1, B_e^1$	0	−1
3	A_3'	$B_o^{1\prime}, B_o^{1\prime}, B_o^{1\prime}, B_o^{1\prime} \ldots, B_o^{1\prime}, B_o^{1\prime}, B_o^{1\prime}$ $B_e^{1\prime}, B_e^{1\prime}, B_e^{1\prime}, B_e^{1\prime} \ldots, B_e^{1\prime}, B_e^{1\prime}, B_e^{1\prime}$	1	1

Table 15. Vertex labeling and edge of $P_k \odot C^3_{n,4r+1}$

Lemma 3.3 $P_k \odot C^3_{n,m}$,$m \equiv 2(mod4)$,i.e. $m = 4r + 2$ then is cordial, except at $r = 1$.

subcase (3-3-1): if $n = 1,2,3$.

The next table (16) illustrate the labeling of the Cone $C^3_{n,4r+2}$.

n	labeling of cone $C^3_{n,4r+2}$	y_0	y_1	b_0	b_1
1	$B_e^2 = 0; L'_{8r}01$ $B_e^{2\prime} = 0; 1_2L'_{8r-8}N'_4010_2$ $B_o^2 = 0; 1L'_{8r}N'_40$ $B_O^{2\prime} = 0; 1L_{8r-8}0N_4$	$4r+2$ $4r+2$ $4r+4$ $4r+4$	$4r+1$ $4r+1$ $4r+3$ $4r+3$	$16r+1$ $16r+2$ $16r+9$ $16r+10$	$16r+2$ $16r+1$ $16r+10$ $16r+9$
2	$B_e^2 = 01; L'_{8r}01$ $B_e^{2\prime} = 01; 1_2L'_{8r-8}N'_4010_2$ $B_o^2 = 01; 1L'_{8r}N'_40$ $B_O^{2\prime} = 01; 1L_{8r-8}0N_4$	$4r+2$ $4r+2$ $4r+4$ $4r+4$	$4r+2$ $4r+2$ $4r+4$ $4r+4$	$20r+2$ $20r+3$ $20r+12$ $20r+13$	$20r+3$ $20r+2$ $20r+13$ $20r+12$
3	$B_e^2 = 010; L'_{8r}01$ $B_e^{2\prime} = 010; 1_2L'_{8r-8}N'_4010_2$ $B_o^2 = 010; 1L'_{8r}N'_40$ $B_O^{2\prime} = 010; 1L_{8r-8}0N_4$	$4r+3$ $4r+3$ $4r+5$ $4r+5$	$4r+2$ $4r+2$ $4r+4$ $4r+4$	$24r+3$ $24r+4$ $24r+15$ $24r+16$	$24r+4$ $24r+3$ $24r+16$ $24r+15$

Table 16. Vertex labeling and edge of a cone $C^3_{n,4r+2}$.

By using table (2), we study the cordiality of $P_k \odot C^3_{n,4r+2}$, where $k = 1,2,3$.

n	P_k	$C^3_{n,4r+2}$	$v_0 - v_1$	$e_0 - e_1$
1	P_1'	$B_o^{2\prime}$ $B_e^{2\prime}$	0	0
2	P_1	$B_o^{2\prime}$ $B_e^{2\prime}$	1	1
3	P_1'	$B_o^{2\prime}$ $B_e^{2\prime}$	0	0
1	P_2''	$B_o^{2\prime}, B_o^{2\prime}$ $B_e^{2\prime}, B_e^{2\prime}$	0	1

2	P_2	$B_o^{2\prime}, B_o^{2\prime}$ $B_e^{2\prime}, B_e^{2\prime}$	0	1
3	P_2''	$B_o^{2\prime}, B_o^{2\prime}$ $B_e^{2\prime}, B_e^{2\prime}$	0	1
1	P_3''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2$	0	0
2	P_3	$B_o^{2\prime}, B_o^{2\prime}, B_o^{2\prime}$ $B_e^{2\prime}, B_e^{2\prime}, B_e^{2\prime}$	1	1
3	P_3''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2$	0	0

Table 17. Vertex labeling and edge of $P_3 \odot C_{n,4r+2}^3$.

By using table (3), we study the cordiality of $P_k \odot C_{n,4r+2}^3$ when $k = i(mode)4\ \forall i = 0,1,2,3$.

n	P_k	$C_{n,4s+2}^3$	$v_0 - v_1$	$e_0 - e_1$
1	A_0''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2, B_o^2 \ldots$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2, B_e^2 \ldots$	0	−1
2	A_0'	$B_o^{2\prime}, B_o^{2\prime}, B_o^{2\prime}, B_o^{2\prime} \ldots$ $B_e^{2\prime}, B_e^{2\prime}, B_e^{2\prime}, B_e^{2\prime} \ldots$	0	1
3	A_0''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2, B_o^2 \ldots$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2, B_e^2 \ldots$	0	−1
1	A_1''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2, B_o^2 \ldots, B_o^{2\prime}$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2, B_e^2 \ldots, B_e^{2\prime}$	0	0
2	A_1'''	$B_o^{2\prime}, B_o^2, B_o^{2\prime}, B_o^2 \ldots, B_o^{2\prime}$ $B_e^{2\prime}, B_e^2, B_e^{2\prime}, B_e^2 \ldots, B_e^{2\prime}$	1	1
3	A_1''	$B_o^{2\prime}, B_o^2, B_o^{2\prime}, B_o^2 \ldots, B_o^{2\prime}$ $B_e^{2\prime}, B_e^2, B_e^{2\prime}, B_e^2 \ldots, B_e^{2\prime}$	0	0
1	A_2''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2, B_o^2 \ldots, B_o^2, B_o^{2\prime}$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2, B_e^2 \ldots, B_e^2, B_e^{2\prime}$	0	−1
2	A_2'''	$B_o^2, B_o^2, B_o^{2\prime}, B_o^{2\prime} \ldots, B_o^{2\prime}, B_o^2$ $B_e^2, B_e^2, B_e^{2\prime}, B_e^{2\prime} \ldots, B_e^{2\prime}, B_e^2$	0	1
3	A_2''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2, B_o^2 \ldots, B_o^{2\prime}, B_o^2$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2, B_e^2 \ldots, B_e^{2\prime}, B_e^2$	0	−1
1	A_3''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2, B_o^2 \ldots, B_o^{2\prime}, B_o^2, B_o^{2\prime}$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2, B_e^2 \ldots, B_e^{2\prime}, B_e^2, B_e^{2\prime}$	0	0
2	A_3'''	$B_o^{2\prime}, B_o^{2\prime}, B_o^2, B_o^2 \ldots, B_o^{2\prime}, B_o^2, B_o^{2\prime}$ $B_e^{2\prime}, B_e^{2\prime}, B_e^2, B_e^2 \ldots, B_e^{2\prime}, B_e^2, B_e^{2\prime}$	1	1
3	A_3''	$B_o^{2\prime}, B_o^2, B_o^{2\prime}, B_o^2 \ldots, B_o^{2\prime}, B_o^2, B_o^{2\prime}$ $B_e^{2\prime}, B_e^2, B_e^{2\prime}, B_e^2 \ldots, B_e^{2\prime}, B_e^2, B_e^{2\prime}$	0	0

Table 18. Vertex labeling and edge of $P_k \odot C_{n,4r+2}^3$

<u>subcase (3-3-2):</u> if $n = i(mod4)$, where $i = 0,1,2,3$.

The next table (19) illustrate the labeling of the Cone $C^3_{n,4r+2}$.

n	$labeling\ of\ cone$ $C^3_{n,4r+2}$	y_0	y_1	b_0	b_1
0	$B_e^2 = F'_{4t}; L'_{8r}01$ $B_e^{2'} = F'_{4t}; 1_2L'_{8r-8}N'_4010_2$ $B_o^2 = F'_{4t}; 1L'_{8r}N'_40$ $B_O^{2'} = F'_{4t}; 1L_{8r-8}0N_4$	$4r+2t+1$ $4r+2t+1$ $4r+2t+3$ $4r+2t+3$	$4r+2t+1$ $4r+2t+1$ $4r+2t+3$ $4r+2t+3$	$12r+16rt+4t$ $12r+16rt+4t+1$ $12r+16rt+12t+6$ $12r+16rt+12t+7$	$12r+16rt+4t+1$ $12r+16rt+4t$ $12r+16rt+12t+7$ $12r+16rt+12t+6$
1	$B_e^2 = F'_{4t}0; L'_{8r}01$ $B_e^{2'} = F'_{4t}0; 1_2L'_{8r-8}N'_4010_2$ $B_o^2 = F'_{4t}0; 1L'_{8r}N'_40$ $B_0^{2'} = F'_{4t}0; 1L_{8r-8}0N_4$	$4r+2t+2$ $4r+2t+2$ $4r+2t+4$ $4r+2t+4$	$4r+2t+1$ $4r+2t+1$ $4r+2t+3$ $4r+2t+3$	$16r+16rt+4t+1$ $16r+16rt+4t+2$ $16r+16rt+12t+9$ $16r+16rt+12t+10$	$16r+16rt+4t+2$ $16r+16rt+4t+1$ $16r+16rt+12t+10$ $16r+16rt+12t+9$
2	$B_e^2 = F'_{4t}01; L'_{8r}01$ $B_e^{2'} = F'_{4t}01; 1_2L'_{8r-8}N'_4010_2$ $B_o^2 = F'_{4t}01; 1L_{8r}N'_40$ $B_0^{2'} = F'_{4t}01; 1L_{8r-8}0N_4$	$4r+2t+2$ $4r+2t+2$ $4r+2t+4$ $4r+2t+4$	$4r+2t+2$ $4r+2t+2$ $4r+2t+4$ $4r+2t+4$	$20r+16rt+4t+2$ $20r+16rt+4t+3$ $20r+16rt+12t+12$ $20r+16rt+12t+13$	$20r+16rt+4t+3$ $20r+16rt+4t+2$ $20r+16rt+12t+13$ $20r+16rt+12t+12$
3	$B_e^2 = F'_{4t}010; L'_{8r}01$ $B_e^{2'} = F'_{4t}010; 1_2L'_{8r-8}N'_4010_2$ $B_o^2 = F'_{4t}010; 1L'_{8r}N'_40$ $B_O^{2'} = F'_{4t}010; 1L_{8r-8}0N_4$	$4r+2t+3$ $4r+2t+3$ $4r+2t+5$ $4r+2t+5$	$4r+2t+2$ $4r+2t+2$ $4r+2t+4$ $4r+2t+4$	$24r+16rt+4t+3$ $24r+16rt+4t+4$ $24r+16rt+12t+15$ $24r+16rt+12t+16$	$24r+16rt+4t+4$ $24r+16rt+4t+3$ $24r+16rt+12t+16$ $24r+16rt+12t+15$

Table 19. Vertex labeling and edge of a cone $C^3_{n,4r+2}$.

By using table (2), we study the cordiality of $P_k \odot C^3_{n,4r+2}$.

i	p_k	$C^3_{n,4r+2}$	$v_0 - v_1$	$e_0 - e_1$
0	P_1	$B_o^{2'}$ $B_e^{2'}$	1	1
1	P'_1	$B_o^{2'}$ $B_e^{2'}$	0	0
2	P_1	$B_o^{2'}$ $B_e^{2'}$	1	1
3	P'_1	$B_o^{2'}$ $B_e^{2'}$	0	0
0	P_2	$B_o^{2'}, B_o^{2'}$ $B_e^{2'}, B_e^{2'}$	0	1
1	P''_2	$B_o^{2'}, B_o^{2'}$ $B_e^{2'}, B_e^{2'}$	0	1
2	P_2	$B_o^{2'}, B_o^{2'}$ $B_e^{2'}, B_e^{2'}$	0	1
3	P''_2	$B_o^{2'}, B_o^{2'}$ $B_e^{2'}, B_e^{2'}$	0	1
0	P_3	$B_o^{2'}, B_o^{2'}, B_o^{2'}$ $B_e^{2'}, B_e^{2'}, B_e^{2'}$	1	1
1	P''_3	$B_o^2, B_o^{2'}, B_o^{2'}$ $B_e^2, B_e^{2'}, B_e^{2'}$	0	0
2	P_3	$B_o^{2'}, B_o^{2'}, B_o^{2'}$ $B_e^{2'}, B_e^{2'}, B_e^{2'}$	1	1
			0	0

3	P_3''	$B_o^{2'}, B_o^{2'}, B_o^2$ $B_e^{2'}, B_e^{2'}, B_e^2$		

Table 20. Vertex labeling and edge of $P_k \odot C_{n,4r+2}^3$.

By using table (3), we study the cordiality of $p_k \odot C_{n,4r+2}^3$ when $k = i(mode)4$ where $i = 0,1,2,3$.

i	p_k	$C_{n,4r+2}^3$	$v_0 - v_1$	$e_0 - e_1$
0	A_0'	$B_o^{2'}, B_o^{2'}, B_o^{2'}, B_o^{2'} \dots$ $B_e^{2'}, B_e^{2'}, B_e^{2'}, B_e^{2'} \dots$	0	1
1	A_0''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots$	0	-1
2	A_0'	$B_o^{2'}, B_o^{2'}, B_o^{2'}, B_o^{2'} \dots$ $B_e^{2'}, B_e^{2'}, B_e^{2'}, B_e^{2'} \dots$	0	1
3	A_0''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots$	0	-1
0	A_1'''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots, B_o^{2'}$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots, B_e^{2'}$	1	1
1	A_1''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots, B_o^{2'}$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots, B_e^{2'}$	0	0
2	A_1'''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots, B_o^{2'}$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots, B_e^{2'}$	1	1
3	A_1''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots, B_o^{2'}$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots, B_e^{2'}$	0	0
0	A_2'''	$B_o^{2'}, B_o^2, B_o^{2'}, B_o^2 \dots, B_o^{2'}, B_o^2$ $B_e^2, B_e^2, B_e^{2'}, B_e^2 \dots, B_e^{2'}, B_e^2$	0	1
1	A_2''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots, B_o^{2'}, B_o^2$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots, B_e^{2'}, B_e^2$	0	-1
2	A_2'''	$B_o^2, B_o^{2'}, B_o^2, B_o^{2'} \dots, B_o^{2'}, B_o^2$ $B_e^2, B_e^{2'}, B_e^2, B_e^{2'} \dots, B_e^{2'}, B_e^2$	0	1
3	A_2''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots, B_o^{2'}, B_o^2$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots, B_e^{2'}, B_e^2$	0	-1
0	A_3'''	$B_o^{2'}, B_o^{2'}, B_o^2, B_o^2 \dots, B_o^{2'}, B_o^2, B_o^{2'}$ $B_e^{2'}, B_e^{2'}, B_e^2, B_e^2 \dots, B_e^{2'}, B_e^2, B_e^{2'}$	1	1
1	A_3''	$B_o^{2'}, B_o^2, B_o^{2'}, B_o^2 \dots, B_o^{2'}, B_o^2, B_o^{2'}$ $B_e^{2'}, B_e^2, B_e^{2'}, B_e^2 \dots, B_e^{2'}, B_e^2, B_e^{2'}$	0	0
2	A_3'''	$B_o^{2'}, B_o^2, B_o^2, B_o^{2'} \dots, B_o^{2'}, B_o^2, B_o^{2'}$ $B_e^{2'}, B_e^2, B_e^2, B_e^{2'} \dots, B_e^{2'}, B_e^2, B_e^{2'}$	1	1
3	A_3''	$B_o^{2'}, B_o^2, B_o^{2'}, B_o^2 \dots, B_o^{2'}, B_o^2, B_o^{2'}$ $B_e^{2'}, B_e^2, B_e^{2'}, B_e^2 \dots, B_e^{2'}, B_e^2, B_e^{2'}$	0	0

Table 21. Vertex labeling and edge of $p_k \odot C_{n,4r+2}^3$

Lemma 3.4 $P_k \odot C_{n,4s+3}^3$, $m \equiv 3(mod4)$, i.e $m = 4r + 3$ is cordial.

subcase (3-4-1): if $n = 1,2,3$.

The next table (22) illustrate the labeling of the Cone $C_{n,4r+3}^3$.

n	labeling of cone $C_{n,4r+3}^3$	y_0	y_1	b_0	b_1

1	$B_o^3 = 0; 01_2L'_{8r-8}N'_4$ $B_o^{3'} = 0; 0L_{8r-8}N_4 1_2$	$4r$ $4r$	$4r$ $4r$	$16r-5$ $16r-4$	$16r-4$ $16r-5$
2	$B_o^3 = 01; 01_2L'_{8r-8}N'_4$ $B_e^3 = 01; 1_2L'_{8r}0$	$4r$ $4r+2$	$4r+1$ $4r+3$	$20r-5$ $20r+5$	$20r-5$ $20r+5$
3	$B_o^3 = 010; 01_2L'_{8r-8}N'_4$ $B_o^{3'} = 010; 0L_{8r-8}N_4 1_2$	$4r+1$ $4r+1$	$4r+1$ $4r+1$	$24r-6$ $24r-5$	$24r-5$ $24r-6$

Table 22. Vertex labeling and edge of a cone $C^3_{n,4r+3}$.

By using table (2), we study the cordiality of $P_k \odot C^3_{n,4r+3}$, where $k = 1,2,3$.

n	P_k	$C^3_{n,4r+3}$	$v_0 - v_1$	$e_0 - e_1$
1	P_1	B_o^3 B_e^3	1	-1
2	P_1	B_o^3 B_e^3	0	-1
3	P_1	B_o^3 B_e^3	1	-1
1	P_2	$B_o^{3'}, B_o^{3'}$	0	1
2	P'_2	B_o^3, B_o^3 B_e^3, B_e^3	0	-1
3	P_2	$B_o^{3'}, B_o^{3'}$	0	1
1	P_3	$B_o^{3'}, B_o^{3'}, B_o^{3'}$	1	1
2	P'_3	B_o^3, B_o^3, B_o^3 B_e^3, B_e^3, B_e^3	0	-1
3	P_3	$B_o^{3'}, B_o^{3'}, B_o^{3'}$	1	1

Table 23. Vertex labeling and edge of $P_k \odot C^3_{n,4r+3}$.

By using table (3), we study the cordiality of $P_k \odot C^3_{n,4r+3}$ when $k = i(mode)4\ \forall i = 0,1,2,3$.

n	P_k	$C^3_{n,4s+3}$	$v_0 - v_1$	$e_0 - e_1$
1	A'_0	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \ldots$	0	1
2	A_0	$B_o^3, B_o^3, B_o^3, B_o^3 \ldots$ $B_e^3, B_e^3, B_e^3, B_e^3 \ldots$	0	-1
3	A'_0	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \ldots$	0	1
1	A'_1	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \ldots, B_o^{3'}$	1	1
2	A_1	$B_o^3, B_o^3, B_o^3, B_o^3 \ldots, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \ldots, B_e^3$	0	-1
3	A'_1	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \ldots, B_o^{3'}$	1	1
1	A'_2	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \ldots, B_o^{3'}, B_o^{3'}$	0	1
2	A_2	$B_o^3, B_o^3, B_o^3, B_o^3 \ldots, B_o^3, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \ldots, B_e^3, B_e^3$	0	-1
3	A'_2	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \ldots, B_o^{3'}, B_o^{3'}$	0	1
1	A'_3	$B_o^{3'}, B_o^{3'}, B_o^3, B_o^3 \ldots, B_o^{3'}, B_o^3, B_o^{3'}$	1	1

2	A_3	$B_o^3, B_o^3, B_o^3, B_o^3 \ldots, B_o^3, B_o^3, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \ldots, B_e^3, B_e^3, B_e^3$	0	-1
3	A_3'	$B_o^{3\prime}, B_o^3, B_o^{3\prime}, B_o^{3\prime} \ldots, B_o^{3\prime}, B_o^{3\prime}, B_o^{3\prime}$	1	1

Table 24. Vertex labeling and edge of $P_k \odot C_{n,4s+3}^3$

subcase (3-4-2): if $n = i(mod4)$, where $i = 0,1,2,3$.

The next table (25) illustrate the labeling of the Cone $C_{n,4r+3}^3$.

n	$labeling\ of\ cone$ $C_{n,4r+3}^3$	y_0	y_1	b_0	b_1
0	$B_o^3 = F_{4t}'; 01_2L_{8r-8}'N_4'$ $B_e^3 = F_{4t}'; 1_2L_{8r}'0$	$4r+2t-1$ $4r+2t+1$	$4r+2t$ $4r+2t+2$	$12r+16rt-2t-4$ $12r+16rt+6t+2$	$12r+16rt-2t-4$ $12r+16rt+6t+2$
1	$B_o^3 = F_{4t}'0; 01_2L_{8r-8}'N_4'$ $B_o^{3\prime} = F_{4t}0; 0L_{8r-8}N_41_2$	$4r+2t$ $4r+2t$	$4r+2t$ $4r+2t$	$16r+16rt-2t-5$ $16r+16rt-2t-4$	$16r+16rt-2t-4$ $16r+16rt-2t-5$
2	$B_o^3 = F_{4t}'01; 01_2L_{8r-8}'N_4$ $B_e^3 = F_{4t}'01; 1_2L_{8r}'0$	$4r+2t$ $4r+2t+2$	$4r+2t+1$ $4r+2t+3$	$20r+16rt-2t-5$ $20r+16rt+6t+5$	$20r+16rt-2t-5$ $20r+16rt+6t+5$
3	$B_o^3 = F_{4t}'010; 01_2L_{8r-8}'N_4$ $B_o^{3\prime} = F_{4t}'010; 0L_{8r-8}N_41_2$	$4r+2t+1$ $4r+2t+1$	$4r+2t+1$ $4r+2t+1$	$24r+16rt-2t-6$ $24r+16rt-2t-5$	$24r+16rt-2t-5$ $24r+16rt-2t-6$

Table 25. Vertex labeling and edge of a cone $C_{n,4r+3}^3$.

By using table (2), we study the cordiality of $P_k \odot C_{n,4r+3}^3$, where $k = 1,2,3$.

i	P_k	$C_{n,4r+3}^3$	$v_0 - v_1$	$e_0 - e_1$
0	P_1	B_o^3 B_e^3	0	-1
1	P_1	B_o^3 B_e^3	1	-1
2	P_1	B_o^3 B_e^3	0	-1
3	P_1	B_o^3 B_o^3	1	-1
0	P_2'	B_o^3, B_o^3 B_e^3, B_e^3	0	-1
1	P_3	$B_o^{3\prime}, B_o^{3\prime}$	0	1
2	P_2'	B_o^3, B_o^3 B_e^3, B_e^3	0	-1
3	P_2	$B_o^{3\prime}, B_o^{3\prime}$	0	1
0	P_3'	B_o^3, B_o^3, B_o^3 B_e^3, B_e^3, B_e^3	0	-1
1	P_3	$B_o^{3\prime}, B_o^{3\prime}, B_o^{3\prime}$	1	1

2	P'_3	B_o^3, B_o^3, B_o^3 B_e^3, B_e^3, B_e^3	0	-1
3	P_3	$B_o^{3'}, B_o^{3'}, B_o^{3'}$	1	1

Table 26. Vertex and edge of $P_k \odot C^3_{n,4r+3}$.

By using table (3), we study the cordiality of $P_k \odot C^3_{n,4r+3}$ when $k = i(mode)4$ where $i = 0,1,2,3$.

i	P_k	$C^3_{n,4r+3}$	$v_0 - v_1$	$e_0 - e_1$
0	A_0	$B_o^3, B_o^3, B_o^3, B_o^3 \dots$ $B_e^3, B_e^3, B_e^3, B_e^3 \dots$	0	-1
1	A'_0	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \dots$	0	1
2	A_0	$B_o^3, B_o^3, B_o^3, B_o^3 \dots$ $B_e^3, B_e^3, B_e^3, B_e^3 \dots$	0	-1
3	A'_0	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \dots$	0	1
0	A_1	$B_o^3, B_o^3, B_o^3, B_o^3 \dots, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \dots, B_e^3$	0	-1
1	A'_1	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \dots, B_o^{3'}$	1	1
2	A_1	$B_o^3, B_o^3, B_o^3, B_o^3 \dots, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \dots, B_e^3$	0	-1
3	A'_1	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \dots, B_o^{3'}$	1	1
0	A_2	$B_o^3, B_o^3, B_o^3, B_o^3 \dots, B_o^3, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \dots, B_e^3, B_e^3$	0	-1
1	A'_2	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \dots, B_o^{3'}, B_o^{3'}$	0	1
2	A_2	$B_o^3, B_o^3, B_o^3, B_o^3 \dots, B_o^3, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \dots, B_e^3, B_e^3$	0	-1
3	A'_2	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \dots, B_o^{3'}, B_o^3$	0	1
0	A_3	$B_o^3, B_o^3, B_o^3, B_o^3 \dots, B_o^3, B_o^3, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \dots, B_e^3, B_e^3, B_e^3$	0	-1
1	A'_3	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \dots, B_o^{3'}, B_o^{3'}, B_o^{3'}$	1	1
2	A_3	$B_o^3, B_o^3, B_o^3, B_o^3 \dots, B_o^2, B_o^3, B_o^3$ $B_e^3, B_e^3, B_e^3, B_e^3 \dots, B_e^3, B_e^3, B_e^3$	0	-1
3	A'_3	$B_o^{3'}, B_o^{3'}, B_o^{3'}, B_o^{3'} \dots, B_o^{3'}, B_o^{3'}, B_o^{3'}$	1	1

Table 27. Vertex and edge of $P_k \odot C^3_{n,4r+3}$.

As As a consequence of the previous Lemmasss one can establish the following theorem.

Theorem 3.1. The corona Product between paths and a third power of Cone graphs denoted by $P_k \odot C_{n,m}^3$ for all k, m, n are cordial.

4. Conclusion

This article is evidence for the presence of labeling for the corona Product between paths and a third power of Cone graphs. It was inspiring to investigate the cordiality of the corona Product between paths and a third power of Cone graphs. This labeling can be extended to various types of graphs and examined in the future.

References

[1]] A.Rosa, On certain valuations of the vertices of a graph, Theory of Graphs(Internat Symposium, Rome, July 1966),Gordon and Breach, N.Y.and Dunod Paris, (1967), pp. 349-355.

[2] B.D. Acharya, S.M.Hegde, Arithmetic Graphs, J. Graph Theory, 14, (3), (1990),pp. 275–299.

[3] I.Cahit, Cordial Graphs: A weaker version of graceful and harmonious Graphs, Ars Combinatoria, 23, (1987), pp. 201-207.

[4] I.Cahit, On cordial and 3-eqtitable labeling of graphs, Utiliuies Math, 37, (1990).

[5] I.Cahit, On cordial labeling of the second power of paths with other graphs , ARS Combinatoria, 97A, (2010), pp.327-343.

[6] Josephm.A.Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, DS6, (2017).

[7] S.Freeda, S.R.Chellathurai, H_1 and H_2 cordial labeling of some graphs, Open J. Discrete Math, 2, (2012), pp 149 – 155.

[8] S.I.Nada, A.T,Diab, A.Elrokh, and D.E.Sabra, The corona between paths and cycles , Ars Combin, 139 (2018)269-281.

[9] V.Yegnanarayanan and P.Vaidhyanathan, "Some interesting applications of graph labellings, Journal of Mathematical and Computational Science, 2, no. 5, (2012), pp. 1522–1531..

[10] ELrokh, A., Ali Al-Shamiri, M. M., Nada, S., & El-hay, A. A. (2022). Cordial and Total Cordial Labeling of Corona Product of Paths and Second Order of Lemniscate Graphs. Journal of Mathematics, 2022.

[11] Badr, E., Nada, S., Ali Al-Shamiri, M. M., Abdel-Hay, A., & ELrokh, A. (2022). A novel mathematical model for radio mean square labeling problem. Journal of Mathematics, 2022.

[12] ELrokh, A., Ali Al-Shamiri, M. M., & El-hay, A. (2022). A Novel Problem to Solve the Logically Labeling of Corona between Paths and Cycles. Journal of Mathematics, 2022.

[13] Badr, E., Abd El-hay, A., Ahmed, H., & Moussa, M. (2021). Polynomial, exponential and approximate algorithms for metric dimension problem. Mathematical Combinatorics, 2, 51-67.