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Research Article

Asymptotic Properties of the Robust Regression Estimator by using the Local Linear Method Case of Spatial Functional Data

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Abstract: A robust nonparametric local linear estimator will be constructed in this research when the regresses are functional spatial random variables. It is calculated by the combination of the technique M-estimation and local linear idea. The main goal is to generalize the result given in **(Ibrahim M. Almanjahie. 2020)**, and to establish the almost complete convergence with robust of this estimator under an α- mixing assumption.

Keywords: Spatial Data, Robust regression, Local linear Method, Asymptotic properties, Strong mixing process.

AMS Mathematics Subject Classification 2020 :

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1. Introduction

Recent years have seen an increase in interest in statistical issues relating to the modeling and analysis of special data. The rise in the quality of real world issues for which data is being obtained spatial order is what drives the importance of this field of research. Such issues are common in a wide range of disciplines, including epidemiology, econometrics, earth and environment sciences, agronomy, etc. Several monographs in spatial data analysis are presented in **(Anselin, L., and R. J. G. M. Florax. 1995)**, **(Cressie, N. A. 1993)** and **(Ripley, B. 1981)**. In addition, some important sources for functional data modeling we refer **(Ramsay, J. O., and B. W. Silverman. 2002)**, **(Bosq, D. 2000)**, **(Ferraty, F., and P. Vieu. 2006)**, **(Geenens, G. 2011)**, **(Horvath, L., and P. Kokoszka. 2012)**, **(Zhang, J. 2014)** or **(Hsing, T., and R. Eubank. 2015)**. The non-parametric functional spatial statistics serve as the overall framework for current contribution. Modeling are association between a scalar response variable and functional covariant has received a lot of attention in this context. The most popular strategies rely on local constant fitting. In this study, we suggest employing local linear M-regression to model this relationship. An alternate method to estimate the spatial regression was put on forth by **(Li, J., Tran, L.T., 2009)**. The authors in **(Xu, R., Wang, J., 2008)** investigated the local linear estimation of the regression model in the spatial situation. The least absolute deviation is minimized to provide the final semi-parametric estimate.

The local linear technique has a variety of advantages over local constant fitting, as is widely known. Particularly, it enables the reduction of the bias term in a diverse range of situations. For further information on the significance of this strategy, see **(Fan, J., and I. Gijbels. 1996)**. Noting that many authors have looked into the robust linear estimation in the multivariate case. For the i.i.d. case we suggest to see **(Fan, J., and Hu, T.C., Troung.Y.K., 1994)**, and **(Cai. Z., and E. Ould-Said. 2003)** for the α-mixing case. For some asymptotic findings using the robust constant method for both techniques (KNN and kernel), we go-back to **(Boente, G., Manteiga, W.G., Péerez. Gonzàlez, A., 2009)**. We shall concentrate on the scenario when the covariant has infinite dimensions in this paper. It should be mentioned that in applied statistics, the importance of statistical analysis of infinite-dimensional data is increasing. These studies all primarily concentrate on the classical regression.

It is well recognized, nonetheless, that the given model is particularly susceptible to outliers and struggles with errors that are heavy-tailed. Such data are frequently seen in many practical domains, including econometrics, finance, and many more. In this study, we propose to robustly the functional local linear regression model to minimize the lack of robustness of this model. The structure of this paper is as follow: In Section 2, we present our model. The main outcomes and necessarily conditions are presented in Section 3. In Section 4, we describe the key aspects of our strategy and contrast it with other strategies. In this part, we also present some perspectives on the current contribution.

2. Presentation of the spatial data and the robust local linear estimator

Let a F×R-valued measurable noted by $Z_{\mathbf{i}}=(X_{\mathbf{i}},Y_{\mathbf{i}})$ for $\mathbf{i} \in □^{N}$ and a strictly stationary spatial process, given in a space of probability (Ω,A,P), where (F,d) denotes a semi-metric space, and N ≥ 1. Let a point $\mathbf{i} = (i_1, \dots . i_n) \epsilon \mathbb{N}^N$ refers a site.

We suppose that the process $Z_{\mathbf{i}}= (X_{\mathbf{i}},Y_{\mathbf{i}})$, is observed over a rectangular domain:

$$I_n = \{\mathbf{i} = (i_1, \dots . i_n) \epsilon \mathbb{Z}^N , \quad 1 \le i_k \le n_k , \quad k = 1, \dots , N \}$$

where $= (n_1, \dots . n_N) \epsilon \mathbb{Z}^N$.

The point i denotes a site, and we shall write:

$$n \to \infty \quad \text{if} \quad \min_{k=1,\dots,N}\{n_k\} \to \infty \quad \text{and} \quad |n_j/n_k| < C$$

where C is a constant in which 0 ≤ C ≤ ∞ for all j, k ∈ {1, ..., N}.

For $n = (n_1, \dots . n_N) \epsilon \mathbb{N}^N$, we set $\hat{n} = n_1 \times \cdot \cdot \cdot \times n_N$.

(Z_i) is assumed to have the same distribution as (X, Y) throughout this paper. We suppose a regular version of the conditional probability of Y given X exists and has a continnous and bounded density function. It is given with respect to the lebesgue measure over ℝ. The conditional distribution (respectively density) function given by F^x(respectively f^x) of Y given X with x ∈□ and N_x denotes a given neighborhood. Our goal in this work is the generalization of the results given in **(Ibrahim M. Almanjahie)**, that we can see in our model by taking $\rho(y) = |y||\alpha - I_{y<0}|$we obtain this last result. Furthermore, by taking $\rho(y) = y^2|\alpha - I_{y<0}|$ we obtain the α^{th} conditional expectile. For x∈□, θ_x, the nonparametric robust regression, defines the unique minimizer of

$$\theta_x = \arg\min_{t\epsilon\mathbb{R}} \mathbb{E}\, [\rho(Y - t)/X = x] \tag{1}$$

where ρ(.) is real-valued Borel function satisfying some regularity condition that will be stated below. This model belongs to the of M-estimates class introduced in **(Huber, P.J., 1964)**. We refer the reader to **(Stone, C. J. 2005)** for more additional examples. Note that the local linear smoothing method relies on the model being approximated by a linear function in a neighborhood of x. Several techniques exist in functional static analysis to extend this strategy (see, **(Baıllo, A., and A. Grane. 2009)** or **(Barrientos, J., F. Ferraty, and P. Vieu. 2010)** for several instances). In this contribution, we use the simple version suggested by **(Barrientos, J., F. Ferraty, and P. Vieu. 2010)** in which the function θ_x is approximately presented by

$$\forall z \in N_x \ , \ \theta_x = a + b\beta(z,x) ,$$

where $\hat{a}$ and $\hat{b}$ denotes the estimates of a and b respectively and they are solution of

$$\min_{(a,b)\epsilon\mathbb{R}^2} \sum_{\mathbf{i}\epsilon I_n} \rho(Y_{\mathbf{i}} - a - b\, \beta(X_{\mathbf{i}} , x))K(h^{-1}\, \delta(x , X_{\mathbf{i}})) . \tag{2}$$

Here, β(. , .) is a known function from (□ ×□) into ℝ such that,∀ξ ∈□, β(ξ, ξ) = 0 with kernel K. Also $h = h_{\mathbf{n}}$ denotes a positive real sequence numbers and **δ(., .)** denotes a function of (□ ×□) such that d(., .)=|δ(., .)|. It obvious that taking this into account, we can write

$$\theta_x = a \,, \text{ and } \widehat{\theta_x} = \hat{a}.$$

We emphasize the fact that the robust linear estimator cannot be defined directly, in contrast to classical regression case investigated by **(Barrientos, J., F. Ferraty, and P. Vieu. 2010)**. As a result, establishing the asymptotic proprieties of our estimate is quite challenging and necessitates the use of some additional techniques.

Note that, the major outcome of this work is to demonstrate the estimator's $\widehat{\theta_x}$ almost complete consistency (with rate) forr the following condition of mixing:

$$\begin{cases} \text{There exists a function } \varphi(t) \downarrow 0 \text{ as } t \to \infty, \text{ such that} \\ \quad \forall E, E' \text{ subsets of } \mathbb{N}^N \text{ with finite cardinals} \\ \alpha\left(B(E), B(E')\right) = \sup_{B \in \mathbf{B}(E),\ C \in \mathbf{B}(E')} |\mathbb{P}(B \cap C) - \mathbb{P}(B)\mathbb{P}(C)| \\ \quad \le \psi(\mathrm{Card}(E), \mathrm{Card}(E'))\varphi(\mathrm{dist}(E, E')) \end{cases} \tag{3}$$

where $B(E)$ (respectively, $B(E')$) denotes the Borel σ-field generated by ($Z_{\mathbf{i}}, \mathbf{i} \in E$) (respectively, ($Z_{\mathbf{i}}$, $\mathbf{i} \in E'$)). The cardinality number of E (respectively, E'), is denoted by Card(E) (respectively, Card(E')) and dist(E, E') defines the Euclidean distance between E and E. A symmetric positive non-decreasing function $\psi : \mathbb{Z}^2 \to \mathbb{R}^+$ in each variable so that:

$$\forall n, m \in \mathbb{Z};\ \psi(n, m) \le C \min(n, m). \tag{4}$$

For some $C > 0$ and

$$\sum_{i=1}^{\infty} i^{\delta} \varphi(i) < \infty, \quad \delta > 0 \tag{5}$$

Remark that, when (3) holds with $\psi \equiv 1$ or N = 1, thus $Z_{\mathbf{i}} = (X_{\mathbf{i}}, Y_{\mathbf{i}})$, the random field is called strongly mixing.

3. Hypotheses and results

When there is no possibility of confusion, a strictly positive generic C and C' will be denoted in what follow. Furthermore, we denotes x a fixed point in □, and by N_x for a fixed neighborhood of x. For $\mathbf{i} \in I_n$, we denote by $K_{\mathbf{i}} = K(h^{-1}\delta(x, X_{\mathbf{i}}))$, and $\beta_{\mathbf{i}} = \beta(X_{\mathbf{i}}, x)$. In addition, we put $\varphi_x(r_1, r_2) = P(r_2 \le \delta(x, X) \le r_1)$, where B (x, r) = $\{x' \in □ / \delta(x, x') \le r\}$ as well as the following hepotheses:

(H1) For any $r > 0$, $\boldsymbol{\phi}_{\mathbf{x}}(\mathbf{r}) := \boldsymbol{\phi}_{\mathbf{x}}(-\mathbf{r}, \mathbf{r}) > 0$ and there exists a function $\boldsymbol{\chi}_{\mathbf{x}}(.)$ so that:

$$\forall \text{constants } t \in (-1, 1), \lim_{h \to \infty} \frac{\boldsymbol{\phi}_{\mathbf{x}}(\mathbf{th}, \mathbf{h})}{\boldsymbol{\phi}_{\mathbf{x}}(\mathbf{h})} = \boldsymbol{\chi}_{\mathbf{x}}(\mathbf{t}).$$

(H2) For all, $\mathbf{i} \neq \mathbf{j}$

$$0 < \sup_{\mathbf{i} \neq \mathbf{j}} □ [(X_{\mathbf{i}}, X_{\mathbf{j}}) \in B(x, h) \times B(x, h)] \le C \left(\phi_x(h)\right)^{\frac{(a+1)}{a}}$$
$$\text{for some } C > 0, \quad 1 < a < N^{-1}.$$

(H3) A continuously differentiable function p in which it is a strictly convex, and has a Lipschitzian derivative $\boldsymbol{\psi}$ such that

$$\mathbb{E}[|\psi(Y - t)|^p \backslash X = x] < C < \infty \text{ almost surely,} \qquad \text{for all } p \ge 2$$

(H4) The function $\Gamma_\lambda(x, .) := \mathbb{E}[\psi^\lambda(Y - .) \backslash X = x]$ is of class C^1 on

$$[\theta_x - \delta, \theta_x + \delta];\ \delta > 0 \text{ and } \lambda \in \{1, 2\}. \text{ We put } \gamma(x, .) := \frac{d}{dt}\Gamma_1(x, .)$$

$$\begin{cases} \text{(i)} \quad \forall (t_1, t_2) \in [\theta_x - \delta, \theta_x + \delta]^2, \quad \forall (x_1, x_2) \in N_x \times N_x, \quad \text{and for } k_1, k_2 > 0 \\ \qquad |\Gamma_\lambda(x_1, t_1) - \Gamma_\lambda(x_2, t_2)| \le C\, d^{k_1}(x_1, x_2) + |t_1 - t_2|^{k_2} \\ \text{(ii)} |\gamma(x_1, t_1) - \gamma(x_2, t_2)| \le C' d^{k_1}(x_1, x_2) + |t_1 - t_2|^{k_2}. \end{cases}$$

(H4') The variable δ (x, X) is σ (β (X, x))-measurable and the two partial derivatives of the function

$$Y_x(s, t) := \mathbb{E}[\Gamma_1(X, t) \backslash \beta(X, x) = s] \text{ at } (0, \theta_x) \text{ exist.}$$

(H5) The function β (., .) is such that:

$$\begin{cases} \text{(i)} \ \forall z \in □, \quad C|\delta(x, z)| \le |\beta(z, x)| \le C'|\delta(x, z)| \\ \text{(ii)} \sup_{u \in B(x, r)} |\beta(u, x) - \delta(x, u)| = o(r). \end{cases}$$

$$\text{where } B(x, r) = \{x' \in \square / \delta(x, x') \leq r\}.$$

(H6) K is a positive kernel, differentiable function that is supported within (−1, 1) so that:

$$\begin{pmatrix} K(1) - \int_{-1}^{1} K'(t)\chi_x(t)\,dt & K(1) - \int_{-1}^{1} (t\,K(t))'\chi_x(t)\,dt \\ K(1) - \int_{-1}^{1} (t\,K(t))'\chi_x(t)\,dt & K(1) - \int_{-1}^{1} (t^2\,K(t))'\chi_x(t)\,dt \end{pmatrix},$$

is a positive definite matrix.

(H7) There exists $\eta_0 > 0$, and the bandwidth h satisfies :

$$\begin{cases} \text{(i) } \lim_{n\to\infty} \dfrac{\log \widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(h)} = 0\ . \\ \text{(ii) } C\,\widehat{\mathbf{n}}^{\frac{5N}{\delta}-1+\eta_0} \leq \phi_x(h), \quad \text{for } C > 0. \end{cases}$$

Comments on hypotheses

The classical concentration property is characterized by condition (H1). Moste recent non-parametric statistical works on functional data use it. As in the i.i.d. case, the same convergence rate can be obtained by the local dependence. Condition (H3) is much lower than that considered by **(Attouch, M., Laksaci, A., Ould Said, E. 2009)**. In order to assess the asymptotic bias, the regularity constraints (H4) and (H4') that define the functional space of our model must be met. Note that, the convexity of loss function ρ and the boundedness of the score function ψ are what regulate the robustness attribute of M-estimators. However, employing the truncation method described in **(Laıb, N., Ould-Saıd, E., 2000)** allows for dropping boundedness condition of ψ which helps to obtain generalize our model. Finally (H5) - (H7) are technical conditions **(Barrientos, J., F. Ferraty, and P. Vieu. 2010)**.

The main result of the paper is given in this following theorem.

4. Main result

Theorem 1. *Under* hypotheses *(H1)–(H7) and if* $\gamma(x, \theta_x) > 0$ *then*

$$\left|\widehat{\theta_x} - \theta_x\right| = O\left(\square^{\min(k_1,k_2)}\right) + O\left(\left(\frac{\log \widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(\square)}\right)^{\frac{1}{2}}\right), \qquad a.co.$$

5. Proofs

To begin, we present the lemmas that are required to prove our asymptotic findings

Lemma 1. Let $V_{\mathbf{n}}$ be a sequence of vectorial function, such that:

i). For all $\lambda \geq 1$ and a vector $\boldsymbol{\delta}$:

$$t_{(\delta)} V_{\mathbf{n}}(\lambda\delta) \leq t_{(\delta)} V_{\mathbf{n}}(\delta)$$

ii). For a positive definite matrix D and vectorial sequences $A_{\mathbf{n}}$, such that: $\|A_{\mathbf{n}}\| = O_{a.co}(1)$ for some A> 0 we have

$$\sup_{\|\delta\| \leq M} \|V_{\mathbf{n}}(\delta) + \lambda_0 D\delta - A_{\mathbf{n}}\| = O_{a.co}(1) \text{ for } \frac{A}{\lambda_0\lambda_1(D)} < M < \infty.$$

where $\lambda_1(D)$ is the minimum eigenvalue ofD and $\lambda_0 > 0$.
So that for any vectorial sequence δ_n, $V_{\mathbf{n}}(\delta_{\mathbf{n}}) = O_{a.co}(1)$, we have

$$\|\delta_{\mathbf{n}}\| \leq M, \ \text{a.co} \qquad (6)$$

Proof of Lemma 1.

The proof of the lemma has the same concepts as presented in **(Koenker, R., and Q. Zhao. 1996)**. In fact, for η > 0, we can write

$$\mathbb{P}\left(\|\delta_{\mathbf{n}}\| \geq \mathrm{M}\right) = \mathbb{P}\left(\|\delta_{\mathbf{n}}\| \geq \mathrm{M}, \|\mathrm{V}_{\mathbf{n}}(\delta_{\mathbf{n}})\| < \eta\right) + \mathbb{P}\left(\|\mathrm{V}_{\mathbf{n}}(\delta_{\mathbf{n}})\| \geq \eta\right)$$

$$\leq \mathbb{P}\left(\inf_{\|\delta\| \geq \mathrm{M}} \|\mathrm{V}_{\mathbf{n}}(\delta)\| < \eta\right) + \mathbb{P}\left(\|\mathrm{V}_{\mathbf{n}}(\delta_{\mathbf{n}})\| \geq \eta\right)$$

Since, $\mathrm{V}_{\mathbf{n}}(\delta_{\mathbf{n}}) = \mathrm{O}_{\mathrm{a.co}}(1)$, then

$$\sum_{\mathbf{n}} \mathbb{P}\left(\|\mathrm{V}_{\mathbf{n}}(\delta_{\mathbf{n}})\| \geq \eta\right) < \infty$$

Consequently, it is enough to prove that

$$\sum_{\mathbf{n}} \mathbb{P}\left(\inf_{\|\delta\| \geq \mathrm{M}} \|\mathrm{V}_{\mathbf{n}}(\delta)\| < \eta\right) < \infty$$

Then, for all δ such that $\|\delta\| \geq \mathrm{M}$, there exists $\lambda \geq 1$ and $\|\delta_1\| = \mathrm{M}$ for which $\delta = \lambda\delta_1$. The first condition of lemma (1) gives us

$$\|\mathrm{V}_{\mathbf{n}}(\delta)\| = \left\| - \frac{t_{\delta_1} \mathrm{V}_{\mathbf{n}}(\delta)}{\mathrm{M}} \right\| \geq - \frac{t_{\delta_1} \mathrm{V}_{\mathbf{n}}(\delta)}{\mathrm{M}} \geq - \frac{t_{\delta_1} \mathrm{V}_{\mathbf{n}}(\delta_1)}{\mathrm{M}}$$

So we have

$$\mathbb{P}\left(\inf_{\|\delta\| \geq \mathrm{M}} \|\mathrm{V}_{\mathbf{n}}(\delta)\| < \eta\right) \leq \mathbb{P}\left(\inf_{\|\delta_{\mathbf{1}}\| = \mathrm{M}} \left[- \frac{t_{\delta_1} \mathrm{V}_{\mathbf{n}}(\delta_1)}{\mathrm{M}}\right] < \eta\right)$$

Therefore, evaluating this last quantity is the only thing left to do. We write about this

$$\begin{aligned}
\mathbb{P}\left(\inf_{\|\delta_{\mathbf{1}}\| = \mathrm{M}} \left[- \frac{t_{\delta_1} \mathrm{V}_{\mathrm{n}}(\delta_1)}{\mathrm{M}}\right] < \eta\right) &\leq \mathbb{P}\Big(\inf_{\|\delta_{\mathbf{1}}\| = \mathrm{M}} \left[- t_{\delta_1} \mathrm{V}_{\mathbf{n}}(\delta_1)\right] < \eta \mathrm{M} \\
&\quad \inf_{\|\delta_{\mathbf{1}}\| = \mathrm{M}} \left[- t_{\delta_{\mathbf{1}}}(- \lambda_0 \mathrm{D}\delta_1 + \mathrm{A}_{\mathbf{n}})\right] \geq 2\,\eta \mathrm{M} \Big) \\
&\quad + \mathbb{P}\left(\inf_{\|\delta_{\mathbf{1}}\| = \mathrm{M}} \left[- t_{\delta_1}(- \lambda_0 \mathrm{D}\delta_1 + \mathrm{A}_{\mathbf{n}})\right] \leq 2\,\eta \mathrm{M}\right) \\
&\leq \mathbb{P}\left(\sup_{\|\delta_{\mathbf{1}}\| = \mathrm{M}} \|\mathrm{V}_{\mathbf{n}}(\delta_1) + \lambda_0 \mathrm{D}\delta_1 - \mathrm{A}_{\mathbf{n}}\| \geq \eta\right) \\
&\quad + \mathbb{P}\left(\|\mathrm{A}_{\mathbf{n}}\| \geq \lambda_0 \lambda_1(\mathrm{D})\mathrm{M} - 2\,\eta\right).
\end{aligned}$$

At last, under the second conditions given in Lemma(1), we can choose η for which

$$\sum_{\mathbf{n}} \mathbb{P}\left(\sup_{\|\delta_1\| = \mathrm{M}} \|\mathrm{V}_{\mathbf{n}}(\delta_1) + \lambda_0 \mathrm{D}\delta_1 - \mathrm{A}_{\mathbf{n}}\| \geq \eta\right) < \infty$$

Also

$$\sum_{\mathbf{n}} \mathbb{P}\left(\|\mathrm{A}_{\mathbf{n}}\| \geq \lambda_0 \lambda_1(\mathrm{D})\mathrm{M} - 2\,\eta\right) < \infty$$

Accordingly,

$$\sum_{\mathbf{n}} \mathbb{P}\left(\|\delta_{\mathbf{n}}\| \geq \mathrm{M}\right) < \infty$$

.

Proof of Theorem

We define, for all $\delta = \binom{c}{d}$,

$$\varphi_{\mathbf{i}}(\delta) = \psi\left(Y_{\mathbf{i}} - \left((c+a) + (h^{-1}d + b)\beta_{\mathbf{i}}\right)\right)$$

and we consider the following vectoriel sequence

$$V_{\mathbf{n}}(\delta) = \frac{1}{\hat{\mathbf{n}}\,\phi_x(h)} \sum_{\mathbf{i} \in I_{\mathbf{n}}} \varphi_{\mathbf{i}}(\delta) \begin{pmatrix} 1 \\ h^{-1}\beta_{\mathbf{i}} \end{pmatrix} K_{\mathbf{i}}$$

$$\text{and} \qquad A_{\mathbf{n}} = V_{\mathbf{n}}(\delta_0) \text{with} \delta_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus, the proof of Theorem is based on the application of Lemma (1)'s second part of to $(V_{\mathbf{n}}, A_{\mathbf{n}}, \delta_{\mathbf{n}})$ with $V_{\mathbf{n}}(\delta_{\mathbf{n}}) = \begin{pmatrix} \hat{a} - a \\ h^{-1}(\hat{b} - b) \end{pmatrix}$

While the second result is obtained by applying the lemma (1)'s first part to $(V_{\mathbf{n}}', A_{\mathbf{n}}', \delta_{\mathbf{n}}')$ where

$$\varphi_{\mathbf{i}}'(\delta) = \psi\left(Y_{\mathbf{i}} - \left(a + \frac{1}{\sqrt{\hat{\mathbf{n}}\,\phi_x(h)}}c\right) + \left(\frac{1}{h\sqrt{\hat{\mathbf{n}}\,\phi_x(h)}}d + b\right)\beta_i\right)$$

$$V_{\mathbf{n}}'(\delta) = \frac{\mathbf{1}}{\sqrt{\hat{\mathbf{n}}\,\phi_x(h)}} \sum_{\mathbf{i} \in I_{\mathbf{n}}} \varphi_{\mathbf{i}}'(\delta) \begin{pmatrix} 1 \\ h^{-1}\beta_{\mathbf{i}} \end{pmatrix} K_{\mathbf{i}}$$

$$\text{and} \qquad A_{\mathbf{n}}' = V_{\mathbf{n}}'(\delta_0) \qquad \text{with} \qquad \delta_{\mathbf{n}}' = \sqrt{\hat{\mathbf{n}}\,\phi_x(h)\,\delta_{\mathbf{n}}}$$

Clearly, the condition (i) given in lemma (1) holds, because $\boldsymbol{\psi}$ is a monotone increasing function. The case of decreasing can be obtained by considering$-\boldsymbol{\psi}$.

The theorem's result is a consequence of the following lemmas:

Lemma 2. Under hypotheses (H1)-(H7), we have

$$\|A_{\mathbf{n}}\| = O\left(h^{\min(k_1,k_2)}\right) + O_{\mathbf{a.co}}\left(\left(\frac{\log\hat{\mathbf{n}}}{\hat{\mathbf{n}}\,\phi_x(h)}\right)^{\frac{1}{2}}\right)$$

Lemma 3. Under hypotheses (H1)-(H7), we have

$$\sup_{\|\delta\| \leq M} \|V_{\mathbf{n}}(\delta) + \lambda_0 D\delta - A_{\mathbf{n}}\| = O\left(h^{\min(k_1,k_2)}\right) + O_{a.co}\left(\left(\frac{\log\hat{\mathbf{n}}}{\hat{\mathbf{n}}\,\phi_x(h)}\right)^{\frac{1}{2}}\right)$$

With

$$D = \begin{pmatrix} K(1) - \int_{-1}^{1} K'(t)\chi_x(t)\,dt & K(1) - \int_{-1}^{1} \left(t\,K(t)\right)' \chi_x(t)\,dt \\ K(1) - \int_{-1}^{1} \left(t\,K(t)\right)' \chi_x(t)\,dt & K(1) - \int_{-1}^{1} \left(t^2\,K(t)\right)' \chi_x(t)\,dt \end{pmatrix}$$

And

$$\lambda_0 = \gamma(x, \theta_x)$$

Proof of Lemma 2.

Firstly, we establish

$$A_{\mathbf{n}} - \mathbb{E}[A_{\mathbf{n}}] = \begin{pmatrix} A_{\mathbf{n}}{}^{1} \\ A_{\mathbf{n}}{}^{2} \end{pmatrix}$$

where

$$\begin{cases} A_{\mathbf{n}}{}^{1} := \dfrac{1}{\widehat{\mathbf{n}}\,\phi_x(h)} \sum_{i \in I_n} \Delta_{\mathbf{i}}{}^{1} \\ A_{\mathbf{n}}{}^{2} := \dfrac{1}{\widehat{\mathbf{n}}\,\phi_x(h)} \sum_{i \in I_n} \Delta_{\mathbf{i}}{}^{2} \end{cases}$$

with

$$A_{\mathbf{n}}{}^{l+1} = \frac{1}{\widehat{\mathbf{n}}\,\phi_x(h)} \sum_{\mathbf{i} \in I_n} \Delta_{\mathbf{i}}{}^{l+1} \text{for} l = 0, 1$$

and

$$\Delta_{\mathbf{i}}{}^{l+1} = \varphi_{\mathbf{i}}(\delta_0) h^{-1} \beta_{\mathbf{i}}{}^{l} K_{\mathbf{i}} - \mathbb{E}\left[\varphi_{\mathbf{i}}(\delta_0) h^{-1} \beta_{\mathbf{i}}{}^{l} K_{\mathbf{i}}\right]$$

Furthermore, under (H1),(H3) and (H5)

$$\left|\boldsymbol{\varphi}_{\mathbf{i}}(\boldsymbol{\delta}_{\mathbf{0}}) \mathbf{h}^{-1} \boldsymbol{\beta}_{\mathbf{i}}{}^{\mathbf{l}} \mathbf{K}_{\mathbf{i}}\right| \leq \mathbf{C}\, \mathbf{h}^{-1} |\boldsymbol{\gamma}(\mathbf{X}_{\mathbf{i}}, \mathbf{x})|^{\mathbf{l}} \mathbf{K}_{\mathbf{i}} \leq \mathbf{C}\, \mathbf{K}_{\mathbf{i}} \leq \mathbf{C}$$

Thus, it is sufficient to demonstrate that

$$A_{\mathbf{n}} - \mathbb{E}[A_{\mathbf{n}}] = O_{a.co}\left(\left(\frac{\log \widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(h)}\right)^{\frac{1}{2}}\right)$$

On the other hand

$$\begin{aligned} \mathbb{E}\left[A_n{}^{l+1}\right] &= \frac{1}{\phi_x(h)} \mathbb{E}\left[\varphi_1(\delta_0) h^{-1} \beta_1{}^{l} K_1\right] \\ &\leq \frac{1}{\phi_x(h)} \mathbb{E}\left[h^{-1} \beta_1{}^{l} K_1 1_{B(x,h)}(X_1) |\Gamma_1(x, \theta_x) - \Gamma_1(X_1, a + b\beta_1)|\right] \\ &= O\left(h^{k_1}\right) + o\left(h^{k_2}\right). \end{aligned}$$

which achieves the proof of Lemma (2).

Now, for the second claimed result, we consider the spatial decomposition of **(Tran, L. T. 1990)** in which it is based on the split of the sum into 2^N random variables sums defined for a fixed integers pn, given as follow

$$T(1, \mathbf{n}, X, \mathbf{j}) = \sum_{\mathbf{i_k = 2j_k p_n + 1 k = 1, \dots N}}^{\mathbf{2j_k p_n + p_n}} \Delta_{\mathbf{i}}$$

$$T(2,\mathbf{n},X,\mathbf{j}\ ) = \sum_{i_k = 2j_kp_n+1 k=1,\dots N-1}^{2j_kp_n+p_n} \sum_{i_n = 2j_np_n+p_n+1}^{(j_n+1)p_n} \Delta_{\mathbf{i}}$$

$$T(3,\mathbf{n},X,\mathbf{j}\ ) = \sum_{i_k = 2j_kp_n+1 k=1,\dots N-2}^{2j_kp_n+p_n} \sum_{i_{N-1} = 2j_{N-1}p_n+p_n+1}^{2(j_{N-1}+1)p_n} \sum_{i_n = 2j_np_n+1}^{2j_np_n+p_n} \Delta_{\mathbf{i}}$$

$$T(4,\mathbf{n},X,\mathbf{j}\ ) = \sum_{i_k = 2j_kp_n+1 k=1,\dots N-2}^{2j_kp_n} \sum_{i_{N-1} = 2j_{N-1}p_n+p_n+1}^{2(j_{N-1}+1)p_n} \sum_{i_n = 2j_np_n+p_n+1}^{2(j_n+1)p_n} \Delta_{\mathbf{i}}$$

Along with others.

Lastly,

$$T(\ 2^{N-1},\mathbf{n},X,\mathbf{j}\ ) = \sum_{i_k = 2j_kp_n+p_n+1 k=1,\dots N-1}^{2(j_k+1)p_n} \sum_{i_n = 2j_np_n+1}^{2j_np_n+p_n} \Delta_{\mathbf{i}}$$

$$T(\ 2^{N},\mathbf{n},X,\mathbf{j}\ ) = \sum_{i_k = 2j_kp_n+p_n+1 k=1,\dots N}^{2(j_k+1)p_n} \Delta_{\mathbf{i}}$$

Where Δ_{i} means either Δ_{i}^{1} or$h^{-1}\Delta_{\mathrm{i}}^{2}$.

Next, put

$$U(\mathbf{n},X,I) = \sum_{\mathbf{j}\epsilon\zeta} T(I,\mathbf{n},X,\mathbf{j})$$

with $\zeta = \{0,\dots,r_1-1\} * \dots * \{0,\dots,r_N-1\}$ and $r_{\mathbf{i}} = 2n_{\mathbf{i}}p_{\mathbf{n}}^{-1}$, $\mathbf{i}$= 1, ….,N.

At the end, From $\Delta_{\mathbf{i}}\boldsymbol{'s}$, we can write,

$$\left|A_{\mathbf{n}}{}^{0} - \mathbb{E}\left[A_{\mathbf{n}}{}^{0}\right]\right| = \frac{1}{\hat{\mathbf{n}}\,\phi_x(h)} \sum_{i=1}^{2^N} U\,(\mathbf{n},X,I) \quad \mathbf{(7)}$$

Where $A_{\mathbf{n}}{}^{0}$ equal $A_{\mathbf{n}}{}^{1}$or $h^{-1}A_{\mathbf{n}}{}^{2}$.

Note that, the term say $U(\mathbf{n},X,\ 2^N+1)$ can be added as raised in **(Biau, G., and B. Cadre. 2004)**, if we don't have equalities $n_{\mathbf{i}} = \mathbf{2}r_{\mathbf{i}}p_{\mathbf{n}}{}^{\mathbf{-1}}$ (which holds the $\Delta_{\mathbf{i}}(x)'$s at the ends and it is not included in the blocks above). The proof will be not greatly changed by this.

So according to (7), for all η > 0, we have,

$$\square\left(\left|\mathbf{A_n}{}^{0} - \mathbb{E}\left[\mathbf{A_n}{}^{0}\right]\right| \geq \eta\right) \leq 2^N \max_{\mathbf{i=1\dots}} \square\left(U(\mathbf{n},X,I) \geq \eta\hat{\mathbf{n}}\,\phi_x(h)\right)$$

Lastly, the only thing left to calculate is

$$\square\left(U(\mathbf{n},X,I) \geq \eta\hat{\mathbf{n}}\,\phi_x(h)\right) \text{ FOR ALL } I = 1,\dots\dots\ 2^N$$

We are just considering the case where **i=1** to maintain generality. For such situation, we enumerate the$M = \prod_{k=1}^{N} r_k = 2^{-N}\hat{\mathbf{n}}p_{\mathbf{n}}^{-\mathbf{N}} \leq \hat{\mathbf{n}}p_{\mathbf{n}}^{-\mathbf{N}}$random variable $T(1,\mathbf{n},X,\mathbf{j}\ )$; $\mathbf{j}\epsilon\zeta$in the arbitrary way$Z_1, \dots \dots, Z_M$. Therfore, for each Z_j it exists some $\mathbf{j}_j$ in ζ such that

$$Z_{\mathbf{j}} = \sum_{\mathbf{i}\epsilon I(1,\mathbf{n},X,\ \mathbf{j}_j)} \Delta_{\mathbf{i}}$$

Where $I(1,\ \mathbf{n},\ X,\ \mathbf{j}_j) = \{\mathbf{i} : 2j_{j_k}P_{\mathbf{n}} + \mathbf{1} \leq I_K \leq 2j_{j_k}P_{\mathbf{n}} + P_{\mathbf{n}}\ ; K = 1, \dots N\}$**.** Straightforward calculations can indicate that these sets contains $p_{\mathbf{n}}^{\mathbf{N}}$ sites and are distant at least by $p_{\mathbf{n}}^{\mathbf{N}}$.

The remain of the proof is from **(Carbon, M., L. T. Tran, and B. Wu. 1997)** which allows to approximate $Z_1, \dots \dots, Z_M$ by some independent random variables $Z_1^*, \dots \dots, Z_M^*$ that has the same low as $Z_{j=1,\dots M}$ and such that

$$\sum_{j=1}^{r} \mathbb{E}|Z_{\mathbf{j}} - Z_{\mathbf{j}}^*| \leq 2CM\, p_{\mathbf{n}}^{\mathbf{N}}\psi((M-1)p_{\mathbf{n}}^{\mathbf{N}}, p_{\mathbf{n}}^{\mathbf{N}})\varphi(P_{\mathbf{n}})$$

Now, taking a look at the quantity$\mathbb{P}(U(\mathbf{n},X,1) \geq \eta)$. In there, we derive from the Bernstein and Markov inequalities that

$$\mathbb{P}(U(\mathbf{n},X,1) \geq \eta\hat{\mathbf{n}}\,\phi_x(h)) \leq B_1 + B_2$$

where

$$B_1 = \mathbb{P}\left(\left|\sum_{j=1}^{M} Z_{\mathbf{j}}^*\right| \geq \frac{M\eta\hat{\mathbf{n}}\,\phi_x(h)}{\mathbf{2}M}\right) \leq 2\exp\left(-\frac{(\eta\hat{\mathbf{n}}\,\phi_x(h))^{\mathbf{2}}}{M\,\mathrm{Var}[Z_{\mathbf{j}}^*] + Cp_{\mathbf{n}}^{\mathbf{N}}\eta\hat{\mathbf{n}}\,\phi_x(h)}\right)$$

and

$$B_2 = \mathbb{P}\left(\left|\sum_{j=1}^{M}|Z_{\mathbf{j}} - Z_{\mathbf{j}}^*|\right| \geq \frac{\eta\hat{\mathbf{n}}\,\phi_x(h)}{2}\right)$$

$$\leq \frac{\mathbf{1}}{\eta\hat{\mathbf{n}}\,\phi_x(h)}\sum_{j=1}^{M}|Z_{\mathbf{j}} - Z_{\mathbf{j}}^*|$$

$$\leq 2Mp_{\mathbf{n}}^{\mathbf{N}}(\eta\hat{\mathbf{n}}\,\phi_x(h))^{-\mathbf{1}}\psi((M-1)p_{\mathbf{n}}^{\mathbf{N}}, p_{\mathbf{n}}^{\mathbf{N}})\varphi(\mathbf{p}_{\mathbf{n}})$$

Since $\hat{\mathbf{n}} = 2^N Mp_{\mathbf{n}}^{\mathbf{N}}and\psi((M-1)p_{\mathbf{n}}^{\mathbf{N}}, p_{\mathbf{n}}^{\mathbf{N}}) \leq p_{\mathbf{n}}^{\mathbf{N}}$we get for $\eta = \eta_0\sqrt{\frac{\log\hat{\mathbf{n}}}{\hat{\mathbf{n}}\,\phi_x(h)}}$

$$B_2 \leq \hat{\mathbf{n}}p_{\mathbf{n}}^{\mathbf{N}}(\log\hat{\mathbf{n}})^{-\frac{\mathbf{1}}{2}}(\hat{\mathbf{n}}\,\phi_x(h))^{-\frac{\mathbf{1}}{2}}\varphi(\mathbf{p}_{\mathbf{n}})$$

Taking $P_{\mathbf{n}} = \left(\frac{\hat{\mathbf{n}}\,\phi_x(h)}{\log\hat{\mathbf{n}}}\right)^{\frac{1}{2N}}$**,** to get

$$B_2 \leq \hat{\mathbf{n}}\varphi(P_{\mathbf{n}}) \qquad \textbf{(8)}$$

So, combining (5) and (H5) together, we may say that

$$\sum_{\mathbf{n}} \hat{\mathbf{n}}\varphi(P_{\mathbf{n}}) < \infty$$

Initially, we asymptotically evaluate $\mathrm{Var}\left[Z_{\mathbf{j}}{}^{*}\right]$ for B_1. Indeed,

$$\mathrm{Var}\left[Z_{\mathbf{j}}{}^{*}\right] = Q_{\mathbf{n}} + R_{\mathbf{n}}$$

Where $Q_{\mathbf{n}} = \sum_{\mathbf{i}\epsilon I(1,\mathbf{n},X,1)} \mathrm{Var}[\Delta_{\mathbf{i}}(x)]$and $R_{\mathbf{n}} = \sum_{\mathbf{i}\neq\mathbf{j}\epsilon I(1,\mathbf{n},X,1)} \left|\mathrm{Cov}\left(\Delta_{\mathbf{i}}(x),\ \Delta_{\mathbf{j}}(x)\right)\right|$.

By the hypothesis (H1) and the first part of (H5), we have

$$\mathrm{Var}[\Delta_{\mathbf{i}}(x)] \leq C\left(\Phi_X(H), \left(\Phi_X(H)\right)^2\right)$$

hence

$$\mathbf{Q_n} = O\left(p_{\mathbf{n}}{}^{\mathbf{N}}\Phi_X(H)\right)$$

Additionally, we introduce the following sets for the term Rn:

$$S_1 = \{\mathbf{i}, \mathbf{j} \in I(1,\mathbf{n},X,1) : 0 < \|\mathbf{i} - \mathbf{j}\| \leq C_N\}, S_2 = \{\mathbf{i}, \mathbf{j} \in I(1,\mathbf{n},X,1) : \|\mathbf{i} - \mathbf{j}\| > C_N\}$$

Where $C_N$is a real sequence that converges to +∞ that will be defined later. Divide this sum into two separate summations over sites in S_1 and S_2

$$R_{\mathbf{n}} = \sum_{(\mathbf{i},\mathbf{j})\epsilon\mathbf{S_1}} \left|\mathrm{Cov}\left(\Delta_{\mathbf{i}}(x), \Delta_{\mathbf{j}}(x)\right)\right| + \sum_{(\mathbf{i},\mathbf{j})\epsilon\mathbf{S_2}} \left|\mathrm{Cov}\left(\Delta_{\mathbf{i}}(x), \Delta_{\mathbf{j}}(x)\right)\right| = R_{\mathbf{n}}{}^{1} + R_{\mathbf{n}}{}^{2}$$

Initially, we have:

$$R_{\mathbf{n}}{}^{1} \leq C \sum_{(\mathbf{i},\mathbf{j})\epsilon\mathbf{S_1}} \left|\mathbb{E}\left[K_{\mathbf{i}}K_{\mathbf{j}}\right]\right| + \left|\mathbb{E}[K_{\mathbf{i}}]\mathbb{E}\left[K_{\mathbf{j}}\right]\right|$$

$$\leq Cp_{\mathbf{n}}{}^{\mathbf{N}}C_{\mathbf{n}}{}^{\mathbf{N}}\Phi_X(H)\left(\left(\Phi_X(H)\right)^{\frac{1}{A}} + \Phi_X(H)\right)$$

$$\leq Cp_{\mathbf{n}}{}^{\mathbf{N}}C_{\mathbf{n}}{}^{\mathbf{N}}\left(\Phi_X(H)\right)^{\frac{A+1}{A}}$$

Rather, we have

$$R_{\mathbf{n}}{}^{2} = \sum_{(\mathbf{i},\mathbf{j})\epsilon\mathbf{S_2}} \left|\mathrm{Cov}\left(\Delta_{\mathbf{i}}, \Delta_{\mathbf{j}}\right)\right|$$

As $K_{\mathbf{j}}$ are bounded, we deduce from that **(Ibragimov, I. A., and Y. V. Linnik. 1971)**

$$\left|\mathrm{Cov}\left(\Delta_{\mathbf{i}}, \Delta_{\mathbf{j}}\right)\right| \leq C\varphi(\|\mathbf{i} - \mathbf{j}\|)$$

Therefore

$$R_{\mathbf{n}}{}^{2} \leq C \sum_{(\mathbf{i},\mathbf{j})\in S_2} \varphi(\|\mathbf{i} - \mathbf{j}\|) \leq C\, p_{\mathbf{n}}{}^{\mathbf{N}} \sum_{\mathbf{i}:\|\mathbf{i}\|\geq C_{\mathbf{n}}} \varphi(\|\mathbf{i}\|)$$

$$\leq C p_{\mathbf{n}}{}^{\mathbf{N}} C_{\mathbf{n}}{}^{-\mathbf{Na}} \sum_{\mathbf{i}:\|\mathbf{i}\|\geq C_{\mathbf{n}}} \|\mathbf{i}\|^{\mathbf{Na}} \varphi(\|\mathbf{i}\|)$$

Let $C_{\mathbf{n}} = (\Phi_X(H))^{\frac{-1}{\mathbf{Na}}}$, then we have

$$R_{\mathbf{n}}{}^{2} \leq C\, p_{\mathbf{n}}{}^{\mathbf{N}}\, C_{\mathbf{n}}{}^{-\mathbf{Na}} \sum_{\mathbf{i}:\|\mathbf{i}\|\geq C_{\mathbf{n}}} \|\mathbf{i}\|^{\mathbf{Na}} \varphi(\|\mathbf{i}\|)$$

$$\leq C\, p_{\mathbf{n}}{}^{\mathbf{N}} \Phi_X(H) \sum_{\mathbf{i}:\|\mathbf{i}\|\geq C_{\mathbf{n}}} \|\mathbf{i}\|^{\mathbf{Na}} \varphi(\|\mathbf{i}\|)$$

Due to (5) and (H3) we obtain

$$R_{\mathbf{n}}{}^{2} \leq C\, p_{\mathbf{n}}{}^{\mathbf{N}} \Phi_X(H)$$

Furthermore, under this choose of $C_{\mathbf{n}}$we have

$$R_{\mathbf{n}}{}^{1} \leq C\, p_{\mathbf{n}}{}^{\mathbf{N}} \Phi_X(H)$$

Consequently

$$\mathrm{Var}[Z_{\mathbf{j}}{}^{*}] = O(p_{\mathbf{n}}{}^{\mathbf{N}} \Phi_X(H))$$

Using this final outcome along with the definitions of $p_{\mathbf{n}}$, M and η, we obtain

$$B_1 \leq \exp(-C(\eta_0)\log\widehat{\mathbf{n}})$$

At the end, we conclude that

$$A_n - \mathbb{E}[A_n] = O_{a.co}\left(\left(\frac{\log\widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(h)}\right)^{\frac{1}{2}}\right)$$

And

$$\mathbb{E}[A_n{}^{l+1}] = \frac{1}{\phi_x(h)} \mathbb{E}[\varphi_1(\delta_0) h^{-1} \beta_1{}^{l} K_1]$$

$$\leq \frac{\mathbf{1}}{\boldsymbol{\phi}_{\mathbf{x}}(\mathbf{h})} \mathbb{E}[h^{-1} \beta_1{}^{l} K_1 1_{B(x,h)}(X_1) |\Gamma_1(x, \theta_x) - \Gamma_1(X_1, a + b\beta_1)|]$$

$$= \mathbf{O}(\mathbf{h}^{\mathbf{k_1}}) + \mathbf{o}(\mathbf{h}^{\mathbf{k_2}}).$$

Thus

$$\| A_n\| = O(h^{\min(k_1,k_2)}) + O_{a.co}\left(\left(\frac{\log\widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(h)}\right)^{\frac{1}{2}}\right)$$

Proof of Lemma 3.

We prove that

$$\sup_{\|\delta\|\leq M} \|\mathbb{E}[V_n(\delta) - A_n] + \gamma(x, \theta_x) D\delta\| = O(h^{\min(k_1,k_2)}) \qquad \textbf{(9)}$$

And

$$\sup_{\|\delta\|\leq M} \|V_n(\delta) - A_n - \mathbb{E}[V_n(\delta) - A_n]\| = O_{\mathbf{a.co}}\left(\left(\frac{\log\hat{\mathbf{n}}}{\hat{\mathbf{n}}\,\phi_x(h)}\right)^{\frac{1}{2}}\right) \qquad \textbf{(10)}$$

The initial outcome is as follow

$$V_{\mathbf{n}} - \mathbb{E}[V_n] = \begin{pmatrix} W_n{}^1(\delta) \\ W_n{}^2(\delta) \end{pmatrix}$$

Where

$$W_{\mathbf{n}}{}^{l+1}(\delta) = \frac{\mathbf{1}}{\hat{\mathbf{n}}h^l\phi_x(h)} \sum_{\mathbf{i}\in I_{\mathbf{n}}} \left(\varphi_{\mathbf{i}}(\delta) - \varphi_{\mathbf{i}}(\delta_0)\right)\beta_{\mathbf{i}}{}^l K_{\mathbf{i}}\,.$$

Observe that, by (H4) we have

$$\mathbb{E}\left[W_{\mathbf{n}}{}^{l+1}(\delta)\right]=\frac{\mathbf{1}}{\mathbf{h}^l\boldsymbol{\phi}_{\mathbf{x}}(\mathbf{h})}\,\mathbb{E}\left[\boldsymbol{\beta}_{\mathbf{1}}{}^{\mathbf{l}}\mathbf{K}_{\mathbf{1}}\mathbf{1}_{\mathbf{B(x,h)}}(\mathbf{X}_{\mathbf{1}})\left(\boldsymbol{\Gamma}(\mathbf{X}_{\mathbf{1}},(\mathbf{a}+\mathbf{c})+(\mathbf{h}^{-1}{}_{\mathbf{n}}\mathbf{d}+\mathbf{b})\boldsymbol{\beta}_{\mathbf{1}}) - \boldsymbol{\Gamma}(\mathbf{X}_{\mathbf{1}},\ \mathbf{a}+\mathbf{b}\boldsymbol{\beta}_{\mathbf{1}})\right)\right]$$

$$= \frac{1}{\mathbf{h}^l\phi_x(h)}\ \mathbb{E}\left[\beta_1{}^l K_1 1_{B(x,h)}(X_1)\,\gamma(X_1, a+b\beta_1)(1,\ h^{-1}\beta_1)\,\delta\right] + o(\|\delta\|)$$

$$= \gamma(x,\ \theta_x)\frac{1}{\mathbf{h}^l\phi_x(h)}\left(\mathbb{E}\left[\beta_1{}^l K_1\right],\ h^{-1}\mathbb{E}\left[\beta_1{}^{l+1}K_1\right]\right)\delta + O\left(\mathbf{h}^{\mathbf{min(k_1,k_2)}}\right) + o(\|\delta\|).$$

Therefore

$$\mathbb{E}[V_n(\delta) - A_n] = \gamma(x,\ \theta_x)\frac{1}{\phi_x(h)}\begin{pmatrix} \mathbb{E}[K_{\mathbf{i}}] & \mathbb{E}[K_{\mathbf{i}}\,h^{-1}\beta_{\mathbf{i}}] \\ \mathbb{E}[K_{\mathbf{i}}\,h^{-1}\beta_{\mathbf{i}}] & \mathbb{E}\left[h^{-2}\,\beta_{\mathbf{i}}{}^2K_{\mathbf{i}}{}^2\right] \end{pmatrix}\delta + O\left(\mathbf{h}^{\mathbf{min(k_1,k_2)}}\right) + o(\|\delta\|).$$

Following **(Demongeot, J., A. Laksaci, F. Madani, and M. Rachdi. 2013)**, under the second part of (H5) so that

$$h_{\mathbf{n}}^{-\mathbf{a}}\mathbb{E}\left[\beta^{\mathbf{a}}K_{\mathbf{i}}{}^{\mathbf{b}}\right] = \phi_x(h)\left(K^b(1) - \int_{-1}^{1}\left(u^aK(u)\right)' + \chi_x(u)du\right) + o\left(\phi_x(h)\right).$$

It follows that

$$\sup_{\|\delta\|\leq M}\|\mathbb{E}[V_n(\delta) - A_n] + \gamma(x,\ \theta_x)D\delta + o(\|\delta\|)\| = O\left(h^{\min(k_1,k_2)}\right).$$

Which complete the proof (9). Regarding (10), we utilize the compactness of the ball $B(0\,,M)$ in $\square^2$and we can write

$$B(0\,,M) \subset \cup_{j=1}^{d_n} B\left(\delta_j\,,\ l_{\mathbf{n}}\right)\quad,\quad \delta_j = \begin{pmatrix} c_j \\ d_j \end{pmatrix}\text{and}\ \ l_{\mathbf{n}} = d_{\mathbf{n}}^{-1} = \frac{1}{\sqrt{\mathbf{n}}}$$

By taking $j(\delta) = \text{argmin}_j\left|\delta - \delta_j\right|$ and use fact that

$$\sup_{\|\delta\|\leq M}\|V_n(\delta) - A_n - \mathbb{E}[V_n(\delta) - A_n]\| \leq \sup_{\|\delta\|\leq M}\left\|V_n(\delta) - V_n(\delta_j)\right\|\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{F_1}$$

$$+ \sup_{\|\delta\|\leq M}\left\|V_n(\delta_j) - A_n - \mathbb{E}\left[V_n(\delta_j) - A_n\right]\right\|\ldots\mathbf{F_2}$$

$$+ \sup_{\|\delta\|\leq M}\left\|\mathbb{E}\left[V_n(\delta) - V_n(\delta_j)\right]\right\|\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{F_3}$$

Concerning $\mathbf{F_1}$ and $\mathbf{F_2}$ utilizing the fact that ψ is Lipshitzian and K is bounded. In addition, under (H5) we get

$$\sup_{\|\delta\|\leq M}\left\|V_n(\delta) - V_n(\delta_j)\right\| \leq \frac{Cl_n}{\phi_x(h)} = o\left(\sqrt{\frac{\log\widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(h)}}\right)$$

it follows that

$$\sup_{\|\delta\|\leq M}\left\|V_n(\delta) - V_n(\delta_j)\right\| = O_{a.co}\left(\sqrt{\frac{\log\widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(h)}}\right)$$

Regarding $\mathbf{F_3}$ by utilizing the same steps used for F1 we have :

$$\sup_{\|\delta\|\leq M}\left\|V_n(\delta) - V_n(\delta_j)\right\| = o\left(\sqrt{\frac{\log\widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(h)}}\right)$$

Presently, we deal with the quantity $\mathbf{F_2}$, and take

$$V_n(\delta_j) - A_n - \mathbb{E}\left[V_n(\delta_j) - A_n\right] = W_n(\delta_j) - \mathbb{E}\left[W_n(\delta_j)\right] = \begin{pmatrix} W_n{}^1(\delta_j) \\ W_n{}^2(\delta_j) \end{pmatrix}$$

where

$$W_n{}^{l+1}(\delta_j) = \frac{1}{\widehat{\mathbf{n}}\,\phi_x(h)}\sum_{\mathbf{i}\in I_n}\Lambda_{\mathbf{i}}{}^{l+1}$$

and

$$\Lambda_{\mathbf{i}}{}^{l+1} = \left(\varphi_i(\delta_j) - \varphi_i(\delta_0)\right) h^{-l}\beta_{\mathbf{i}}{}^{l}K_{\mathbf{i}} - \mathbb{E}\left[\left(\varphi_i(\delta_j) - \varphi_i(\delta_0)\right) h^{-l}\beta_{\mathbf{i}}{}^{l}K_{\mathbf{i}}\right].$$

Once more we present the same arguments such given in Lemma (2). We replace the classical spatial block decomposition with precision, $\Delta_{\mathbf{i}}{}^{\mathbf{k}}$; $k = 0, 1$ by Γ_i^k ; $k = 0, 1$ .Hence $\eta > 0$ exists so that

$$\sum_n \square\left(\sup_{\|\delta\|\leq M}\left\|V_n(\delta_j) - A_n - \mathbb{E}\left[V_n(\delta_j) - A_n\right]\right\| \geq \eta\sqrt{\frac{\log\widehat{\mathbf{n}}}{\widehat{\mathbf{n}}\,\phi_x(h)}} \leq \sum_n \widehat{\mathbf{n}}^{\frac{1}{2}}(B_1 + B_2)\right).$$

When combining both (8) and (H6), we can show that

$$\sum_n \widehat{\mathbf{n}}^{\frac{1}{2}} B_2 < \infty.$$

When choosing an appropriate η_0, it allows obtaining that

$$\sum_n \widehat{\mathbf{n}}^{\frac{1}{2}} B_1 < \infty,$$

which implies the result (10).

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