Ramanujan's Three Famous Partition Congruences

Abhijit Konch
Dhemaji College, Dhemaji, Assam, India
Email:abhijitkonch100@gmail.com

Abstract

Let p(n) denote the number of partitions of n where n is a positive integer. In this paper, we study Ramanujan's congruences for the Partition function $p(n)$, especially $p(5m+4)$, $p(7m+5)$ and $p(11m+6) \equiv 0(mod\ 5,7\ and\ 11)$ respectively and explore the different types of partition congruences along with proof of some partition congruences.

Keywords: Ramanujan, Partitions, Congruences, Pentagonal Number Theorem, Eisenstein series.
2010 Mathematics Subject Classification: 11P84, 11P83

1 Introduction : In 1919, Ramanujan [1], [2, pp.210-213] announced that he had found three simple congruences satisfied by $p(n)$ namely,

$$p(5n+4) \equiv 0(mod\ 5) \qquad 1.1$$
$$p(7n+5) \equiv 0(mod\ 7) \qquad 1.2$$
$$p(11n+6) \equiv 0(mod\ 11) \qquad 1.3$$

In their most general form they can be stated as follows

$$p(5^n m + l_n) \equiv 0(mod\ 5^n) \qquad 1.4$$
$$p(7^n m + k_n) \equiv 0\left(mod\ 7^{\left\lfloor \frac{n+2}{2} \right\rfloor}\right) \qquad 1.5$$
$$p(11^n m + t_n) \equiv 0(mod\ 11^n) \qquad 1.6$$

where for $p = 5{,}7$ and 11, the numbers l_n, k_n and t_n are the least positive solutions of $24\,x \equiv 1\ (mod\ p^n)$ respectively.

He provided the proofs for the first two (1.1) and (1.2) of the aforementioned congruences in [1] and [2]. The congruence (1.3) was stated for the first time by Ramanujan in [2] and [3].After Ramanujan died in 1920, G.H. Hardy [3], [2] extracted proofs of (1.1)—(1.3) from an unpublished manuscript of Ramanujan on $p(n)$ and $\tau(n)$. This manuscript was published for the first time in handwritten form in a volume [5] containing Ramanujan's Lost notebook. An expanded and annotated version was prepared by Bruce C. Berndt and K. Ono [6]. The proofs in [4] employ Eisenstein series. The simplest of the proofs provided for the third identity above is credited to L. Winquist [7] and makes use of Winquist's Identity. A new identity for $(q;q)_\infty^{10}$ has been established by Berndt, S. H. Chan, Z.-G. Liu, and H. Yesilyurt [8], who also developed a simple method of establishing the third identity. . Hirschhorn [9] has developed a method for demonstrating all three of them.

Ramanujan also sketched proofs of (1.4) and (1.5) with $n = 2$. In 1938, G. N. Watson [10] proved (1.4) and (1.5). Ramanujan's original formulation of (1.5) was in fact incorrect. The congruence (1.6) has remained unproven until A.O.L. Atkin [11] gave a proof in 1967. The works of Newman [12], and of Atkin and J. N. O'Brien [13], and of Atkin and H. P. F.

Swinnerton-Dyer [14] have shown that there are many other congruences for the partition function. For example, Atkin and O'Brien [13] found that

$$p(594.13n + 111247) \equiv 0 \ (mod\ 13)$$

In 2000, K. Ono [15] proved that if $m \geq 5$ is a prime, then there are infinitely many integers a and b such that $p(an + b) \equiv 0 \ (mod\ m)$ for all n. Ramanujan stated that other than those he found there seemed to be no other congruence in the form $p(an + b) \equiv 0 \ (mod\ n)$ with n prime. His guess was proven to be correct by M. Boylan and Ahlgren [16] in 2003.
In 1944, F. J. Dyson [17] gave the first combinatorial interpretation of Ramanujan's partition congruences for the modulus 5 and 7. He also conjectured another partition statistic which he called "crank" that would divide partitions of $11n + 6$ into 11 equinumerous classes. His conjectures about the rank was proven by Atkin and Swinnerton-Dyer in 1958 [18]. The existence of a "crank" was first proved by F. G. Garvan [19] in terms of vector partitions. Later in the same year G. E. Andrews and Garvan [20] discovered another "crank" in terms of regular partitions. The methods of Garvan and of Atkin and Swinnerton-Dyer were purely analytical but in 2003 Garvan, D. Kim and D. Stanton [21] found yet another crank along with explicit bijections between equinumerous classes. Several identities stated in Ramanujan's "Lost Notebook" were very influential in Garvan's discovery of crank. These identities and their relation to the works of Atkin and Swinnerton-Dyer on Dyson's rank together with further contributions of Ramanujan to partition congruences with numerous references can be found in [22].

Our objective in this paper is to explore the different types of partition congruences along with proof of some partition congruences introduced by S. Ramanujan. To do so, we have to mention some preliminaries whose proof can be found in [23].

2 Preliminaries

2.1 Partition of an integer: The partition function $p(n)$ is defined as follows. For a positive integer n, $p(n)$ is the number of partitions of n into positive integral parts. Here, in a partition, the parts are not necessarily distinct and the order in which the parts are arranged is irrelevant.

2.2 Some basic Definitions:

$p(n)$: It is the number of partition of a positive integer n in which each summand is positive.
$p^d(n)$: It denotes the number of distinct partitions of n.
$p^o(n)$: It denotes the number of odd partitions of n.
$\sigma_k(n)$: It denotes the sum of the k-th powers of the positive divisors of n.

2.3 Euler's identity: The number of distinct partitions of a positive integer n is equal to the number of odd partitions of that integer.

2.4 Generating function for $p(n)$: Define,

$$(a)_n \coloneqq (a;q)_n \coloneqq \prod_{k=0}^{n-1}(1 - aq^k), \qquad n \geq 1,$$

$$(a)_\infty \coloneqq (a;q)_\infty \coloneqq \prod_{k=0}^{\infty}(1 - aq^k), \qquad |q| < 1,$$

The generating function of $p(n)$ is given by,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q,q)_\infty}, \qquad |q| < 1$$

2.5 Ramanujan's general theta function: Ramanujan's general theta function $f(a,b)$ is defined by,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \qquad |ab| < 1$$

Three special cases are defined by in Ramanujan's notation,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

2.6 Jacobi's identity:

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n(n+1)}{2}} = (q;q)_\infty^3, \qquad |q| < 1$$

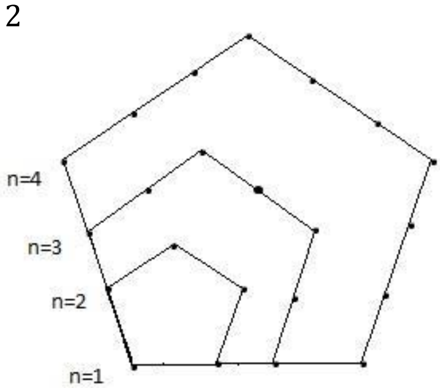
2.7 Euler's pentagonal number theorem:

$$\sum_{k=0}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}} = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k+1)}{2}} = (q;q)_\infty, \qquad |q| < 1$$

2.8 Pentagonal numbers: The formula for the n-th pentagonal number P_n is given by,

$$P_n = \frac{n(3n-1)}{2}$$

On counting the number of dots in and inside the pentagons we get the pentagonal numbers for positive value of n.



Values of $p(n)$ upto $n = 80$

1 1	17 297	33 10143	49 173525	65 2012558
2 2	18 385	34 12310	50 204226	66 2323520
3 3	19 490	35 14883	51 239943	67 2679689

4 5	20 627	36 17977	52 281589	68 3087735
5 7	21 792	37 21637	53 329931	69 3554345
6 11	22 1002	38 26015	54 386155	70 4087968
7 15	23 1255	39 31185	55 451276	71 4697205
8 22	24 1575	40 37338	56 526823	72 5392783
9 30	25 1958	41 44583	57 614154	73 6185689
10 42	26 2436	42 53174	58 715220	74 7089500
11 56	27 3010	43 63261	59 831820	75 8118264
12 77	28 3718	44 75175	60 966467	76 9289091
13 101	29 4565	45 89134	61 1121505	77 10619863
14 135	30 5604	46 105558	62 1300156	78 12132164
15 176	31 6842	47 124754	63 1505499	79 13848650
16 231	32 8349	48 147273	64 1741630	80 15796476

A similar table was constructed by MacMohan, who found and listed the values for $p(n)$ upto $p(200)$. From MacMahon's table for $p(n)$ for $1 \leq n \leq 200$, Ramanujan conjectured his three famous congruences (1.1)—(1.3) for the partition function $p(n)$. In the later part of his stay in England, Ramanujan wrote his famous papers on congruences for $p(n)$.
We begin with some elementary definitions and results that will be required in proving the congruences.

2.9 Classical Eisenstein Series: For positive integer n,

$$E_{2n}(q) := 1 - \frac{4n}{B_{2n}} \sum_{k=1}^{\infty} \frac{k^{2n-1}}{1-q^k} = -\frac{4n}{B_{2n}} S_{2n-1}, \qquad n \geq 1 \tag{1.1.1}$$

Let

$$P := 1 - 24\left(\frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \cdots\right) \tag{1.1.2}$$

$$Q := 1 + 240\left(\frac{q}{1-q} + \frac{2^3q^2}{1-q^2} + \frac{3^3q^3}{1-q^3} + \cdots\right) \tag{1.1.3}$$

$$R := 1 - 504\left(\frac{q}{1-q} + \frac{2^5 q^2}{1-q^2} + \frac{3^5 q^3}{1-q^3} + \cdots\right) \tag{1.1.4}$$

Let $f(q) = (1-q)(1-q^2)(1-q^3)\ldots$ (1.1.5)

Then it is well known that

$$f(q) = 1 - q - q^2 + q^5 + q^7 - \cdots = 1 + \sum_{n=1}^{\infty}(-1)^n\left(q^{\frac{1}{2}n(3n-1)} + q^{\frac{1}{2}n(3n+1)}\right) \tag{1.1.6}$$

$$\Phi_{r,s}(q) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s q^{mn} = \sum_{n=1}^{\infty} n^r\,\sigma_{s-r}(n) q^n \tag{1.1.7}$$

Then

$$\Phi_{0,s}(q) = \frac{q}{1-q} + \frac{2^s q^2}{1-q^2} + \frac{3^s q^3}{1-q^3} + \cdots \tag{1.1.8}$$

And in particular

$$P = 1 - 24\Phi_{0,1}(q), \qquad Q = 1 + 240\Phi_{0,3}(q), \qquad R = 1 - 504\Phi_{0,5}(q)$$

Some other basic results are:

$$Q^2 = 1 + 480\Phi_{0,7}(q) = 1 + 480\left(\frac{q}{1-q} + \frac{2^7 q^2}{1-q^2} + \cdots\right) \tag{1.1.9}$$

$$QR = 1 - 264\Phi_{0,9}(q) = 1 - 264\left(\frac{q}{1-q} + \frac{2^9 q^2}{1-q^2} + \cdots\right) \tag{1.1.10}$$

$$\begin{aligned} 441Q^3 + 250R^2 &= 691 + 65520\Phi_{0,11}(q) \\ &= 691 + 65520\left(\frac{q}{1-q} + \frac{2^{11} q^2}{1-q^2} + \cdots\right) \end{aligned} \tag{1.1.11}$$

$$Q^3 - R^2 = 1728q\big(f(q)\big)^{24} \tag{1.1.12}$$

$$Q - P^2 = 288\Phi_{1,2}(q) \tag{1.1.13}$$

$$PQ - R = 720\Phi_{1,4}(q) \tag{1.1.14}$$

$$Q^2 - PR = 1008\Phi_{1,6}(q) \tag{1.1.15}$$

$$Q(PQ - R) = 720\Phi_{1,8}(q) \tag{1.1.16}$$

$$3PQ - 2R - P^3 = 1728\Phi_{2,3}(q) \tag{1.1.17}$$

$$P^2Q - 2PR + Q^2 = 1728\Phi_{2,5}(q) \tag{1.1.18}$$

$$2PQ^2 - P^2R - QR = 1728\Phi_{2,7}(q) \tag{1.1.19}$$

$$6P^2Q - 8PR + 3Q^2 - P^4 = 6912\Phi_{3,4}(q) \tag{1.1.20}$$

$$P^3Q - 3P^2R + 3PQ^2 - QR = 3456\Phi_{3,6}(q) \tag{1.1.21}$$

$$15PQ^2 - 20P^2R + 10P^3R - 4QR - P^5 = 20736\Phi_{4,5}(q) \tag{1.1.22}$$

Theorem 3.1: For each non negative integer n,

$$p(5n+4) \equiv 0 (mod\ 5) \tag{1.2.1}$$

Proof: We begin by writing,

$$q(q;q)_\infty^4 \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} = q\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m) q^{m+1} \tag{1.2.2}$$

By the binomial theorem, $(q;q)_\infty^5 \equiv (q^5;q^5)_\infty (mod\ 5)$

$$or, \qquad \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} \equiv 1(mod\ 5) \tag{1.2.3}$$

Hence by equation (1.2.2) and (1.2.3) we have

$$q(q;q)_\infty^4 \equiv (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^{m+1}\ (mod\ 5) \tag{1.2.4}$$

We now see from (1.2.4) that in order to show that $p(5n+4)$ is divisible by 5 we must show that the coefficients of q^{5n+5} on the left side of (1.2.4) are multiples of 5.

By the pentagonal number theorem and Jacobi's identity we have,

$$q(q;q)_\infty^4 = q(q;q)_\infty(q;q)_\infty^3$$

$$\Rightarrow \qquad q(q;q)_\infty^4 = q\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{j(3j+1)}{2}} \sum_{k=0}^{\infty}(-1)^k\,(2k+1)q^{\frac{k(k+1)}{2}}$$

$$\Rightarrow \qquad q(q;q)_\infty^4 = \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\,(2k+1)q^{1+\frac{j(3j+1)}{2}+\frac{k(k+1)}{2}} \tag{1.2.5}$$

Our objective is to determine when the exponents on the right side of (1.2.5) are multiples of 5. We have

$$2(j+1)^2+(2k+1)^2 = 8\left\{1+\tfrac{1}{2}j(3j+1)+\tfrac{1}{2}(k+1)k\right\}-10j^2-5.$$

Thus, $\quad 1+\frac{1}{2}j(3j+1)+\frac{1}{2}k(k+1)$ is a multiple of 5 if and only if,

$$2(j+1)^2+(2k+1)^2 \equiv 0(mod\ 5).$$

We have, $2(j+1)^2 \equiv 0, 2,\ or\ 3\ modulo\ 5$ and, $(2k+1)^2 \equiv 0, 1\ or\ 4\ modulo\ 5$, and none of the combinations lead to $0\ modulo\ 5$, except when both are congruent to $0\ modulo\ 5$.

In particular, $2k+1 \equiv 0(mod\ 5)$, which by (1.2.5) implies that the coefficient of q^{5n+5}, $n \geq 0$ in $q(q;q)_\infty^4$ is a multiple of 5. The co-efficient of q^{5n+5} on the right side of (1.2.4) is also a multiple of 5, i.e., $p(5n+4)$ is a multiple of 5.

Theorem 3.2: For each non-negative integer n,

$$p(7n+5) \equiv 0(mod\ 7) \tag{1.3.1}$$

Proof: First by the binomial theorem $(q;q)_\infty^7 \equiv (q^7;q^7)_\infty(mod\ 7)$,

So, $\qquad q^2(q^7;q^7)_\infty \sum_{n=0}^{\infty} p(n)q^n = q^2\frac{(q^7;q^7)_\infty}{(q;q)_\infty} = q^2(q;q)_\infty^6\frac{(q^7;q^7)_\infty}{(q;q)_\infty^7}$

$$\equiv q^2(q;q^6)_\infty(mod\ 7) \tag{1.3.2}$$

We now see from (1.3.2) that in order to show that $p(7n+5)$ is divisible by 7, we must show that the coefficients of q^{7n+7} on the left side of (1.3.2) are multiples of 7.

Applying Jacobi's identity, we find that

$q^2(q;q)_\infty^6 = q^2\{(q;q)_\infty^3\}^2$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}(2j+1)(2k+1)\,q^{2+\frac{1}{2}j(j+1)+\frac{1}{2}k(k+1)} \tag{1.3.3}$$

We have $\qquad (2j+1)^2+(2k+1)^2 = 8\left\{2+\frac{1}{2}j(j+1)+\frac{1}{2}k(k+1)\right\}-14$

$$\Rightarrow \qquad (2j+1)^2+(2k+1)^2 \equiv \left\{2+\frac{1}{2}j(j+1)+\frac{1}{2}k(k+1)\right\}(mod\ 7)$$

So, $2+\frac{1}{2}j(j+1)+\frac{1}{2}k(k+1)$ is a multiple of 7 if and only if

$$(2j+1)^2+(2k+1)^2 \equiv 0(mod\ 7) \tag{1.3.4}$$

We easily see that $(2j+1)^2 \equiv 0,\ 1,\ 2, or\ \ 4(mod\ 7)$ and $(2k+1)^2 \equiv 0,\ 1,\ 2, or\ \ 4(mod\ 7)$, and so the only way (1.3.4) can hold is if both $(2j+1)^2,\ \ (2k+1)^2 \equiv 0(mod\ 7)$.

So the coefficients on the right side of (1.3.3) are multiples of 7. Hence, the coefficient of $q^{7n+7},\ n \geq 1$, on the left side of (1.3.3) is a multiple of 7.

$$\text{So,}\quad p(7n+5) \equiv 0(mod\ 7).$$

Theorem 3.3: For each non-negative integer n,

$$p(11n+6) \equiv 0(mod\ 11) \tag{1.4.1}$$

Proof: Let us denote by J an integral power series in x whose coefficients are integers.
It is obvious from (1.1.10) that

$$QR = 1+11J \tag{1.4.2}$$

Also $n^{11}-n \equiv 0(mod\ 11)$, and so from (1.1.2) and (1.1.11),

$$Q^3-3R^2 = 441Q^3+250R^2+11J = 691+65520\left(\frac{x}{1-x}+\frac{2^{11}x^2}{1-x^2}+\cdots\right)+11J$$

$$= -2+48\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\cdots\right)+11J = -2P+11J \tag{1.4.3}$$

It can be deduced that

$$(Q^3-R^2)^5 = (Q^3-3R^2)^5 - Q(Q^3-3R^2)^3 - R(Q^3-3R^2)^2 + 6QR + 11J$$

$$= P^5-3P^3Q-4P^2R+6QR+11j \tag{1.4.4}$$

On multiplying (1.1.16), (1.1.19), (1.1.21) , and (1.1.22) by $-1,\ 3, -4$, and -1, and adding, we obtain, on rejecting multiples of 11,

$$P^5-3P^3Q-4P^2R+6QR = -5\Phi_{1,8}+3\Phi_{2,7}+3\Phi_{3,6}-\Phi_{4,5}+11j$$

and from this and (1.4.4) follows

$$(Q^3-R^2)^5 = -\sum_{n=1}^{\infty}\left(5n\sigma_7(n)-3n^2\sigma_3(n)-3n^3\sigma_5(n)+n^4\sigma_1(n)\right)x^n+11J \tag{1.4.5}$$

Again we have

$$\left(f(x)\right)^{120} = \frac{f(x^{121})}{f(x)}+11J \tag{1.4.6}$$

From (1.4.5) and (1.4.6)

$$x^5\frac{f(x^{121})}{f(x)} = x^5\left(f(x)\right)^{120}+11J = 1728^5x^5\left(f(x)\right)^{120}+11J = (Q^3-R^2)^5+11J$$

$$= -\sum_{n=1}^{\infty}\left(5n\sigma_7(n)-3n^2\sigma_5(n)-3n^3\sigma_3(n)+n^4\sigma_1(n)\right)x^n+11J$$

So

$$p(n-5)-p(n-126)-p(n-247)+p(n-610)+p(n-852)-p(n-1457)-\cdots$$
$$\equiv -n^4\sigma_1(n)+3n^3\sigma_3(n)+3n^2\sigma_5(n)-5n\sigma_7(n)(mod\ 11)$$

$5, 126, 247, \ldots$ being the numbers of the forms

$$\frac{1}{2}(11n-2)(33n-5),\qquad \frac{1}{2}(11n+2)(33n+5);$$

And in particular that $p(11n-5) \equiv 0(mod\ 11)$,

So, $p(11n+6) \equiv 0(mod\ 11)$.

Theorem 3.4: For each non- negative integer n, $p(25n+24) \equiv 0(mod\ 25)$.

Proof: We have,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty} = 5(q^5;q^5)_\infty^4 \sum_{n=0}^{\infty} p(n)q^n (mod\ 25). \quad (1.5.1)$$

From theorem 3.1, we know that the coefficients of $q^4, q^9, q^{14}, \dots q^{5n+4}, \dots$ on the far right side of (1.5.1) are all multiple of 25. It follows that the coefficients of $q^{5n+4}, n \geq 0$, on the far left side of (1.5.1) are also multiples of 25, i.e, $p(25n+24) = 0(mod\ 25)$.

This completes the proof.

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