On signed product cordial of lemniscate graph and its second power

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Abstract: A graph $G = (V, E)$ is called signed product cordial graph if it is possible to label the vertex by the function $f: V \to \{-1,1\}$ and label the edges by $f^*: E \to \{-1,1\}$, where $f^*(uv) = f(u).f(v)$, $u, v \in V$ so that $|v_{-1} - v_1| \le 1$ and $|e_{-1} - e_1| \le 1$. In this paper we present necessary and sufficient conditions for which lemniscate and its second power are signed product cordial graph.
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1. Introduction

Labeling graphs are used widely in different subjects including astronomy and communication networks. The concept of graph labeling was introduced during the sixties of the last century by Rosa [17]. Many researchers have been working with different types of graph labeling [3-16]. Cordial labelings were investigated by Cahit [3] who called a graph $G(v, E)$ is cordial if there is a vertex labeling $f: V \to 0,1$ such that the induced labeling $f^*: E \to 0,1$, defined by $f^*(uv) = |f(u) - f(v)|$, for all edges $uv \in E(G)$ and with the following inequalities hold: $|v_0 - v_1| \le 1$ and $|e_0 - e_0| \le 1$, where v_i (respectively e_i) is the number of vertices (respectively, edges) labeled with i. He showed that each tree is cordial; an Euerlian graph is not cordial if its size is congruent to $2(mod \quad 4)$. Diab [7] [18] reported several results concerning the sum and union of the cycles C_n and paths P_m together and with other specific graphs. An excellent reference for this purpose is the survey written by Gallian [8]. The name "signed graph" and the notion of balance appeared first in a mathematical paper of Frank Harary in 1953 [11]. The concept of signed product cordial labeling was introduced by Baskar Babujee [12]. A graph is called *signed product cordial graphs* if it has a signed product cordial graph labeling. A graph $G = (V, E)$ is called signed product cordial graph if it is possible to label the vertex by the function $f: V \to \{-1,1\}$ and label the edges by $f^*: E \to \{-1,1\}$, where $f^*(uv) = f(u).f(v)$, $u, v \in V$ so that $|v_{-1} - v_1| \le 1$ and $|e_{-1} - e_1| \le 1$. Kirchherr [13], discussed the cordiality of a cactus which is a connected graph all whose blocks are cycles and a similar result is given by Lee and Liu [14]. All graphs considered in this paper are finite, simple and undirected. In this paper, we define the lemniscate graph denoted by $L_{n,m} := C_n * C_m$, as the graph which is obtained from two cycles C_n and C_m having a vertex in common. It easy to see that order of $L_{n,m}$ is $n + m - 1$ and its size is $n + m$, obviously, $L_{n,m}$ is an Euerlian graph and $L_{n,m}$ is isomorphic to $L_{m,n}$ [1]. In section 3, we prove that any lemniscate graph is a signed product cordial graph if and only if its size is not congruent to $2(mod \ 4)$. Finally, we show that the second power of the lemniscate graph, $L^2_{n,m}$, is cordial for all $n \ge 3$ and $m \ge 3$.

2. Terminologies and Notations

We let L_{4r} denote the labeling $(-1)_2(1)_2 \ldots (-1)_2(1)_2$ (repeated r-times), let L'_{4r} denote the labeling $(1)_2(-1)_2 \ldots (1)_2(-1)_2$ (repeated r times). We denote the labeling $1(-1)_2 1 \ 1(-1)_2 1 \ldots 1(-1)_2 1$ (repeated r times) and $(-1)(1)_2(-1) \ldots (-1)(1)_2(-1)$ (repeated r times) by S_{4r} and S'_{4r}, respectively. Sometimes, we modify this by adding symbols at one end or the other (or both), thus $L_{4r}1(-1)1$ denotes the labeling $(-1)_2(1)_2 \ldots (-1)_2(1)_2 \quad 1(-1)1$ when $r \ge 1$ and $1(-1)1$ when $r = 0$. Similarly, $1L'_{4r}$ is the labeling $1 \ (1)_2(-1)_2 \ldots (1)_2(-1)_2$ when $r \ge 1$ and 1 when $r = 0$. The labeling $(-1)(1) \ldots (-1)(1)$ (repeated r times) denoted by M_r if r is even and $(-1)(1) \ldots (-1)(1)(-1)$ if r is odd. Likewise $(1)(-1) \ldots (1)(-1)$, is denoted by M'_r if r is even and $(1)(-1) \ldots (1)(-1)(1)$, if r is odd. For a given labeling of the graph $G \sharp H$, we let v_i and e_i (for $i = -1,1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H. It follows that $v_{-1} = x_{-1} + y_{-1}$; $v_1 = x_1 + y_1$; $e_{-1} = a_{-1} + b_{-1}$ and $e_1 = a_1 + b_1$, thus, $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1)$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1)$. The labeling of lemniscate $L_{n,m}$ and its second power are denoted by $[A;B]$; where the labeling A is given to C_n or C_n^2 and the labeling B is given to C_m or C_m^2.

3. The signed product cordial of lemniscate graphs

In this section, we show that the lemniscate $L_{n,m}$ is signed product cordial graph for all $n,m \geq 3$. Through out this section the labeling of the common vertex is considered as a part of the first cycle C_n. This target will be achieved after the following series of lemmas.

Lemma 3.1. The lemniscate graph $L_{3,m}$ is signed product cordial graph if and only if $m \not\equiv 3(mod4)$.

Proof. We prove the easy direction first. Let $m \equiv 3(mod4)$, then it is obvious that $L_{3,4t+3}$, $t \geq 0$, is an Eulerian graph with size congruent to $2(mod4)$; and consequently $L_{3,4t+3}$ is not signed product cordial graph. Now, let $m \not\equiv 3(mod4)$. Then we consider three cases:

<u>**Case (1)**</u> Suppose that $m \equiv 0(mod4)$, i.e. $m = 4t, t \geq 1$, then, we choose the labeling $[1(-1)(-1);(-1)11L_{4t-4}]$ for $L_{3,4t}$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = 2, a_1 = 1, y_{-1} = 2t-1, y_1 = 2t, b_{-1} = 2t$ and $b_1 = 2t$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. Thus $L_{3,4t}$ is signed product cordial graph for all $t \geq 1$.

<u>**Case (2)**</u> Suppose that $m \equiv 1(mod4)$, i.e. $m = 4t+1, t \geq 1$, then, we choose the labeling $[11(-1);L'_{4t}]$ for $L_{3,4t+1}$. Therefore $x_{-1} = 1, x_1 = 2, a_{-1} = 2, a_1 = 1, y_{-1} = 2t, y_1 = 2t, b_{-1} = 2t$ and $b_1 = 2t+1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L_{3,4t+1}$ is signed product cordial graph for all $t \geq 1$.

<u>**Case (3)**</u> Suppose that $m \equiv 2(mod4)$, i.e. $m = 4t+2, t \geq 1$, then, one can choose the labeling $[(-1)1(-1);L_{4t}1]$ for $L_{3,4t+2}$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = 2, a_1 = 1, y_{-1} = 2t, y_1 = 2t+1, b_{-1} = 2t$ and $b_1 = 2t+2$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = -1$. Thus $L_{3,4t+2}$ is signed product cordial graph for all $t \geq 1$. Thus the lemma is proved.□

Lemma 3.2. The lemniscate graph $L_{n,m}$, where $n \equiv 0(mod4)$ is signed product cordial graph if and only if $m \not\equiv 2(mod4)$.

Proof. We prove the easy direction first. Let $n = 4r, r \geq 1$, and let $m = 4t+j, t \geq 1$, and $j = 2$. $L_{4r,4t+2}$ is not signed product cordial graph since it is an Eulerian graph with size congruent to $2(mod4)$ [4]. Now, we consider three cases:

<u>**Case (1)**</u> At $j = 0$ i.e. $m = 4t, t \geq 1$. One can select the labeling $[L_{4r};1(-1)(-1)L'_{4t-4}]$ for $L_{4r,4t)}$. Therefore $x_{-1} = x_1 = a_{-1} = a_1 = 2r, y_{-1} = 2t, y_1 = 2t-1, b_{-1} = 2t$ and $b_1 = 2t$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L_{4r,4t}$ is signed product cordial graph for all $r,t \geq 1$.

<u>**Case (2)**</u> At $j = 1$ i.e. $m = 4t+1, t \geq 1$. One can select the labeling $[L_{4r};L'_{4t}]$ for $L_{4r,4t+1}$. Therefore $x_{-1} = x_1 = a_{-1} = a_1 = 2r, y_{-1} = 2t, y_1 = 2t, b_{-1} = 2t$ and $b_1 = 2t+1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = -1$. Thus $L_{4r,4t+1}$ is signed product cordial graph for all $r,t \geq 1$.

<u>**Case (3)**</u>. At $j = 3$ i.e. $m = 4t+3, t \geq 1$. One can select the labeling $[L_{4r};(-1)L'_{4t}1]$ for $L_{4r,4t+3}$. Therefore $x_{-1} = x_1 = a_{-1} = a_1 = 2r, y_{-1} = 2t+1, y_1 = 2t+1, b_{-1} = 2t+2$ and $b_1 = 2t+1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. Thus $L_{4r,4t+3}$ is signed product cordial graph for all $r,t \geq 1$. Finally at $t = 0$; $L_{4r,3}$ is isomorphic to $L_{3,4r}$. Using Lemma (3.1), we conclude that $L_{4r,3}$ is signed product cordial graph. Thus the lemma is proved.□

Lemma 3.3. The lemniscate graph $L_{n,m}$, where $n \equiv 1(mod4)$ is signed product cordial graph if and only if $m \not\equiv 1(mod4)$.

Proof. Let $n = 4r+1$, $r \geq 1$ and $m = 4t+j$, $0 \leq j \leq 3$. We prove the easy direction first. $L_{4r+1,4t+1}$ is not signed product cordial graph since this is an Eulerian graph with size congruent to $2(mod4)$ [4]. Now, we consider three cases:

<u>**Case (1)**</u> At $m = 4t, t \geq 1$. We select the labeling $[L_{4r}1;1(-1)(-1)L'_{4t-4}]$ for $L_{4r+1,4t}$. Therefore $x_{-1} = 2r, x_1 = 2r+1, a_{-1} = 2r, a_1 = 2r+1, y_{-1} = 2t, y_1 = 2t-1, b_{-1} = 2t$ and $b_1 = 2t$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. Thus $L_{4r+1,4t}$ is signed product cordial graph for all $r,t \geq 1$.

<u>**Case (2)**</u> At $j = 2$ i.e. $m = 4t+2, t \geq 1$. One can select the labeling $[L_{4r}1;S'_{4t}(-1)]$ for $L_{4r+1,4t+2}$. Therefore $x_{-1} = 2r, x_1 = 2r+1, a_{-1} = 2r, a_1 = 2r+1, y_{-1} = 2t+1, y_1 = 2t, b_{-1} = 2t+2$ and $b_1 = 2t$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. Thus $L_{4r+1,4t+2}$ is signed product cordial graph for all $r,t \geq 1$.

<u>**Case (3)**</u>. At $j = 3$ i.e. $m = 4t+3, t \geq 1$. We choose the labeling $[L_{4r}1;(-1)L'_{4t}1]$ for $L_{4r+1,4t+3}$. Therefore $x_{-1} = 2r, x_1 = 2r+1, a_{-1} = 2r, a_1 = 2r+1, y_{-1} = 2t+1, y_1 = 2t+1, b_{-1} = 2t+2$ and $b_1 = 2t+1$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L_{4r+1,4t+3}$ is signed product cordial graph for all $r,t \geq 1$. Finally; since $L_{4r+1,3}$ is isomorphic to $L_{3,4r+1}$, $L_{4r+1,3}$ is

signed product cordial graph. Thus the lemma follows.

Lemma 3.4. The lemniscate graph $L_{n,m}$, where $n \equiv 2(mod4)$ is signed product cordial graph if and only if $m \not\equiv 0(mod4)$.

Proof. Let $n = 4r + 2$, $r \geq 1$ and $m = 4t + j$, $0 \leq j \leq 3$. We prove the easy direction first. Let $L_{4r+2,4t}$ i.e. $j = 0$. This graph is not signed product cordial graph since it is an Eulerian graph and its size congruent to $2(mod4)$ [4]. Now, we consider three cases:

Case (1) At $j = 1$, i.e. $m = 4t + 1, t \geq 1$. We choose the labeling $[(-1)S_{4r}1; L'_{4t}]$ for $L_{4r+2,4t+1}$. Therefore $x_{-1} = 2r + 1, x_1 = 2r + 1, a_{-1} = 2r + 2, a_1 = 2r, y_{-1} = 2t, y_1 = 2t, b_{-1} = 2t$ and $b_1 = 2t + 1$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. Here, we conclude that the lemniscate graph $L_{4r+2,4t+1}$ is isomorphic to $L_{4t+1,4r+2}$ which is signed product cordial graph by lemma 3.3. Thus $L_{4r+1,4t+2}$ is signed product cordial graph for all $r, t \geq 1$.

Case (2) At $j = 2$, i.e. $m = 4t + 2, t \geq 1$. We choose the labeling $[L_{4r}1(-1); S_{4t}(-1)]$ for $L_{4r+2,4t+2}$. Therefore $x_{-1} = 2r + 1, x_1 = 2r + 1, a_{-1} = 2r, a_1 = 2r + 2, y_{-1} = 2t + 1, y_1 = 2t, b_{-1} = 2t + 2$ and $b_1 = 2t$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L_{4r+1,4t+2}$ is signed product cordial graph for all $r, t \geq 1$.

Case (3) At $j = 3$ i.e. $m = 4t + 3, t \geq 1$. We choose the labeling $[L_{4r}1(-1); (-1)L'_{4t}1]$ for $L_{4r+2,4t+3}$. Therefore $x_{-1} = 2r + 1, x_1 = 2r + 1, a_{-1} = 2r, a_1 = 2r + 2, y_{-1} = 2t + 1, y_1 = 2t + 1, b_{-1} = 2t + 2$ and $b_1 = 2t + 1$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = -1$. Since $L_{4r+2,3}$ is isomorphic to $L_{3,4r+2}$, $L_{4r+2,3}$ is signed product cordial graph. Thus $L_{4r+2,4t+3}$ is signed product cordial graph for all $r, t \geq 1$. Thus the lemma is proved.

Lemma 3.5. The lemniscate graph $L_{n,m}$, where $n = 3(mod4)$ is signed product cordial graph if and only if $m \not\equiv 3(mod4)$.

Proof. Let $n = 4r + 3$ and $m = 4t + j$, $0 \leq j \leq 3$. We prove the easy direction first $L_{4r+3,4t+3}$ is not signed product cordial graph since it is an Eulerian graph and its size congruent to $2(mod4)$ [4]. Now, we consider three cases:

Case (1) At $j = 0$ i.e. $m = 4t, t \geq 1$. We choose the labeling $[1L_{4r}(-1)1; 1(-1)(-1)L'_{4t-4}]$ for $L_{4r+3,4t}$. Therefore $x_{-1} = 2r + 1, x_1 = 2r + 2, a_{-1} = 2r + 2, a_1 = 2r + 1, y_{-1} = 2t, y_1 = 2t - 1, b_{-1} = 2t$ and $b_1 = 2t$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = -1$. The lemniscate graph $L_{4r+3,4t}$ is isomorphic to $L_{4t,4r+3}$ which is signed-cordial by lemma 3.2. Thus $L_{4r+3,4t}$ is signed product cordial graph for all $r, t \geq 1$.

Case (2) At $j = 1$ i.e. $m = 4t + 1, t \geq 1$. We choose the labeling $[1L_{4r}(-1)(-1); L'_{4t}]$ for $L_{4r+3,4t+1}$. Therefore $x_{-1} = 2r + 2, x_1 = 2r + 1, a_{-1} = 2r + 2, a_1 = 2r + 1, y_{-1} = 2t, y_1 = 2t, b_{-1} = 2t$ and $b_1 = 2t + 1$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. The lemniscate graph $L_{4r+3,4t+1}$ is isomorphic to $L_{4t+1,4r+3}$ which is signed product cordial graph by lemma 3.3. Thus $L_{4r+3,4t+1}$ is signed product cordial graph for all $r, t \geq 1$.

Case (3) At $j = 2$ i.e. $m = 4t + 2, t \geq 1$. We choose the labeling $[1L_{4r}(-1)(-1); 1L'_{4t}]$ for $L_{4r+3,4t+2}$. Therefore $x_{-1} = 2r + 2, x_1 = 2r + 1, a_{-1} = 2r + 2, a_1 = 2r + 1, y_{-1} = 2t, y_1 = 2t + 1, b_{-1} = 2t$ and $b_1 = 2t + 2$. Hence $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = -1$. The lemniscate graph $L_{4r+3,4t+2}$ is isomorphic to $L_{4t+2,4r+3}$ which is signed product cordial graph by lemma 3.4. Thus $L_{4r+3,4t+2}$ is signed product cordial graph for all $r, t \geq 1$. Finally; since $L_{4r+3,3}$ is isomorphic to $L_{3,4r+3}$. By lemma(3.1), $L_{4r+1,3}$ is not signed product cordial graph and thus the lemma follows.

As a consequence of the previous Lemmas (Lemma 3.1,…,3.5) one can establish the following theorem.

Theorem 3.1. The lemniscate graph $L_{n,m}$ is signed product cordial graph for all n and all m if and only if $L_{n,m}$ is not an Eulerian graph with size congruent to $2(mod4)$.

4. Signed product cordial of the second power of lemniscate graphs

In this section, we study and investigate the signed product cordial graph of the second power of lemniscate $L^2_{n,m}$ for all $n, m \geq 3$. Through out this section the labeling of the first cycle is starting from the vertex that follows the common vertex. This target will be achieved after the following series of lemmas.

Lemma 4.1. If $n = 3$, then $L^2_{3,m}$ is signed product cordial graph for all m except at.$m = 3$.

Proof. Suppose that $n = 3$. The following cases for m will be examined.

Case (1). $m \equiv 0(mod4)$.

Suppose that $m = 4t, t > 1$. One can label the vertices of $L^2_{3,4t}$ by $[01; L_4M'_{4t-4}]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 2, a_1 = 1, y_{-1} = y_1 = 2t$ and $b_{-1} = b_1 = 4t - 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. For the case $L^2_{3,4}$, the labeling $[1_2; (-1)_3 1]$ is sufficient and thus $L^2_{3,4t}$ is signed product cordial graph for all $t \geq 1$.

Case (2). $m \equiv 1(mod4)$.

Suppose that $m = 4t + 1, t > 1$. Then, one can label the vertices of $L^2_{3,4t+1}$ by $[(-1)1; (-1)L_4 1M_{4t-6}1]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 2, a_1 = 1, y_{-1} = 2t, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. For the special case $L^2_{3,5}$. The labeling $[1(-1), L_4 1]$ is sufficient and thus $L^2_{3,4t+1}$ is signed product cordial graph for all $t \geq 1$.

Case (3). $m \equiv 2(mod4)$.

Suppose that $m = 4t + 2, t > 1$. Then, one can label the vertices of $L^2_{3,4t+2}$ by $[(-1)1; (-1)L_4 1M_{4t-4}]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 2, a_1 = 1, y_{-1} = y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. For the special case $L^2_{3,6}$, the labeling $[(-1)1; L_4 1(-1)]$ which satisfies the conditions of signed product cordial graph. Thus $L^2_{3,4t+2}$ is signed product cordial graph for all $t \geq 1$.

Case (4). $m \equiv 3(mod4)$.

Suppose that $m = 4t + 3, t \geq 1$. Then, we can label the vertices of $L^2_{3,4t+3}$ by $[1_2; M_{4t+3}]$. Here, we label the common vertex as a part of the second cycle. So, when we label the first cycle we neglect the common vertex. Therefore $x_{-1} = 0, x_1 = 2, a_{-1} = 2, a_1 = 1, y_{-1} = 2t + 2, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 2$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 1$. From [9] $l_{3,3}$ is not signed product cordial graph and thus $L^2_{3,4t+3}$ is signed product cordial graph for all $t \geq 1$ and the lemma follows.▯

Lemma 4.2. If $n \equiv 0(mod4)$, then $L^2_{n,m}$ is signed product cordial graph.

Proof. Suppose that $n = 4r, r \geq 1$. We divide our study into two cases.

Case (1). At $n = 4$. We consider the four subcases.

Subcase (1.1). $m \equiv 0(mod4)$.

Suppose that $m = 4t, t > 1$. Then, one can label the vertices of $L^2_{4,4t}$ by $[1(-1)(-1); L_4 M'_{4t-4}]$. The labeling $L_4 M'_{4t-4}$ is starting from the common vertex. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 3, y_{-1} = y_1 = 2t$ and $b_{-1} = b_1 = 4t - 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. For the special case $L^2_{4,4}$. The labeling $[(-1)_3; 1_3(-1)]$ is sufficient for $L^2_{4,4}$ and thus $L^2_{4,4t}$ is signed product cordial graph for all $t \geq 1$.

Subcase (1.2). $m \equiv 1(mod4)$.

Suppose that $m = 4t + 1, t > 1$. Then, one can label the vertices of $L^2_{4,4t+1}$ by $[1(-1)_2; (-1)L_4 1M_{4t-6}1]$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 3, y_{-1} = 2t, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. For the special case $L^2_{4,5}$, the labeling $[1(-1)_2; L_4 1]$ is sufficient and thus $L^2_{3,4t+1}$ is signed product cordial graph for all $t \geq 1$.

Subcase (1.3). $m \equiv 2(mod4)$.

Suppose that $m = 4t + 2, t > 1$. Then, one can label the vertices of $L^2_{4,4t+2}$ by $[1(-1)_2; (-1)L_4 1M_{4t-4}]$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 3, y_{-1} = y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. For the special case $L^2_{4,6}$, the labeling $[1(-1)_2; L_4 1(-1)]$ is sufficient and thus $L^2_{4,4t+2}$ is signed product cordial graph for all $t \geq 1$.

Subcase (1.4). $m \equiv 3(mod4)$.

Suppose that $m = 4t + 3, t \geq 1$. Then, one can label the vertices of $L^2_{4,4t+3}$ by $[1(-1)_2; (-1)M_{4t}11]$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 3, y_{-1} = 2t + 1, y_1 = 2t + 2$ and $b_{-1} = b_1 = 4t + 2$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L^2_{4,4t+3}$ is signed product cordial graph for all $t \geq 1$.

Case (2). At $n \equiv 0(mod4), n > 4$. We consider the four subcases.

Subcase (2.1). $m \equiv 0(mod4)$.

Suppose that $m = 4t, t > 1$. Then, one can label the vertices of $L^2_{4r,4t}$ by $[L_4 M'_{4r-5};\ L_4 M'_{4t-4}]$. Therefore $x_{-1} = 2r - 1, x_1 = 2r, a_{-1} = a_1 = 4r - 1, y_{-1} = y_1 = 2t$ and $b_{-1} = b_1 = 4t - 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Here, we notice that the second power of the lemniscate $L^2_{4r,4}$ is isomorphic to $L^2_{4,4r}$ and hence it is signed product cordial graph by previous case. Thus $L^2_{4r,4t}$ is signed product cordial graph for all $r, t \geq 1$.

Subcase (2.2). $m \equiv 1(mod4)$.

Suppose that $m = 4t + 1, t > 1$. Then, one can label the vertices of $L^2_{4r,4t+1}$ by $[L_4 M_{4r-5};\ 1M'_{4t-6}1L'_4(-1)]$. Therefore $x_{-1} = 2r, x_1 = 2r - 1, a_{-1} = a_1 = 4r - 1, y_{-1} = 2t, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. For the special case $L^2_{4r,5}$ the labeling $[L_4 M'_{4r-5}; (-1)L_4]$ is sufficient. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L^2_{4r,4t+1}$ is signed product cordial graph for all $r, t \geq 1$.

Subcase (2.3). $m \equiv 2(mod4)$.

Suppose that $m = 4t + 2, t > 1$. Then, one can label the vertices of $L^2_{4r,4t+2}$ by $[L_4 M'_{4r-5}; (-1)L_4 1\ M_{4t-4}]$. Therefore $x_{-1} = 2r - 1, x_1 = 2r, a_{-1} = a_1 = 4r - 1, y_{-1} = y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. It remains to study $L^2_{4r,6}$. The labeling $[L_4 M'_{4r-5}; L_4 1(-1)]$ is sufficient for $L^2_{4r,6}$. So, $x_{-1} = 2r - 1, x_1 = 2r, a_{-1} = a_1 = 4r - 1, y_{-1} = y_1 = 3$ and $b_{-1} = b_1 = 5$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$ and consequently $L^2_{4r,6}$ is signed product cordial graph. Thus $L^2_{4r,4t+2}$ is signed product cordial graph for all $r, t \geq 1$.

Subcase (2.4). $m \equiv 3(mod4)$.

Suppose that $m = 4t + 3, t \geq 1$. Then, one can label the vertices of $L^2_{4r,4t+3}$ by $[L_4 M'_{4r-5}; M_{4t+3}]$. Therefore $x_{-1} = 2r - 1, x_1 = 2r, a_{-1} = a_1 = 4r - 1, y_{-1} = 2t + 2, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 2$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Finally; The graphs $L^2_{4,3}$ and $L^2_{4r,3}$ are isomorphic to $L^2_{3,4}$ and $L^2_{3,4r}$, respectively. By using lemma 4.1, we conclude that $L^2_{4,3}$ and $L^2_{4r,3}$ are signed product cordial graph. Thus $L^2_{4r,4t+3}$ is signed product cordial graph for all $r, t \geq 1$ and the lemma follows.□

Lemma 4.3. If $n \equiv 1(mod4)$ then $L^2_{4r+1,m}$ is signed product cordial graph.

Proof. Suppose that $n = 4r + 1, r \geq 1$. The following cases for n will be examined.

Case (1). At $n = 5$. We see that $L^2_{5,4}$ and $L^2_{5,4t}, t \geq 1$ are signed product cordial graph. This is clear since these graphs are isomorphic to $L^2_{4,5}$ and $L^2_{4t,5}$ respectively. So, by lemma 4.2, we conclude that $L^2_{5,4}$ and $L^2_{5,4t}$ are signed product cordial graph. Now, we shall study the following subcases for m.

Subcase (1.1). $m \equiv 1(mod4)$.

Suppose that $m = 4t + 1, t > 1$. Then, one can label the vertices of $L^2_{5,4t+1}$ by $[L_4; (-1)L_4 1M_{4t-6}1]$. Therefore $x_{-1} = x_1 = 2, a_{-1} = a_1 = 4, y_{-1} = 2t, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. At $t = 1$ the labeling $[L'_4; L_4 1]$ is sufficient for $L^2_{5,5}$ and thus $L^2_{5,4t+1}$ is signed product cordial graph for all $t \geq 1$.

Subcase (1.2). $m \equiv 2(mod4)$.

Suppose that $m = 4t + 2, t > 1$. Then, one can label the vertices of $L^2_{5,4t+2}$ by $[L_4; (-1)L_4 1M_{4t-4}]$. Therefore $x_{-1} = x_1 = 2, a_{-1} = a_1 = 4, y_{-1} = y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. For the special case $L^2_{5,6}$. The labeling $[L'_4; L_4 1(-1)]$ is sufficient and thus $L^2_{5,4t+2}$ is signed product cordial graph for all $t \geq 1$.

Subcase (1.3). $m \equiv 3(mod4)$.

Suppose that $m = 4t + 3, t \geq 1$. Then, one can label the vertices of $L^2_{5,4t+3}$ by $[1_3(-1); M_{4t+3}]$. Therefore $x_{-1} = 1, x_1 = 3, a_{-1} = a_1 = 4, y_{-1} = 2t + 2, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 2$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ nd $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L^2_{5,4t+3}$ is signed product cordial graph for all $t \geq 1$.

Case (2). Suppose that $n = 4r + 1$. We see that $L^2_{4r+1,4}$ and $L^2_{4r+1,4t}, r, t \geq 1$ are signed product cordial graph. This is clear since these graphs are isomorphic to $L^2_{4,4r+1}$ and $L^2_{4t,4r+1}$ respectively. Then, by lemma 4.2, $L^2_{4r+1,4}$ and $L^2_{4r+1,4t}$ are signed product cordial graph. Now, we shall study the following subcases for m will be examined.

Subcase (2.1). $m \equiv 1(mod4)$.

Suppose that $m = 4t + 1, t > 1$. Then, one can label the vertices of $L^2_{4r+1,4t+1}$ by $[(-1)L_4 1 M_{4r-6}; 1_2 M'_{4t-6} 1 L'_4(-1)]$ where $t > 1$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 4r, y_{-1} = 2t, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. The labeling $[(-1)L_4 M_{4r-6}; 1L_4]$ is sufficient for $L^2_{4r+1,5}$ and thus $L^2_{4r+1,4t+1}$ is signed product cordial graph for all $r, t \geq 1$.

Subcase (2.2). $m \equiv 2(mod4)$.

Suppose that $m = 4t + 2, t > 1$. Then, one can label the vertices of $L^2_{4r+1,4t+2}$ by $[(-1)L_4 1 M_{4r-6}; M'_{4t-4} 1 L'_4(-1)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 4r, y_{-1} = y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. It remains to study $L^2_{4r+1,6}$. For this purpose choose the labeling $[(-1)L_4 1 M_{4r-6}; L'_4(-1)1]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 4r, y_{-1} = y_1 = 3$ and $b_{-1} = b_1 = 5$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L^2_{4r+1,4t+2}$ is signed product cordial graph for all $r, t \geq 1$.

Subcase (2.3). $m \equiv 3(mod4)$.

Suppose that $m = 4t + 3, t \geq 1$. Then, one can label the vertices of $L^2_{4r+1,4t+3}$ by $[(-1)L_4 1 M_{4r-6};\ M'_{4t+3}]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 4r, y_{-1} = 2t + 1, y_1 = 2t + 2$ and $b_{-1} = b_1 = 4t + 2$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. The graphs $L^2_{5,3}$ and $L^2_{4r+1,3}$ are isomorphic to $L^2_{3,5}$ and $L^2_{3,4r+1}$, respectively. By using lemma (4.1), we conclude that $L^2_{5,3}$ and $L^2_{4r+1,3}$ are signed product cordial graph. Thus $L^2_{4r+1,4t+3}$ is signed product cordial graph for all $r, t \geq 1$ and hence the lemma follows.▯

Lemma 4.4. If $n \equiv 2(mod4)$, then $L^2_{4r+2,m}$ is signed product cordial graph.

Proof. Suppose that $n = 4r + 2, r \geq 1$. The following cases for n will be examined.

Case (1). At $r = 1$. We consider the following subcases.

Subcase (1.1). $m \equiv 0(mod4)$.

The graphs $L^2_{6,4}$ and $L^2_{6,4t}$ are isomorphic to $L^2_{4,6}$ and $L^2_{4t,6}$, respectively. Using lemma (4.2), we conclude that $L^2_{6,4}$ and $L^2_{6,4t}$ are signed product cordial graph.

Subcase (1.2). $m \equiv 1(mod4)$.

The graphs $L^2_{6,5}$ and $L^2_{6,4t+1}$ are isomorphic to $L^2_{5,6}$ and $L^2_{4t+1,6}$, respectively. Using lemma (4.3), we conclude that $L^2_{6,5}$ and $L^2_{6,4t+1}$ are signed product cordial graph.

Subcase (1.3). $m \equiv 2(mod4)$.

Suppose that $m = 4t + 2, t > 1$. Then, one can label the vertices of $L^2_{6,4t+2}$ by $[L_4 1; (-1)L_4 1 M_{4t-4}]$. Therefore $x_{-1} = 2, x_1 = 3, a_{-1} = a_1 = 5, y_{-1} = y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. For $L^2_{6,6}$ the labeling $[L_4 1; L_4 1(-1)]$ is sufficient and thus $L^2_{6,4t+2}$ is signed product cordial graph for all $t \geq 1$.

Subcase (1.4). $m \equiv 3(mod4)$.

Suppose that $m = 4t + 3, t \geq 1$. Then, one can label the vertices of $L^2_{6,4t+3}$ by $[L_4 1; M_{4t+3}]$. Therefore $x_{-1} = 2, x_1 = 3, a_{-1} = a_1 = 5, y_{-1} = 2t + 2, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 2$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Thus $L^2_{6,4t+3}$ is signed product cordial graph for all $t \geq 1$.

Case (2). At $n = 4r + 2, r > 1$. We consider the following subcases.

Subcase (2.1). $m \equiv 0(mod4)$.

The graphs $L^2_{4r+2,4}$ and $L^2_{4r+2,4t}$ are isomorphic to $L^2_{4,4r+2}$ and $L^2_{4t,4r+2}$, respectively. Using lemma (4.2), we conclude that $L^2_{4r+2,4}$ and $L^2_{4r+2,4t}$ are signed product cordial graph.

Subcase (2.2). $m \equiv 1(mod4)$.

The graphs $L^2_{4r+2,5}$ and $L^2_{4r+2,4t+1}$ are isomorphic to $L^2_{5,4r+2}$ and $L^2_{4t+1,4r+2}$, respectively. Using lemma (4.3), we conclude that $L^2_{4r+2,5}$ and $L^2_{4r+2,4t+1}$ are signed product cordial graph.

Subcase (2.3). $m \equiv 2(mod4)$.

Suppose that $m = 4t + 2, t > 1$. Then, one can label the vertices of $L^2_{4r+2,4t+2}$ by $[M'_{4r-4}1L'_4; (-1)\ L_4 1M_{4t-4}]$. Therefore $x_{-1} = 2r, x_1 = 2r + 1, a_{-1} = a_1 = 4r + 1, y_{-1} = y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 1$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = -1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. At $t = 1$ the lemniscate $L^2_{4r+2,6}$ is isomorphic to $L^2_{6,4r+2}$ which is signed product cordial graph by previous case and thus $L^2_{4r+2,4t+2}$ is signed product cordial graph for all $r, t \geq 1$.

Subcase (2.4). $m \equiv 3(mod4)$.

Suppose that $m = 4t + 3, t \geq 1$. Then, one can label the vertices of $L^2_{4r+2,4t+3}$ by $[(-1)L_4 1M_{4r-5};\ M'_{4t+3}]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 4r + 1, y_{-1} = 2t + 1, y_1 = 2t + 2$ and $b_{-1} = b_1 = 4t + 2$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. The graphs $L^2_{6,3}$ and $L^2_{4r+2,3}$ are isomorphic to $L^2_{3,6}$ and $L^2_{3,4r+2}$, respectively. Using lemma (4.1), we conclude that $L^2_{6,3}$ and $L^2_{4r+2,3}$ are signed product cordial graph. Thus $L^2_{4r+2,4t+3}$ is signed product cordial graph for all $r, t \geq 1$ and the lemma follows.□

Lemma 4.5. If $n \equiv 3(mod4)$. Then $L^2_{4r+3,4t+3}$ is signed product cordial graph.

Proof. Suppose that $n = 4r + 3, r \geq 1$. The following cases for m will be examined.

Case (1). $m \equiv 0(mod4)$.

The graphs $L^2_{4r+3,4}$ and $L^2_{4r+3,4t}$ are isomorphic to $L^2_{4,4r+3}$ and $L^2_{4t,4r+3}$, respectively. Using lemma (4.2), we conclude that $L^2_{4r+3,4}$ and $L^2_{4r+3,4t}$ are signed product cordial graph.

Case (2). $m \equiv 1(mod4)$.

The graphs $L^2_{4r+3,5}$ and $L^2_{4r+3,4t+1}$ are isomorphic to $L^2_{5,4r+3}$ and $L^2_{4t+1,4r+3}$, respectively. Using lemma (4.3), we conclude that $L^2_{4r+3,5}$ and $L^2_{4r+3,4t+1}$ are signed product cordial graph.

Case (3). $m \equiv 2(mod4)$.

The graphs $L^2_{4r+3,6}$ and $L^2_{4r+3,4t+2}$ are isomorphic to $L^2_{6,4r+3}$ and $L^2_{4t+2,4r+3}$, respectively. Using lemma (4.4), we conclude that $L^2_{4r+3,6}$ and $L^2_{4r+3,4t+2}$ are signed product cordial graph.

Case (4). $m \equiv 3(mod4)$.

Suppose that $m = 4t + 3, t \geq 1$. Then, one can label the vertices of $L^2_{4r+3,4t+3}$ by $[M_{4r+2}; M_{4t+3}]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = a_1 = 4r + 2, y_{-1} = 2t + 2, y_1 = 2t + 1$ and $b_{-1} = b_1 = 4t + 2$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 1$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. The graph $L^2_{4r+3,3}$ is isomorphic to $L^2_{3,4r+3}$. Using lemma (4.1), we conclude that $L^2_{4r+3,3}$ is signed product cordial graph. Thus $L^2_{4r+3,4t+3}$ is signed product cordial graph for all $r, t \geq 1$ as we wanted to prove.□

As consequence of all lemmas mentioned before we conclude the following theorem.

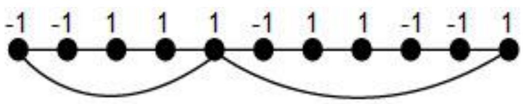
Theorem 4.1. The second power of lemniscate graph $L^2_{n,m}$ is signed product cordial graph for all n, m except at $n = 3$ and $m = 3$.

5. Examples

The signed product cordial graph of $L_{4,7}, L_{5,7}, L^2_{4,5}$ and $L^2_{5,10}$ are illustrated in Figures (5.1, ..., 5.4).

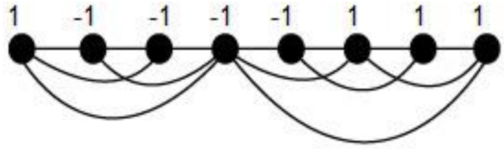
$x_{-1} = x_1 = a_{-}1 = a_1 = 2, y_{-}1 = y_1 = 3, b_{-}1 = 4, b_1 = 3$

Figure (5.1). $L_{4,7}$ is signed product cordial graph.



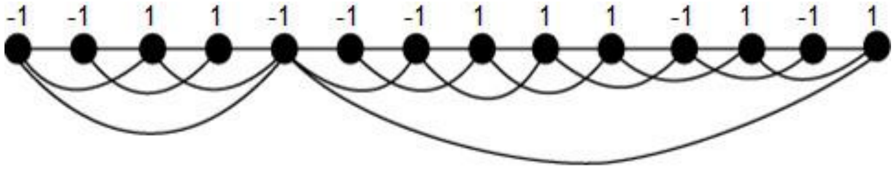
$x_{-1} = 2, x_1 = 3, a_{-}1 = 2, a_1 = 3, y_{-}1 = y_1 = 3, b_{-}1 = 4, b_1 = 3$

Figure (5.2). $L_{5,7}$ is signed product cordial graph



$x_{-1} = 2, x_1 = 1, a\_1 = a_1 = 3, y\_1 = 2, y_1 = 3, b\_1 = b_1 = 4$

Figure (5.3). $L^2_{4,5}$ is signed product cordial graph



$x_{-1} = x_1 = 2, a\_1 = a_1 = 4, y\_1 = y_1 = 5, b\_1 = b_1 = 9$

Figure (5.4). $L^2_{5,10}$ is signed product cordial graph.

6. Applications on Signed graph Conclusion

6.1. Social psychology.

In social psychology, signed graphs have been used to model social situations, with positive edges representing friendships and negative edges enmities between nodes, which represent people. Then, for example, a positive 3-cycle is either three mutual friends, or two friends with a common enemy; while a negative 3-cycle is either three mutual enemies, or two enemies who share a mutual friend. According to balance theory, positive cycles are balanced and supposed to be stable social situations, whereas negative cycles are unbalanced and supposed to be unstable. According to the theory, in the case of three mutual enemies, this is because sharing a common enemy is likely to cause two of the enemies to become friends. In the case of two enemies sharing a friend, the shared friend is likely to choose one over the other and turn one of his or her friendships into an enemy.

6.2. Data clustering.

Correlation clustering looks for natural clustering of data by similarity. The data points are represented as the vertices of a graph, with a positive edge joining similar items and a negative edge joining dissimilar items.

7. Conclusion

In this work, we proposed two contributions. The first contribution is that we proved the lemniscate graph is a signed product cordial graph. The second contribution is that we investigate the conditions under which the second power of the lemniscate graph is signed product cordial graph. An example is introduced in section 5. Finally, an application of a signed graph is introduced.

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