Soc-QP2- Absorbing Submodules

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Abstract
Let R be a unitary left R-module and T be a self-identifiable commutative ring We introduce and analyze the idea of socle-quasi-primary-2-absorbing submodules, which is a combination of primary and 2-absorbing submodules that considers a proper submodule L of an R-module. Socle-quasi-primary is abbreviated as T. T is made up of two absorbing submodules. Soc-QP2-absorbing), if whenever $rst \in L$ for $r, s \in R$, $t \in T$, implies one of two possibilities$rt \in T - rad(L) + soc(T)$ or $st \in T - rad(L) + soc(T)$ or $rs \in \sqrt{[L + soc(T):_R T]}$. The qualities, characterizations, and examples of this innovative notion are described.

Keywords: Prime submodules, Primary submodules, 2-absorbing submodules, Socle of modules. Radical of submodules, Multiplication modules, Non-singular modules.

1. Introduction

To begin, a prime submodule is a proper submodule L of an R-module, a well-known concept in the field of modules theory. When rtL stands for rR, tT stands for either. tL or rTL [1], T is referred to as prime. A generalization of prime submodule is the term main submodule. A correct submodule L of an R-module is as follows: T is referred to as a major submodule if it is used whenever $rt \in L$, for $r \in R$, $t \in T$, implies that either $t \in L$ or $r^nT \subseteq L$ for some $n \in Z^+$ [2]. Recently 2-absorbing Darani and Soheilinia prime submodule was developed as a generalization of prime submodule in [3], a suitable submodule is referred to as 2-absorbing submodule L of an R-module T., if whenever $rst \in L$ for $r, s \in R$, $t \in T$, implies that either $rt \in L$ or $st \in L$ or $rsT \subseteq L$. Weakly 2-absorbing Only a few generalizations of 2-absorbing submodules have been proposed, including submodules, semi-2-absorbing submodules, and 2-absorbing primary submodules. , submodules that are approximately semi-2-absorbing, and pseudo-2-absorbing submodules [3,4,5,6,7]. Tekir U. et. in [8] created the notion of a 2-absorbing quasi-primary ideal, which replaces a proper ideal with a 2-absorbing quasi-primary ideal..I of a ring R is 2-absorbing quasi-primary ideal if and also only when, and only if, and only if, and only if, and only if, and $abc \in I$, then $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$ for each $a, b, c \in R$, where $\sqrt{I}$ is defined as the intersection of all prime ideals of Rcontining I, or $\sqrt{I} = \{a \in R: a^n \in I, for\ some\ n \in Z^+\}$. In 2017 Koc, S. et. A The 2-absorbing quasi-primary ideal is a generalization of the appropriate submodule, and the 2-absorbing quasi-primary submodule is a generalization of the 2-absorbing quasi-primary ideal. An R-L module's T is referred to be a 2-absorbing-quasi-primary submodule of T if rstL, where r,sR, tT, denotes that either $rt \in T - rad(L)$ or $st \in T - rad(L)$ or $rs \in \sqrt{[L:_R T]}$ [9], where $T - rad(L)$ T's intersection with all prime submodules containing L [9]. We presented the notion of socle quasi-primary 2-absorbing submodule as an extension of the 2-absorbing submodule. This concept, socle of an R-module, has numerous basic properties, characterizations, and instances. The point T, indicated by soc(T), is where all of the necessary submodules come together of T [10]. An R-module T is Multiplication is the process of multiplying each submodule.L of T is of the form $L = IT$ for some ideal I of R [13]. An R-module T is called non-singular if $Z(T) = T$ where $Z(T) = \{t \in T: tJ = (0)\ for\ some\ essential\ J\ of\ R\}$ [10]. Finally, throughout this analysis, we assume that all rings are commutative and that all R-modules are left unitary.

2. Socle-quasi-primary 2-absorbing Submodules

In this part, we define the term "socle quasi-primary 2-absorbing submodule" and define some of its fundamental features and characterizations.

Definition (2.1)

A suitable submodule Obtain an R-module. T is a 2-absorbing submodule of the socle-quasi-primary T (for short Soc-QP2-absorbing), if whenever $rst \in L$ for $r, s \in R, t \in T$, implies that either $rt \in T - rad(L) + soc(T)$ or $st \in T - rad(L) + soc(T)$ or $rs \in \sqrt{[L + soc(T):_R T]}$. And a proper ideal J of a ring R is said to be a Soc-QP2-absorbing ideal of R if J is a Soc-QP2-absorbing submodule of an R-module R.

Remarks and Examples (2.2)

1) It is clear that every 2-absorbing submodule of an R-module T is Soc-QP2- absorbing submodule, but not the other way around. The following example demonstrates this:
Consider the Z-module Z_{12} , the submodule $L = \langle\bar{0}\rangle$ is 2-absorbing submodule of Z-module Z_{12} because $2.3.\bar{2} \in L$, where $2,3 \in Z, \bar{2} \in Z_{12}$, then $2.\bar{2} = \bar{4} \notin L$ and $3.\bar{2} = \bar{6} \notin L$ and $2.3 = 6 \notin [L:_Z Z_{12}] = [\langle\bar{0}\rangle:_Z Z_{12}] = 12Z$. But L is Soc-QP2-absorbing submodule of Z_{12} since $soc(Z_{12}) = \langle\bar{2}\rangle$ and $Z_{12} - rad(L) = \langle\bar{6}\rangle$ and for all $r, s \in Z,\ t \in Z_{12}$ with $rst \in L$ implies that either $rt \in Z_{12} - rad(L) + soc(Z_{12}) = \langle\bar{6}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle$ or $st \in Z_{12} - rad(L) + soc(Z_{12}) = \langle\bar{2}\rangle$ or $rs \in \sqrt{[\langle\bar{0}\rangle + soc(Z_{12}):_Z Z_{12}]} = \sqrt{[\langle\bar{2}\rangle:_Z Z_{12}]} = \sqrt{2Z} = 2Z$. That is if $2.3.\bar{2} \in L$, implies that $2.\bar{2} = \bar{4} \in \langle\bar{2}\rangle$ or $3.\bar{2} = \bar{6} \in \langle\bar{2}\rangle$ or $2.3 = 6 \in 2Z$.

2) Every primary submodule of an R-module is obvious.T is Soc-QP2-absorbing submodule of T, IIn general, however, this is not the case. This is demonstrated in the example below. Take a look at the Z-module. Z_{12}, the submodule. $L = \langle\bar{0}\rangle$ is a Soc-QP2-absorbing submodule of Z_{12} by (1), but $L = \langle\bar{0}\rangle$ is not primary, since $3.\bar{4} \in L$, for $3 \in Z$ and $\bar{4} \in Z_{12}$, but $\bar{4} \notin L$ and $3 \notin \sqrt{[\langle\bar{0}\rangle:_Z Z_{12}]} = \sqrt{12Z} = 6Z$.

3) Every prime submodule of an R-module is obvious. T is is Soc-QP2-absorbing submodule of T, But not the other way around. The example below demonstrates this.

Consider the Z-module Z_4 , the submodule $L = \langle\bar{0}\rangle$ is not prime submodule of Z_4, since $2.\bar{2} \in L$, for $2 \in Z$, $\bar{2} \in Z_4$, but $\bar{2} \notin L$ and $2 \notin [\langle\bar{0}\rangle :_Z Z_4] = 4Z$. While L is Soc-QP2-absorbing submodule of Z_4, since $soc(Z_4) = \langle\bar{2}\rangle$ and for all $r, s \in Z$, $t \in Z_4$ such that $rst \in L$, implies that either $rt \in Z_4 - rad(L) + soc(Z_4) = \langle\bar{2}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle$ or $st \in \langle\bar{2}\rangle$ or $rs \in \sqrt{[\langle\bar{0}\rangle :_Z Z_4]} = \sqrt{4Z} = 2Z$. That is if $2.1.\bar{2} \in L$, for $2,1 \in Z$, $\bar{2} \in Z_4$ implies that either $2.\bar{2} = \bar{0} \in Z_4 - rad(L) + soc(Z_4) = \langle\bar{2}\rangle$ and $1.\bar{2} \in \langle\bar{2}\rangle$ and $2.1 \in \sqrt{[\langle\bar{0}\rangle + soc(Z_4) :_Z Z_4]} = \sqrt{[\langle\bar{2}\rangle :_Z Z_4]} = \sqrt{2Z} = 2Z$.

The assertions that follow are descriptions of Soc-QP2-absorbing submodules.

Proposition (2.3)

Let L be a proper submodule of an R-module T. Then L is Soc-QP2-absorbing submodule of T if and only if for each $r, s \in R$ with $rs \notin \sqrt{[L + soc(T) :_R T]}$ and $[L :_T rs] \subseteq [T - rad(L) + soc(T) :_T r] \cup [T - rad(L) + soc(T) :_T s]$.

Proof

(⇒⇒) Let $t \in [L :_T rs]$ for $r, s \in R$ and $rs \notin \sqrt{[L + soc(T) :_R T]}$, implies that $rst \in L$. Since Then L is a submodule of $Soc - QP2$ that absorbs QP2 of T and $rs \notin \sqrt{[L + soc(T) :_R T]}$ implies that either $rt \in T - rad(L) + soc(T)$ or $st \in T - rad(L) + soc(T)$. It follows that either $t \in [T - rad(L) + soc(T) :_T r]$ or $t \in [T - rad(L) + soc(T) :_T s]$. Thus $t \in [T - rad(L) + soc(T) :_T r] \cup [T - rad(L) + soc(T) :_T s]$. Hence $[L :_T rs] \subseteq [T - rad(L) + soc(T) :_T r] \cup [T - rad(L) + soc(T) :_T s]$.

(⇐==) Suppose that $rst \in L$ where $r, s \in R$, $t \in T$ with $rs \notin \sqrt{[L + soc(T) :_R T]}$. Thus $t \in [L :_T rs]$, but by hypothesis $[L :_T rs] \subseteq [T - rad(L) + soc(T) :_T r] \cup [T - rad(L) + soc(T) :_T s]$, it follows that $t \in [T - rad(L) + soc(T) :_T r]$ or $t \in [T - rad(L) + soc(T) :_T s]$. Hence $rt \in T - rad(L) + soc(T)$ or $st \in T - rad(L) + soc(T)$. That is L isSoc-QP2-absorbing submodule of T.

Proposition (2.4)

Let L be a proper submodule of an R-module T. Then L is Soc-QP2-absorbing submodule of T if and only if $rsN \subseteq L$, for $r, s \in R$ and N is a submodule of T with $rs \notin \sqrt{[L + soc(T) :_R T]}$ implies that either $rN \subseteq T - rad(L) + soc(T)$ or $sN \subseteq T - rad(L) + soc(T)$.

Proof

(⇒⇒) Suppose $rsN \subseteq L$, for $r, s \in R$ and N is a submodule of T with $rs \notin \sqrt{[L + soc(T) :_R T]}$, implies that $N \subseteq [L :_T rs]$. Since L is Soc-QP2- If T's absorbing sub module is true, then proposition is true. (2.3) $[L :_T rs] \subseteq [T - rad(L) + soc(T) :_T r] \cup [T - rad(L) + soc(T) :_T s]$, it follows that either $N \subseteq [T - rad(L) + soc(T) :_T r]$ or $N \subseteq [T - rad(L) + soc(T) :_T s]$. That is either $rN \subseteq T - rad(L) + soc(T)$ or $sN \subseteq T - rad(L) + soc(T)$.

(⇐==) Let $rst \in L$, for $r, s \in R, t \in T$, with $rs \notin \sqrt{[L + soc(T) :_R T]}$. Since $rs(t) \subseteq L$, then by hypothesis $r(T) \subseteq T - rad(L) + soc(T)$ or $s(T) \subseteq T - rad(L) + soc(T)$. That is $rT \in$

$T - rad(L) + soc(T)$ or $T \in T - rad(L) + soc(T)$. Thus L is Soc-QP2-absorbing sub module of T.

Proposition (2.5)

Let L be a proper sub module of an R-module T. Then L is Soc-QP2-absorbing sub module of T if and only if $rIN \subseteq L$, for $r \in R$, I is A sub module is an ideal of R. and N. of T, implies that either $rN \subseteq T - rad(L) + soc(T)$ or $IN \subseteq T - rad(L) + soc(T)$ or $rI \subseteq \sqrt{[L + soc(T):_R T]}$

Proof

(⇒⇒) Suppose that $rIN \subseteq L$, $rI \nsubseteq \sqrt{[L + soc(T):_R T]}$ and $IN \nsubseteq T - rad(L) + soc(T)$ it follows that $ra \notin \sqrt{[L + soc(T):_R T]}$ and $bN \nsubseteq T - rad(L) + soc(T)$ for some $a, b \in I$. We must show that $rN \subseteq T - rad(L) + soc(T)$. Suppose that $rN \nsubseteq T - rad(L) + soc(T)$. Since $raN \subseteq L$, and L is a Soc-QP2-absorbing sub module of T, then by proposition (2.4) $aN \subseteq T - rad(L) + soc(T)$, and also $(a + b)N \nsubseteq T - rad(L) + soc(T)$. Now $r(a + b)N \subseteq L$, and again by proposition (2.4) we have $r(a + b) = ra + rb \in \sqrt{[L + soc(T):_R T]}$, and $ra \notin \sqrt{[L + soc(T):_R T]}$, we get $ra = b \notin \sqrt{[L + soc(T):_R T]}$. Since $rbN \subseteq L$ again by proposition (2.4) we get $bL \subseteq T - rad(L) + soc(T)$ or $rN \subseteq T - rad(L) + soc(T)$ which is contradiction.

(⇐==) Suppose that $rsN \subseteq L$, for $r, s \in R$, and N is a submodule of T then $r(s)N \subseteq L$, it follows by hypothesis either $rN \subseteq T - rad(L) + soc(T)$ or $(s)N \subseteq T - rad(L) + soc(T)$ or
$r(s) \subseteq \sqrt{[L + soc(T):_R T]}$. That is $rN \subseteq T - rad(L) + soc(T)$ or $sN \subseteq T - rad(L) + soc(T)$ or $rs \subseteq \sqrt{[L + soc(T):_R T]}$. Hence by proposition (2.4) L is Soc-QP2-absorbing submodule of T.

Proposition (2.6)

Let L be a proper submodule of an R-module T. Then L is Soc-QP2-absorbing submodule of T if and only if $IJN \subseteq L$, for I, J are examples of R and N is a subunit of T, implies that either $IN \subseteq T - rad(L) + soc(T)$ or $JN \subseteq T - rad(L) + soc(T)$ or $IJ \subseteq \sqrt{[L + soc(T):_R T]}$

Proof

(⇒⇒) Suppose that $IJN \subseteq L$, where I, J are examples of R and N is a subunit of T and suppose that $IJ \nsubseteq \sqrt{[L + soc(T):_R T]}$. We want to prove that $IN \subseteq T - rad(L) + soc(T)$ or $JN \subseteq T - rad(L) + soc(T)$. Assume that $IN \nsubseteq T - rad(L) + soc(T)$ and $JN \nsubseteq T - rad(L) + soc(T)$, then there exists $s_1 \in I$ and $s_2 \in J$ such that $s_1N \nsubseteq T - rad(L) + soc(T)$ and $s_2N \nsubseteq T - rad(L) + soc(T)$. Now $s_1s_2N \subseteq L$ with $s_1N \nsubseteq T - rad(L) + soc(T)$ and $s_2N \nsubseteq T - rad(L) + soc(T)$. and L is Soc-QP2-absorbing submodule of T, then by proposition (2.4) $s_1s_2 \in \sqrt{[L + soc(T):_R T]}$. Since $IJ \nsubseteq \sqrt{[L + soc(T):_R T]}$, then there is$a \in I$, $b \in J$ such that $ab \notin \sqrt{[L + soc(T):_R T]}$. Since $abN \subseteq L$, and $ab \notin \sqrt{[L + soc(T):_R T]}$, then again by proposition (2.4) either $aN \subseteq T - rad(L) + soc(T)$ or $bN \subseteq T - rad(L) + soc(T)$.

Now we have these cases:

Case one: Suppose that $aN \subseteq T - rad(L) + soc(T)$ but $bN \nsubseteq T - rad(L) + soc(T)$. Since $s_1bN \subseteq L$ and L is Soc-QP2-absorbing submodule of T with $bN \nsubseteq T - rad(L) + soc(T)$ and $s_1N \nsubseteq T - rad(L) + soc(T)$, then by proposition(2.4) $s_1b \in \sqrt{[L + soc(T):_R T]}$. Also since $aN \subseteq T - rad(L) + soc(T)$ but $s_1N \nsubseteq T - rad(L) + soc(T)$, then $(s_1 + a)N \nsubseteq T - rad(L) + soc(T)$. Now since $(s_1 + a)bN \subseteq L$ and $bN \nsubseteq T - rad(L) + soc(T)$ and $(s_1 + a)N \nsubseteq T - rad(L) + soc(T)$, then by proposition(2.4) $(s_1 + a)b \in \sqrt{[L + soc(T):_R T]}$. That is $(s_1 + a)b = s_1b + ab \in \sqrt{[L + soc(T):_R T]}$ and $s_1b \in \sqrt{[L + soc(T):_R T]}$, implies that $ab \in \sqrt{[L + soc(T):_R T]}$ a contradiction.

Case two: If $bN \subseteq T - rad(L) + soc(T)$ but $aN \nsubseteq T - rad(L) + soc(T)$ in similarly steps of Case one we get a contradiction.

Case three: Suppose that $aN \subseteq T - rad(L) + soc(T)$ and $bN \subseteq T - rad(L) + soc(T)$. Now since $bN \subseteq T - rad(L) + soc(T)$ and $s_2N \nsubseteq T - rad(L) + soc(T)$, then $(s_2 + b)N \nsubseteq T - rad(L) + soc(T)$. Also we have $s_1(s_2 + b)N \subseteq L$ and $s_1N \nsubseteq T - rad(L) + soc(T)$ and $(s_2 + b)N \nsubseteq T - rad(L) + soc(T)$, then by proposition(2.4) $s_1(s_2 + b) = s_1s_2 + s_1b \in \sqrt{[L + soc(T):_R T]}$, then $s_1b \in \sqrt{[L + soc(T):_R T]}$. Now, since $aN \subseteq T - rad(L) + soc(T)$ and $s_1N \nsubseteq T - rad(L) + soc(T)$, then $(s_1 + a)N \nsubseteq T - rad(L) + soc(T)$. Also, since $(s_1 + a)s_2N \subseteq L$ and $s_2N \nsubseteq T - rad(L) + soc(T)$ and $(s_1 + a)N \nsubseteq T - rad(L) + soc(T)$, then by proposition (2.4) $(s_1 + a)s_2 = s_1s_2 + as_2 \in \sqrt{[L + soc(T):_R T]}$. Now, since $s_1s_2 \in \sqrt{[L + soc(T):_R T]}$ and $s_1s_2 + as_2 \in \sqrt{[L + soc(T):_R T]}$, then $as_2 \in \sqrt{[L + soc(T):_R T]}$. Also , since $(s_1 + a)(s_2 + b)N \subseteq L$ and $(s_1 + a)N \nsubseteq T - rad(L) + soc(T)$ and $(s_2 + b)N \nsubseteq T - rad(L) + soc(T)$, then by proposition (2.4) $(s_1 + a)(s_2 + b) = s_1s_2 + s_1b + as_2 + ab \in \sqrt{[L + soc(T):_R T]}$. Again since $s_1s_2,\ s_1b,\ as_2 \in \sqrt{[L + soc(T):_R T]}$, we have $ab \in \sqrt{[L + soc(T):_R T]}$ a contradiction. Thus either $IN \subseteq T - rad(L) + soc(T)$ or $JN \subseteq T - rad(L) + soc(T)$.

($\Leftarrow$) By proposition, the evidence is direct. (2.5).

The following lemmas must be remembered before we may go to the next step.

Lemma (2.7) [11, lemma (2.3.15)]

Let A, B and C are submodules of an R-module T with $B \subseteq C$, then $(A+B) \cap C = (A \cap C) + B = (A \cap C) + (B \cap C)$.

Lemma (2.8) [12, Coro. (9.9)]

Let N be a submodule of an R-module T, then $soc(N) = N \cap soc(T)$.

Proposition (2.9)

Let A and B are proper submodules of an R-module T with $A \subsetneq B$ and $soc(T) \subseteq B$. If A is a Soc-QP2-absorbing submodule of T, then A is a Soc-QP2-absorbing submodule of B.

Proof

Let $abt \in A$, with $r, s \in R$, $t \in B$. Since A is a Soc-QP2-absorbing submodule of T, then either $at \in T - rad(A) + soc(T)$ or $bt \in T - rad(A) + soc(T)$ or $(ab)^2T \subseteq A + soc(T)$, for some $n \in Z^+$. That is either $at \in (T - rad(A) + soc(T)) \cap B$ or $bt \in (T - rad(A) + soc(T)) \cap B$ or $(ab)^nT \subseteq (A + soc(T)) \cap B$. But since $soc(T) \subseteq B$, then by lemma (2.7) we have either $at \in (\ T - rad(A) \cap B) + (soc(T) \cap B)$ or $bt \in (\ T - rad(A) \cap B) + (soc(T) \cap B)$ or $(ab)^nT \subseteq (A \cap B) + (soc(T) \cap B)$. By lemma (2.8) $soc(T) \cap B = soc(B)$, so either $at \in (\ T - rad(A) \cap B) + soc(B) \subseteq T - rad(A) + soc(B)$ or $bt \in (\ T - rad(A) \cap B) + soc(B) \subseteq T - rad(A) + soc(B)$ or $(ab)^nT \subseteq (A \cap B) + soc(B) \subseteq A + soc(B)$. Hence A is a Soc-QP2-absorbing submodule of B.

Lemma (2.10) [12, Theo. (2.12)]

Let N be a T over commutative ring proper submodule of a multiplication R-module R. Then $T - rad(N) = \sqrt{[N:_R T]}T$.

Lemma (2.11) [13, Cor. (2.14)(i)]

Let T be faithful multiplication R-module, then $soc(T) = soc(R)T$.

Proposition (2.12)

Let T be a A appropriate sub-model of T is the faithful multiplication of the R-module and L. Then L is a Soc-QP2-absorbing submodule of T if and only if $[L:_R T]$ is a Soc-QP2-absorbing ideal of R.

Proof

($\Longrightarrow$) Let $rsI \subseteq [L:_R T]$, where $r, s \in R$, I is an ideal of R and $rs \notin \sqrt{[[L:_R T] + soc(R): R]} = \sqrt{[L:_R T] + soc(R)}$, that is $(rs)^n \notin [L:_R T] + soc(R)$ for some $n \in Z^+$, it follows that $(rs)^nT \nsubseteq [L:_R T]T + soc(R)T$. But T is faithful multiplication then by lemma (2.11) $soc(R)T = soc(T)$. Thus $(rs)^nT \nsubseteq L + soc(T)$. That is $rs \notin \sqrt{[L + soc(T):_R T]}$. Now, we have $rsI \subseteq [L:_R T]$, then $rs(IT) \subseteq L$, and $rs \notin \sqrt{[L + soc(T):_R T]}$. Since L is a Soc-QP2- Absorbed subunit of T, then by motion (2.4) $r(IT) \subseteq T - rad(L) + soc(T)$ or $s(IT) \subseteq T - rad(L) + soc(T)$. Since T is multiplication then by lemma (2.10) $T - rad(L) = \sqrt{[L:_R T]}T$. Hence $r(IT) \subseteq \sqrt{[L:_R T]}T + soc(R)T$ or $s(IT) \subseteq \sqrt{[L:_R T]}T + soc(R)T$, it follows that $rI \subseteq \sqrt{[L:_R T]} + soc(R)$ or $sI \subseteq \sqrt{[L:_R T]} + soc(R)$. That is by proposition (2.4) $[L:_R T]$ is a Soc-QP2-absorbing ideal of R.

($\Longleftarrow$) Suppose that $[L:_R\ T]$ is Soc-QP2-absorbing ideal of R, and $rsN \subseteq L$, for $r,s \in R$, N is a submodule of T with $rs \notin \sqrt{[L + soc(T):_R T]}$, it follows that $(rs)^nT \nsubseteq L + soc(T)$ for some $n \in Z^+$. But T is faithful multiplication, then by lemma (2.11) $soc(R)T = soc(T)$. Hence $(rs)^nT \nsubseteq [L:_R T]T + soc(R)T$ for some $n \in Z^+$. It follows that $(rs)^n \notin [L:_R T] + soc(R) = [[L:_R T] + soc(R): R]$, hence $rs \notin \sqrt{[[L:_R T] + soc(R): R]}$. Now, since $rsN \subseteq L$, and T is a multiplication, then $N = JT$ for some ideal J of R, that is $rsJT \subseteq L$, it follows that $rsJ \subseteq [L:_R T]$. Since $[L:_R T]$ is a Soc-QP2-absorbing ideal of R and $rs \notin \sqrt{[[L:_R T] + soc(R): R]}$, then by proposition (2.4) either $rJ \subseteq \sqrt{[L:_R T]} + soc(R)$ or $sJ \subseteq \sqrt{[L:_R T]} + soc(R)$. That is $rJT \subseteq \sqrt{[L:_R T]T} + soc(R)T$ or $sJT \subseteq \sqrt{[L:_R T]}T + soc(R)T$. Thus by lemma (2.10) and lemma (2.11) we get $rN \subseteq T - rad(L) + soc(T)$ or $sN \subseteq T - rad(L) + soc(T)$. Hence by proposition (2.4) L is a Soc-QP2-absorbing submodule of T.

The following lemma must be remembered..

Lemma (2.13) [10, Coro. (1.26)]

If T be is a non-singular R-modules, then $soc(R)T = soc(T)$.

Proposition (2.14)

Let L be a a non-singular multiplication R-propoer module's submodule T. Then L is a Soc-QP2-absorbing submodule of T if and only if $[L:_R T]$ is a Soc-QP2-absorbing ideal of R.

Proof

Follows as in proposition (2.12) using lemma (2.10) and lemma (2.13).

We need to remember the following lemmas.

Lemma (2.15) [14, Coro. of Theo. 9]

Let I and J are ideals of ring R, and T be an R-module with a finitely produced multiplication. Then $IT \subseteq JT$ if and only if $I \subseteq J + ann_R(T)$.

Lemma (2.16)

1- Let T be a faithful multiplication R-module, then $T - rad(IT) = \sqrt{I}T$ for any ideal I of R [15, Theo.1(3)].

2- Let T be a multiplication R-module and I is an ideal of R such that $ann(T) \subseteq I$, then $T - rad(IT) = \sqrt{I}T$ [16, Prop. (2.4)].

The. Proposition (2.17)

Suppose T is a terminally generated R multiplication unit, and I is a Soc-QP2 absorber model of R. Then IT is a Soc-QP 2 absorber subunit of T.

Proof

Let $rsN \subseteq IT$ for $r, s \in R$, and N is a submodule of T with $rs \notin \sqrt{[IT + soc(T):_R T]}$, that is $(rs)^nT \nsubseteq IT + soc(T)$ for some $n \in Z^+$. Since T is faithful multiplication then by lemma (2.11) $soc(T) = soc(R)T$, that is $(rs)^nT \nsubseteq IT + soc(R)T$ for some $n \in Z^+$, it follows that $(rs)^n \notin I + soc(R) = [I + soc(R):_R R]$ implies that $rs \notin \sqrt{[I + soc(R):_R R]}$, Now, since $rsN \subseteq IT$ and T is a multiplication then $N = JT$ for some ideal J of R, thus $rsJT \subseteq IT$. Hence by lemma (2.15) $rsJ \subseteq I + ann_R(T)$, but T is a faithful, then $abJ \subseteq I + (0) = I$. Since I is a Soc-QP2-absorbing ideal of R and $rs \notin \sqrt{[I + soc(R):_R R]}$ then by Proposition

(2.4) either $rJ \subseteq \sqrt{I} + soc(R)$ or $sJ \subseteq \sqrt{I} + soc(R)$, hence either $rJT \subseteq \sqrt{I}T + soc(R)T$ or $sJT \subseteq \sqrt{I}T + soc(R)T$. It follows by lemma(2.10) and lemma (2.16), $rJT \subseteq T - rad(IT) + soc(T)$ or $sJT \subseteq T - rad(IT) + soc(T)$. That is $rN \subseteq T - rad(IT) + soc(T)$ or $sN \subseteq T - rad(IT) + soc(T)$. Hence by proposition (2.4) IT is a Soc-QP2-absorbing submodule of T.

Proposition (2.18)

Let T be a finitely produced non-singular multiplication R-module and I is a Soc-QP2-absorbing ideal of R with $ann_R(T) \subseteq I$. Then IT is a Soc-QP2-absorbing submodule of T.

Proof

Follows the same pattern as the proposition. (2.17) and using lemma (2.13) and lemma (2.16)(2).

The .Proposition (2.19)

Suppose T is an exact generated multiplication that is an exact R unit and L is an appropriate subunit of T. Then the following statements are equivalent.

1) L is Soc-QP2 absorber subunit from T.

2) $[L:_R T]$ is a Soc-QP2- Perfect Absorption of R.

3) $L = JT$ For some ideal absorbing Soc-QP2J of R.

Proof

(1) ⇐⇒ (2) It follows by proposition (2.12).

(2) ⇒⇒ (3) It is clear.

(3) ⇒⇒ (2) Suppose that $L = JT$ for some Soc-QP2-absorbing ideal J of R. Since T is a multiplication, then $L = [L:_R T]T = JT$. But T is faithful finitely generated multiplication, then $L = [L:_R T]$, It follows this$[L:_R T]$ is a Soc-QP2- Perfect assimilation of R.

Proposition (2.20)

Suppose T is a finite non-singular multiplier and L is an appropriate subunit of T. Then the following statements are equivalent.

1) L is Soc-QP2 absorber sub-model from$T$.

2) $[L:_R T]$ It is a Soc-QP2 absorber model of R.

3) $L = JT$ For some Soc-QP2 perfect absorption J for R with$ann_R(T) \subseteq J$.

Proof

It goes in the same direction as the suggestion. (2.19) using motion (2.14) and lemma (2.16)(2).

The following lemmas are required.

Lemma (2.21) [11. Theo. (9.1.4)(a)]

Let $f: T \longrightarrow T'$ be an R-homomorphism, then $f(soc(T)) \subseteq soc(T')$.

Lemma (2.22) [17. Coro. (1.3)]

Let $f: T \longrightarrow T'$ be an R-epimorphism and N is a submodule of T' with $ker\ (f) \subseteq N$, then $f(T - rad(N)) = T' - rad(f(N))$.

Proposition (2.23)

Let $f \in Hom_R(T, T')$ and L' is a Soc-QP2-absorbing sub-module of T' with $f^{-1}(L') \neq T$. Then $f^{-1}(L')$ is a Soc-QP2-absorbing sub-module of T.

Proof

Let $\in f^{-1}(L')$, for $r, s \in R$ $t \in T$, then $f(rst) = rsf(t) \in L'$. But L' is a Soc-QP2-absorbing submodule of T', implies that $rf(t) \in T' - rad(L') + soc(T')$ or $sf(t) \in T' - rad(L') + soc(T')$ or $sr \in \sqrt{[L' + soc(T'):_R T']}$. That is either $f(rt) \in T' - rad(L') + soc(T')$ or $f(st) \in T' - rad(L') + soc(T')$ or $(sr)^n T' \subseteq L' + soc(T')$ for some $n \in Z^+$. It follows by lemma (2.21) and lemma (2.22) that either $rt \in f^{-1}(T' - rad(L')) + f^{-1}(soc(T')) \subseteq T - rad(f^{-1}(L')) + soc(T)$ or $st \in f^{-1}(T' - rad(L')) + f^{-1}(soc(T')) \subseteq T - rad(f^{-1}(L')) + soc(T)$ or $(sr)^n T \subseteq f^{-1}(L') + soc(T)$. Hence $f^{-1}(L')$ be a Soc-QP2-absorbing submodule of T.

Proposition (2.24)

Let $f: T \longrightarrow T'$ be an R-epimorphism and L is Soc-QP2-absorbing sub-module of T with ker $(f) \subseteq L$. Then $f(L)$ is a Soc-QP2-absorbing sub-module of T'.

Proof

Let $rst' \in f(L)$, for $r, s \in R$, $t' \in T'$, it follows that $rsf(t) \in f(L)$ for some $t \in T$ (since f onto) that is $f(rst) \in f(L)$, implies that $f(rst) = f(t_1)$ for some $t_1 \in L$, then $f(rst - t_1) = 0$, it follows that $rst - t_1 \in ker\,(f) \subseteq L$, hence $rst \in L$. But L is a Soc-QP2-absorbing submodule of T, then either $rt \in T - rad(L) + soc(T)$ or $st \in T - rad(L) + soc(T)$ or $(rs)^n T \subseteq L + soc(T)$ for some $n \in Z^+$, it follows that either $rf(t) \in f(T - rad(L)) + f(soc(T)) \subseteq T' - rad(f(L)) + soc(T')$ or $sf(t) \in f(T - rad(L)) + f(soc(T)) \subseteq T' - rad(f(L)) + soc(T')$ or $(rs)^n T' \subseteq f(L) + soc(T')$ by using lemma (2.21), lemma(2.22). That is either $rt' \in T' - rad(f(L)) + soc(T')$ or $st' \in T' - rad(f(L)) + soc(T')$ or $sr \in \sqrt{[f(L) + soc(T'):_R T']}(rs)^n T' \subseteq f(L) + soc(T')$. Hence $f(L)$ is a Soc-QP2-absorbing submodule of T'.

Remark (2.25)

An R-module T intersection of two Soc-QP2-absorbing submodules does not have to be a Soc-QP2-absorbing submodule of T. The example below demonstrates this.:

Consider the Z-module Z and the submodules $5Z$, $6Z$ are Soc-QP2-absorbing submodules of Z-modules Z, but $5Z \cap 6Z = 30Z$is not SocQP2-absorbing submodule of Z-module Z (because if $2.3.5 \in 30Z$, but $2.5 \notin Z - rad(30Z) + soc(Z) = 30Z + (0) = 30Z$ and $3.5 \notin Z - rad(30Z) + soc(Z) = 30Z$ and $2.3 \notin \sqrt{[30Z:_Z Z]} = \sqrt{[30Z + soc(Z):_Z Z]} = \sqrt{30Z} = 30Z$.

We need to recall the following lemma.

Lemma (2.26) [18, Theo. 15(3)]

Let T be a multiplication R-module and K, N be a submodules of T. Then $T - rad(K \cap N) = T - rad(K) \cap T - rad(K)$.

Proposition (2.27)

Let L and K be a R-module appropriate submodules for multiplication$T$ with $soc(T) \subseteq L$ or $soc(T) \subseteq K$. If L and K are Soc-QP2-absorbing submodule of T, then $L \cap K$ is a Soc-QP2-absorbing submodule of T.

Proof

Suppose that L and K are Soc-QP2-absorbing submodule of T, and suppose that $rst \in L \cap K$ for $r, s \in R, t \in T$, then $rst \in L$ and $rst \in K$. But both L and K are Soc-QP2-absorbing submodule of T, then either $rt \in T-rad(L)+soc(T)$ or $st \in T-rad(L)+soc(T)$ or $rs \in \sqrt{[L+soc(T):_R T]}$ and either $rt \in T-rad(K)+soc(T)$ or $st \in T-rad(K)+soc(T)$ or $rs \in \sqrt{[K+soc(T):_R T]}$. Hence either $rt \in (T-rad(L)+soc(T)) \cap (T-rad(K)+soc(T))$ or $st \in (T-rad(L)+soc(T)) \cap (T-rad(K)+soc(T))$ or $(rs)^n T \subseteq (L+soc(T)) \cap (K+soc(T))$. If $soc(T) \subseteq K \subseteq T-rad(K)$, then $K+soc(T)=K$ and $soc(T)+T-rad(K)=T-rad(K)$. Thus either $rt \in (T-rad(L)+soc(T)) \cap T-rad(K)$ or $st \in (T-rad(L)+soc(T)) \cap T-rad(K)$ or $(rs)^n T \subseteq (L+soc(T)) \cap K$. It follows that by lemma (2.7) either $rt \in (T-rad(L) \cap T-rad(K))+soc(T)$ or $st \in (T-rad(L) \cap T-rad(K))+soc(T)$ or $(rs)^n T \subseteq (L \cap K)+soc(T)$. Hence by lemma (2.26) either $rt \in T-rad(L\cap K)+soc(T)$ or $st \in T-rad(L\cap K)+soc(T)$ or $rs \in \sqrt{[(L\cap K)+soc(T):_R T]}$. That is $L\cap K$ is a Soc-QP2-absorbing submodule of T. Similarly if $soc(T) \subseteq L$, we got $L \cap K$ is a Soc-QP2-absorbing submodule of T.

References

[1] Lu, C. P. , Prime Submodules of Modules. Comm. Math. , University Spatula, (1989), 33(1), 61-69.
[2] Lu, C. P. , M-radical of Submodules in Modules. Math. Japan, (1989), 34, 211-219.
[3] Darani, A.Y. and Soheilniai. F. ,2-Absorbing and Weakly 2-Absorbing Submodules, Tahi Journal. Math. (2011), 9, 577-584.
[4] Innam, M. A and Abdulrahman, A. H. , Semi- 2-Absorbing Submodules and Semi-2- absorbing Modules, international Journal of Advanced Scientific and Technical Research, (2015), 5(3), 521-530.
[5] Hojjat, M. , Ece, Y. , Unsal, T. and Darani, A.Y. , On 2-Absorbing Primary Submodules of Modules Over Commutative Rings, An. St. Univ. Ovidius constanta. (2016), 24(1), 339-351.
[6] Haibat, K. Mohammadali and Akram, S. Mohammed, Nearly semi -2-absorbing submodules and related concepts. Italian Journal of Pure and Applied Mathematics, (2019), (41), 620-627.
[7] Haibat, K. Mohammedali and Omar, A. Abdalla, Pseudo-2-Absorbing and Pseudo Semi-2-Absorbing Submodules, AIP Conference Proceeding, 2069, 020006(2019), 1-9. Scopus.
[8] Tekir, U. , Koc, S. , Oral, K. H. and Shum, K. P. , On 2-Absorbing Quasi-primary Ideals in commutative Rings, Communication in Mathematics and statistics, (2016), 4(1), 55-62.
[9] Kos, S. , Uregen, R. N. and Tekir, U. , On 2-Absorbing Quasi-primary Submodules, Faculty of Science and Mathematics, (2017), 31(10), 2943-2950.
[10] Gooderal, K. R. , Ring Theory, Non-singular Ring and Modules, Marcel. Dekker, Inc. New York and Basel, (1976).
[11] Kash, F. , Modules and Rings, London Math. Soc. Monographs, New York, Academic Press, (1982).
[12] Anderson, E. W and Fuller, K. R. , Rings and Categories of Modules, Springer-verloge, New York, (1992).
[13] El-Bast, Z. A. and Smith, P. F. , Multiplication Modules, Comm. In Algebra, (1988), 16(4), 755-779.
[14] Smith, P.F. , Some Remarks On Multiplication Module, Arch. Math. , (1988), 50, 223-225.
[15] Ali, M. M. , Invertibilty of Multiplication Modules III, New Zeland J. Math. , (2009), 39, 192-213.
[16] Ahamed, A. A. , On Submodules of Multiplication Modules, M.Sc. Thesis, University of Baghdad, (1992).
[17] Mc Casland, R.L and Moore, M. E., On Radical of Submodules, Comm. Algebra, (1991),19 (5), 1327-1341.
[18] Ali, M. M., Idempotent and Nilpotent Submodules of Multiplication Modules, Comm. Algebra, (2008), 36, 4620-4642.