Some Fixed point results related to G-metric spaces

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Abstract: In the present paper some fixed-point theorems for single mapping in complete G-metric spaces are established. Obtained results are the generalization of the previous results.
Key Words: Complete G-metric space, G-convergent, G-continuous.

1. Introduction:

Fixed point theory is an interesting subject, with an extensive application in various fields of mathematics. The fixed point of a function has certain contractive conditions due to its dynamic area of research. It has an enormous range of application in different areas such as variational, linear inequalities and parametric estimation problems.

Fixed point theory deals with the investigations leading to the existence and approximation of fixed points and had been a prominent branch of non – linear Analysis as well as Topology and the study of fixed point theorems and their applications initiated long ago and still continue to be a highly fascinating and useful area of research.

Some generalizations of the notion of a metric space have been proposed by some authors. In 2005, Z. Mustafa and B. Sims [2] & [4] introduced a more appropriate generalization of metric spaces (Q, D), that of G- metric spaces, wherein the authors discussed the topological properties of this space and proved the analog of the Banach contraction principle in the context of G- metric spaces. Following these results, many authors have studied and developed several common fixed point theorems in this framework. Some other papers dealing with G- metric spaces are those in [2, 3, 5].

2. Preliminaries:

Definition 2.1: Consider Q be a nonempty set and let $G: Q \times Q \times Q \rightarrow R^{+}$ be a function satisfying the following axioms.

$(G_1)\ G(\alpha,\beta,\gamma) = 0\ if\ \alpha = \beta = \gamma,$

$(G_2)\ 0 < G\ (\alpha,\alpha,\beta) for\ all\ \alpha,\beta \in Q\ with\ \alpha \neq \beta,$

$(G_3) G(\alpha,\alpha,\beta) \leq G(\alpha,\beta,\gamma)\ for\ all\ \alpha,\beta,\gamma \in Q\ with\ \gamma \neq \beta,$

$(G_4) G(\alpha,\beta,\gamma) = G\ (\alpha,\gamma,\beta) = G(\beta,\gamma,\alpha)(symmetry\ in\ all\ three\ variable),$

$(G_5) G(\alpha,\beta,\gamma) \leq G(\alpha,n,n) + G(n,\beta,\gamma) for\ all\ \alpha,\beta,\gamma,n \in Q$ $(rectangle\ inequality)$.

Then the function G is called a generalized metric or G- metric on $Q$and (Q,G) is called a G-metric space.

Definition 2.2 [5]: Let (Q, G) be a G- metric space, let $\{\alpha_i\}$ be a sequence of points of Q, a point $\alpha \in Q$ is said to the limit of the sequence $\{\alpha_i\}$, if $\lim_{i\to\infty} G(\alpha,\alpha_i,\alpha_p) = 0$. Then $\{\alpha_i\}$ is G- convergent to Q.

Proposition 2.3 [5]: Let (Q,G) be a G- metric space. Then for any $\alpha,\beta,\gamma,n \in Q$, it follows that

(i) $If\ G(\alpha,\beta,\gamma) = 0\ then\ \alpha = \beta = \gamma,$
(ii) $G(\alpha,\beta,\gamma) \leq G(\alpha,\alpha,\beta) + G(\alpha,\alpha,\gamma),$
(iii) $G(\alpha,\beta,\beta) \leq 2G(\beta,\alpha,\alpha),$
(iv) $G(\alpha,\beta,\gamma) \leq G(\alpha,n,\gamma) + G(n,\beta,\gamma)$
(v) $G(\alpha,\beta,\gamma) \leq \frac{2}{3}(G(\alpha,\beta,n) + G(\alpha,n,\gamma) + G(\alpha,n,\gamma))$
(vi) $G(\alpha,\beta,\gamma) \leq \left(G(\alpha,n,n) + G(\beta,n,n)G(\gamma,n,n)\right)$

Proposition 2.4 [5]: Let (Q,G) be a G- metric space. Then for a sequence $\{\alpha_i\} \subseteq Q$ and a point $\alpha \in Q$.

The following are equivalent

(i) $\{\alpha_i\}$ is G- metric to α;
(ii) $G(\alpha_i,\alpha_i,\alpha) \to 0\ as\ i \to \infty$;
(iii) $G(\alpha_i,\alpha,\alpha) \to 0\ as\ i \to \infty$;
(iv) $G(\alpha_p,\alpha_i,\alpha) \to 0\ as\ p,i \to \infty$.

Definition 2.5 [5]: Let (Q,G) be a G- metric space. Then the sequence $\{\alpha_i\}$ is said to be G- Cauchy if for every $\varepsilon > 0$, there exists $N\ such\ that\ G(\alpha_i,\alpha_p,\alpha_n) < \varepsilon$, for all $i,p,n \geq N$ i.e. $G(\alpha_i,\alpha_p,\alpha_n) \to 0\ as\ i,p,n \to \infty$

Definition 2.6 [5]: A G- metric space (Q,G) is said to be G-complete if every G-Cauchy sequence in (Q,G) is G-convergent in (Q,G).

Preposition 2.7 [5]: Let $(Q,G),(Q'.G')$ be G-metric space, then a function $T: Q \to Q'$ is G- continuous at a point $\alpha \in Q$ if and only if it is G-sequentially continuous at $\alpha; i.e.$ whenever $\{\alpha_i\}$ is G-convergent to $\alpha, \{T(\alpha_i)\}$ is G-convergent to $T(\alpha)$.

3. Main results:

Theorem 3.1. Let (Q,G)be a complete G-metric space and let $S : Q \to Q$ be a mapping which satisfies the following conditions:

$$G(S\alpha,S\beta,S\gamma) \leq \sigma \frac{[G(\alpha,S\beta,S\beta)+G(\beta,S\alpha,S\alpha)+G(S\alpha,\beta,\gamma)]}{1+G(\beta,S\alpha,S\alpha)G(S\alpha,\beta,\gamma)} + \mu \frac{[G(\alpha,S\gamma,S\gamma)+G(\gamma,S\alpha,S\alpha)+G(S\alpha,\beta,\gamma)}{1+G(\gamma,S\alpha,S\alpha)G(S\alpha,\beta,\gamma)} + \vartheta[G(\beta,S\gamma.S\gamma) + G(\gamma,S\beta,S\beta)] + \rho G(\alpha,\beta,\gamma) \quad 3.1$$

Foe all $\alpha,\beta,\gamma \in Q$. Then S has a unique fixed point and S is G-continuous at that point.

Proof: Let $\alpha_0 \in Q$ be a arbitrary point and define a sequence $\{\alpha_i\}$ with $\alpha_i = S\alpha_{i-1}$, then by 3.1, we get

$$G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) = G(S\alpha_{i-1}, S\alpha_i, S\alpha_i)$$

$$\leq \sigma \frac{[G(\alpha_{i-1}, S\alpha_i, S\alpha_i) + G(\alpha_i, S\alpha_{i-1}, S\alpha_{i-1}) + G(S\alpha_{i-1}, \alpha_i, \alpha_i)]}{1 + G(\alpha_i, S\alpha_{i-1}, S\alpha_{i-1})G(S\alpha_{i-1}, \alpha_i, \alpha_i)} + \mu \frac{[G(\alpha_{i-1}, S\alpha_i, S\alpha_i) + G(\alpha_i, S\alpha_{i-1}, S\alpha_{i-1}) + G(S\alpha_{i-1}, \alpha_i, \alpha_i)]}{1 + G(\alpha_i, S\alpha_{i-1}, S\alpha_{i-1})G(S\alpha_{i-1}, \alpha_i, \alpha_i)} + \vartheta[G(\alpha_i, S\alpha_i, S\alpha_i) + G(\alpha_i, S\alpha_i, S\alpha_i)] + \rho G(\alpha_{i-1}, \alpha_i, \alpha_i)$$

$$\leq \sigma \frac{[G(\alpha_{i-1}, \alpha_{i+1}, \alpha_{i+1}) + G(\alpha_i, \alpha_i, \alpha_i) + G(\alpha_i, \alpha_i, \alpha_i)]}{1 + G(\alpha_i, \alpha_i, \alpha_i)G(\alpha_i, \alpha_i, \alpha_i)} + \mu \frac{[G(\alpha_{i-1}, \alpha_{i+1}, \alpha_{i+1}) + G(\alpha_i, \alpha_i, \alpha_i) + G(\alpha_i, \alpha_i, \alpha_i)]}{1 + G(\alpha_i, \alpha_i, \alpha_i)G(\alpha_i, \alpha_i, \alpha_i)} + \vartheta[G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) + G(\alpha_i, \alpha_{i+1}, \alpha_{i+1})] + \rho G(\alpha_{i-1}, \alpha_i, \alpha_i)$$

$$\leq \sigma G(\alpha_{i-1}, \alpha_{i+1}, \alpha_{i+1}) + \mu G(\alpha_{i-1}, \alpha_{i+1}, \alpha_{i+1}) + 2\vartheta G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) + \rho G(\alpha_{i-1}, \alpha_i, \alpha_i) \tag{3.2}$$

But by (G_3), we have

$$G(\alpha_{i-1}, \alpha_{i+1}, \alpha_{i+1}) \leq G(\alpha_{i-1}, \alpha_i, \alpha_i) + G(\alpha_i, \alpha_{i+1}, \alpha_{i+1})$$

So, 3.2 becomes

$$G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) \leq (\sigma + \mu)[G(\alpha_{i-1}, \alpha_i, \alpha_i) + G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) + 2\vartheta G\left(\alpha_{i,}\alpha_{i+1}\alpha_{i+1}\right) + \rho G(\alpha_{i-1}, \alpha_i, \alpha_i)$$

$$G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) \leq \frac{\sigma + \mu + \rho}{1 - (\sigma + \mu + 2\vartheta)} G(\alpha_{i-1}, \alpha_i, \alpha_i)$$

$$G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) \leq MG(\alpha_{i-1}, \alpha_i, \alpha_i)$$

$where\ M = \dfrac{\sigma + \mu + \rho}{1 - (\sigma + \mu + 2\vartheta)} < 1$

Continuing in the same logic, we get

$$G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) \leq M^i G(\alpha_0, \alpha_1, \alpha_1) \tag{3.3}$$

Thus for all $i, p \in N. i < p$, we have by rectangular inequality that

$$G\left(\alpha_i, \alpha_p, \alpha_p\right) \leq G(\alpha_i, \alpha_{i+1}, \alpha_{i+1}) + G(\alpha_{i+1}, \alpha_{i+2}, \alpha_{i+2}) + \_\_\_\_\ \_ + G(\alpha_{p-1}, \alpha_p, \alpha_i)$$

$$\leq (M^I + M^{i+1} + \_\_\_\_\_ + M^{p-1})G(\alpha_0, \alpha_1, \alpha_1)$$

$$\leq \frac{M^i}{M-1} G(\alpha_0, \alpha_1, \alpha_1)$$

Taking limit as $i, p \ \to \infty$, we get

$$limG(\alpha_i, \alpha_p, \alpha_p) = 0$$

Thus $\{\alpha_i\}$ is a Cauchy sequence.

By the completeness of (Q, G), there exists a point $r \in Q$ such that $\{\alpha_i\}$ is a G-convergent to r. Suppose that $St \neq t$, then

$$G(\alpha_i, Sr, Sr) \leq \sigma \frac{[G(\alpha_{i-1}, Sr, Sr) + G(r, S\alpha_{i-1}, S\alpha_{i-1}) + G(S\alpha_{i-1}, r, r)]}{1 + G(r, S\alpha_{i-1}, S\alpha_{i-1})G(S\alpha_{i-1}, r, r)} + \mu \frac{[G(\alpha_{i-1}, Sr, Sr) + G(r, S\alpha_{i-1}, S\alpha_{i-1}) + G(S\alpha_{i-1}, r, r)]}{1 + G(r, S\alpha_{i-1}, S\alpha_{i-1})G(S\alpha_{i-1}, r, r)} + \vartheta[G(r, Sr, Sr) + G(r, Sr, Sr)] + \rho G(\alpha_{i-1}, r, r)$$

$$= \sigma \frac{[G(\alpha_{i-1}, Sr, Sr) + G(r, \alpha_i, \alpha_i) + G(\alpha_i, r, r)]}{1 + G(r, \alpha_i, \alpha_i)G(\alpha_i, r, r)} + \mu \frac{[G(\alpha_{i-1}, Sr, Sr) + G(r, \alpha_i, \alpha_i) + G(\alpha_i, r, r)]}{1 + G(r, \alpha_i, \alpha_i)G(\alpha_i, r, r)} + 2\vartheta G(r, Sr, Sr) + \rho G(\alpha_{i-1}, r, r)$$

Taking the limit as $i \to \infty$ and using the fact that G is continuous. Then

$$G(r, Sr, Sr) \leq \sigma G(r, Sr, Sr) + \mu G(r, Sr, Sr) + 2\vartheta G(r, Sr, Sr)$$

$$\leq (\sigma + \mu + 2\vartheta)G(r, Sr, Sr)$$

This is a contradiction. Hence $r = Sr$. Since $\sigma + \mu + 2\vartheta < 1$.

Uniqueness: Let c be another fixed point in Q. Suppose that $c \neq r$ such that $Sc = c$ then (2.1) implies that

$$G(r, c, c) = G(Sr, Sc, Sc)$$

$$\leq \sigma \frac{[G(r, Sc, Sc) + G(c, Sr, Sr) + G(Sr, c, c)]}{1 + G(c, Sr, Sr)G(Sr, c, c)} + \mu \frac{[G(r, Sc, Sc) + G(c, Sr, Sr) + G(Sr, c, c)]}{1 + G(c, Sr, Sr)G(Sr, c, c)} + \vartheta[G(c, Sc, Sc) + G(c, Sc, Sc)] + \rho G(r, c, c)$$

$$\leq \sigma \frac{[G(r, c, c) + G(c, r, r) + G(r, c, c)]}{1 + G(c, r, r)G(r, c, c)} + \mu \frac{[G(r, c, c) + G(c, r, r) + G(r, c, c)]}{1 + G(c, r, r)G(r, c, c)} + \rho G(r, c, c)$$

$(1 - \rho)G(r, c, c) \leq (\sigma + \mu)[2G(r, c, c) + G(c, r, r)]$

$$G(r, c, c) \leq \frac{(\sigma + \mu)}{(1 - \rho)}[2G(r, c, c) + G(c, r, r)]$$

Or $\quad G(r, c, c) \leq B[2G(r, c, c) + G(c, r, r)] \qquad (3.4)$

This implies that $G(r, c, c) \leq BG(c, r, r)$ and by repeated use of the same logic we will find

$$G(c, r, r) \leq BG(r, c, c).$$

Therefore, from (3.4), we get

$$G(r,c,c) \leq B^2 G(r,c,c)$$

Which is a contradiction then $r = c$ as $B < 1$.

To show that S is G- continuous at r. Let $\{\beta_i\}$ be a sequence such that $\lim \beta_i = r$.

Then

$$G(S\beta_i, Sr, Sr) \leq \sigma \frac{[G(\beta_i, Sr, Sr) + G(r, S\beta_i, S\beta_i) + G(S\beta_i, r, r)]}{1 + G(r, S\beta_i, S\beta_i)G(S\beta_i, r, r)} + \mu \frac{[G(\beta_i, Sr, Sr) + G(r, S\beta_i, S\beta_i) + G(S\beta_i, r, r)]}{1 + G(r, S\beta_i, S\beta_i)G(S\beta_i, r, r)} + \vartheta[G(r, Sr, Sr) + G(r, Sr, Sr)] + \rho G(\beta_i, r, r)$$

$$G(S\beta_i, r, r) \leq \sigma \frac{[G(\beta_i, r, r) + G(r, S\beta_i, S\beta_i) + G(S\beta_i, r, r)]}{1 + G(r, S\beta_i, S\beta_i)G(S\beta_i, r, r)} + \mu \frac{[G(\beta_i, r, r) + G(r, S\beta_i, S\beta_i) + G(S\beta_i, r, r)]}{1 + G(r, S\beta_i, S\beta_i)G(S\beta_i, r, r)} + \rho G(\beta_i, r, r)$$

$$\leq \sigma[G(\beta_i, r, r) + G(r, S\beta_i, S\beta_i) + G(S\beta_i, r, r)] + \mu[G(\beta_i, r, r) + G(r, S\beta_i, S\beta_i) + G(S\beta_i, r, r)] + \rho G(\beta_i, r, r)$$

$$G(S\beta_i, r, r) \leq (\sigma + \mu + \rho)G(\beta_i, r, r) + (\sigma + \mu)[(r, S\beta_i, \beta_i) + (S\beta_i, r, r)] \quad (3.5)$$

But by proposition 2.3(3) we have

$$G(r, S\beta_i, \beta_i) \leq 2G(S\beta_i, r, r)$$

Therefore (3.5) implies that

$$G(S\beta_i, r, r) \leq (\sigma + \mu + \rho)G(\beta_i, r, r) + 3(\sigma + \mu)(S\beta_i, r, r)$$

We deduce that

$$G(S\beta_i, r, r) \leq \frac{\sigma+\mu+\vartheta}{1-3(\sigma+\mu)} G(\beta_i, \mathrm{r}, \mathrm{r})$$

Taking the limit as $i \to \infty$. We observe that

$G(S\beta_i, r, r) \to 0$ and so by proposition 2.4, we have

$S\beta_i$, is G- convergent to $r = Sr$. This implies that S is G- continuous at r.

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