# Common Fixed Point Theorems Satisfying Contractive Type Conditions in Complex Valued Metric Spaces

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## ABSTRACT

In this paper we prove common fixed point theorems satisfying contractive conditions involving rational expressions and product for four mappings that satisfy property (E.A) along with weak compatibility of pairs are proved property are obtained in complex valued metric spaces which generalize various results of ordinary metric spaces.

## 1. Introduction :

Fixed point theory is one of the fundamental theories in nonlinear analysis which has various applications in different branches of mathematics. In this theory, to prove the existence and the uniqueness of a fixed point of operators or mappings has been a valuable research area by using the Banach contraction principle. There are many generalizations of the Banach contraction principle particularly in metric spaces. Generalizations of the concept of metric spaces such as 2-metric spaces, D-metric spaces, G-metric spaces, K-metric spaces, cone metric spaces, and probabilistic metric space.

Recently, Huang and Zhang [1] generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space; hence they have defined the cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality condition of a cone. Azam et al. [2] introduced and studied the notion of complex valued metric space and established some common fixed point theorems for mappings involving rational expressions which are not meaningful in cone metric spaces.

In 2002, Aamri and Moutawakil [3] introduced the property (E.A) and pointed out that this property buys containment of ranges without any continuity requirements besides minimizing the commutativity conditions of the maps to the commutativity at their points of coincidence. Further, property (E.A) allows replacing the completeness condition of the space with a natural condition of closeness of the range..

The aim of this paper is to establish common fixed point theorems for two pairs of weakly compatible self-mappings of a complex valued metric space satisfying contractive condition involving product and rational expressions Moreover, we give some results using the property common limit in the range of one of the mappings.

## 2. Basic Facts and Definitions :

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$ . Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $R(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. (2.1)

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $R(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(2) $R(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(3) $R(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$ and

(4) $R(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

**Remark 2.1:** We can easily check the followings:

(i) $a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim bz, \forall\ z \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$. Azam et al. [4] defined the complex valued metric space in the following way:

**Definition 2.2 ([12]):**

Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

($C_1$) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(C3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$. Then $d$ is called a complex valued metric on X and $(X, d)$ is called a complex valued metric space.

**Definition 2.3.([9]):** Let $(X, d)$ be a complex valued b-metric space. Then

(i) A sequence $\{x_n\}$ in $X$ is said to converge to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $(x_n , x ) \prec r$, $\forall\ n > N$. We denote this by $\lim n{\to}\infty\ x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) If for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $(x_n, x_{n+m}) \prec r$ for all $n > N$, $m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in $(X, d)$.

(iii) If every Cauchy sequence in $X$ is convergent in $X$ then $(X, d)$ is called a complete complex valued b-metric space.

**Lemma 2.4.** Let $X$ be a complex valued metric space and $\{x_n\}$ a sequence in $X$. Then

(i) $\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$;

(ii) $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $n, m \to \infty$.

**Lemma 2.5.** Let X be a complex valued b- metric space and sequences $\{x_n\}$, $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} |d(x_n, y_n| = |d(x, y).|$

**Lemma 2.6.** Let X be a complex valued b- metric space then $|d(x, z)| \leq |d(x, y) + d(y, z)|$

And $|d(x, z)| \leq |d(x, y) + d(z, y)|$

For all $x, y, z \in X$ . Also $d(x,y) = d(y,x)$ for all $x, y \in X$.

**Lemma 2.7.** Let X be a complex valued b- metric space and sequences $\{x_n\}$, $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} |d(x_n, y_n| = 0$. Whenever $x_n$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ and $\lim_{n\to\infty} y_n = y$ for some $t \in X$

**Definition 2.8.** The 'max' function for the partial order $\precsim$ is defined as follows:

(1) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.

(2) $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$ .

(3) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$.

**Lemma 2.9 (see [12]).**

Let $z_1, z_2, z_3, \ldots \in \mathbb{C}$ and the partial order relation $\preceq$ is defined on $\mathbb{C}$. Then the following statements are easy to prove.

(i) If $z_1 \preceq \max\{z_2, z_3\}$, then $z_1 \preceq z_2$ if $z_3 \preceq z_2$.

(ii) If $z_1 \preceq \max\{z_2, z_3, z_4\}$, then $z_1 \leq z_2$ if $\max\{z_3, z_4\} \preceq z_2$.

(iii) If $z_1 \preceq \max\{z_2, z_3, z_4, z_5\}$, then $z_1 \leq z_2$ if $\max\{z_3, z_4, z_5\} \preceq z_2$.

Now we give the definition of complex valued metric space which has been introduced by Azam et al. [2].

**Definition 2.10 (see [14]).**

A pair of self-mappings $S$, $T$ : $X \to X$ is called weakly compatible if they commute at their coincidence point; that is, if there is a point $z \in X$ such that $Sz = Tz$, then $STz = TSz$, for each $z \in X$. The definition of property (E.A) has been introduced by Aamri and Moutawakil in [3] and redefined by Verma and Pathak [12] in complex valued metric spaces.

**Definition 2.11.**

Let $S, T : X \to X$ be two self-mappings of a complex valued metric space $(X, d)$. The pair $(S, T)$ is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$in $X$ such that

$$\lim_{n\to\infty} d\,(Sx_n, u) = \lim_{n\to\infty} d\,(Tx_n, u) = 0, \qquad (2.2)$$

for some $u \in X$.

**Example 2.12.** Let $X$ = C be endowed with the complex valued metric $d$ : C × C → C as

$$d\,(z_1, z_2) = |x_1 - x_2| + \mathrm{i}|y_1 - y_2|\ , (2.3)$$

where $z_1 = x_1 + iy_1$ and $x_2 + iy_2$. Then (C, $d$) is a complete complex valued metric space. Define the mappings $T, S : X \to X$ as $Tz = 2z - 1$, $Sz = z^2$ for all $z \in X$ and consider the sequence

$\{z_n\} = \{1 + i/2^n\}$. Thus we obtain

$$\lim_{n\to\infty} d\,(Tz_n, z) = \lim_{n\to\infty} d\,(Sz_n, z)\ ,$$

$$\lim_{n\to\infty} d\,(2\,(1 + i\,\tfrac{1}{2^n}) - 1, 1) = \lim_{n\to\infty} d\,((1\,(1 + \mathrm{i}\,\tfrac{1}{2^n})\,^2, 1) = 0, \qquad (2.4)$$

where $z$=1 is the limit of sequence $\{z_n\}$. Hence the pair$(S, T)$ satisfies property (E.A).

**Definition 2.12 (see [15]).** Let $S$ and $T$ be two self-mappings of complex valued metric space $X$. $S$ and $T$ are said to satisfy the common limit in the range of $S$ property if

$\lim n\to\infty d\,(Sx_n, Sx) = \lim n\to\infty d\,(Tx_n, Sx) = 0$, (2.5) for some $x \in X$.

**Note:** some common fixed point in a complex valued metric space . let $\emptyset : R^+ \to R^+$such that $\emptyset$

Is non decreasing condition and $\sum_{n=1}^{\infty} \emptyset^n(\mathrm{t}) < \infty$ for all t > 0 . it is clear that $\emptyset^n(\mathrm{t}) \to 0$ as n → ∞for all t >0 and hence we have $\emptyset(t) <$ t for all t > 0.

## 3. Main Results

In this section, initially, some common fixed point results for the pairs, which are weakly compatible and satisfy property (E.A), have been proved, by reconstructing the contractive conditions given in [16].

**Theorem 3.1.**

Let (X, d) be a complex valued metric space and let A, B, U, T : X → X be four self-mappings satisfying the following conditions:

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) for all x, y, z ∈ X, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2, k_3 < 1$ such that
$S(Ax, Bz) \leq \alpha(\max\{d(Tx,Uz), k_1d(Ax,Tx), k_2d(Bz,Uz), k_3S(Ay,Bz)\})$. (3.1)
(iii) the pair (B, U) and (A, T) are weaklycompatible
(iv) one of the pairs (B, U) and (A, T) satisfies property (E.A)

if the range of one of the mapping U(X), T(X) is complete subspace of X;Then the maps A, B, U and T have a unique common fixed point in X.

**Proof:** Let $x_0 \in X$ be arbitrary point of X. From condition (i) we can construct a sequence $\{y_n\}$ in X as follows:

$y_{2n} = Ax_{2n} = Ux_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$, $n \geq 0$.

Now, we show that $\{y_n\}$ is a Cauchy sequence. Let $d_{n+1} = S(y_n, y_{n+1})$. Then we have

$d_{2n+1}= d(y_{2n,}y_{2n+1})$

$= d(Ay_{2n,}By_{2n+1}))$

$\leq \alpha$ (max{ $d(Tx_{2n,}BU_{2n+1})$

$k_1\, d(Ax_{2n,}Tx_{2n})\ k_2\, d(Bx_{2n+1,}Ux_{2n,})\ k_3d(Ax_{2n+1,}Bx_{2n+1})\}$

$= \phi(\max\{\{\ d(y_{2n-1,}y_{2n})$

$k_1\, d(y_{2n,}y_{2n-1})\ k_2\, d(y_{2n+1,}y_{2n})\ k_3d(y_{2n,}y_{2n+1})\}$

$= \phi(\max\{d_{2n}, k_1d_{2n}, k_2d_{2n+1}, k_3d_{2n+1}\})$.

Thus $d_{2n+1} \leq \phi(d_{2n})$. By comparable point of view we have,

$d_{2n} = d(y_{2n-1}, y_{2n}) =d(y_{2n}, y_{2n-1}) = d(Ax_{2n}, Bx_{2n-1})$

$\leq \phi(\max\{d(Tx_{2n}, Ux_{2n-1}), k_1d(Ax_{2n}, Tx_{2n}), k_2d(Bx_{2n-1}, Ux_{2n-1}), k_3d(Ax_{2n},Bx_{2n-1})\})$

$= \phi(\max\{\{\ d(y_{2n-1,}y_{2n-2})\ k_1\, d(y_{2n,}y_{2n-1})\ k_2\, d(y_{2n+1,}y_{2n-2})\ k_3d(y_{2n,}y_{2n-1})\}$

$= \phi(\max\{d_{2n-1}, k_1d_{2n}, k_2d_{2n-1}, k_3d_{2n}\})$.

Thus $d_{2n} \leq \phi(d_{2n-1})$. Hence, for all $n \geq 2$, we have,

$d(y_n, y_{n+1}) \leq \phi(d(y_{n-1}, y_n)$

$\leq \phi^2 d(y_{n-2}, y_{n-1}) \cdot \cdot \cdot \cdot \cdot \cdot$

$\leq \phi^{n-1} (d(y_1, y_2)).$

for m > n we have

$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_m)$

$\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) d(y_{n+2} y_m,) \cdot \cdot \cdot \cdot \cdot \cdot$

$\leq \sum_{i=n}^{m-2} d(y_i\, y_{i+1}) + d(y_{i-1}, y_m)$

$\leq [\alpha^{n-1} d(y_1, y_2) + \alpha^n d(y_1, y_2) + \cdot \cdot \cdot + \alpha^{m-2} d(y_1, y_2)]$

$= \sum_{i=n-1}^{m-2} \alpha^i d(y_1, y_2)$ .

Since $\sum_{n=1}^{m-2} \alpha^n(t) < \infty$ for all $t > 0$,

so $d(y_n, y_m) \to 0$ as $n \to \infty$. Therefore, for each $\varepsilon > 0$, there exists $n_0 \in N$ such that for each n, $m \geq n_0$, $d(y_n, y_m) < \varepsilon$. Hence, $\{y_n\}$ is a Cauchy sequence in X.

Since X is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Ux_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = u$.

Since T is continuous, so we have

$\lim_{n\to\infty} T^2 x_{2n+2} = u$ and $\lim_{n\to\infty} TAx_{2n} = Tu$., then

$\lim_{n\to\infty} d(ATx_{2n}, TAx_{2n}) = 0$. So we have $\lim_{n\to\infty} ATx_{2n} = Tu$.

by condition (3.1), we obtain $d(ATx_{2n}, Bx_{2n+1})$

$\leq \phi(\max\{d(T^2x_{2n}, Ux_{2n+1}) k_1 d(ATx_{2n}, T^2x_{2n}), k_2 d(Bx_{2n+1}, Ux_{2n+1}), k_3 d(ATx_{2n}, Bx_{2n+1},)\})$. (3.2) Taking the upper limit as $n \to \infty$ in (3.2),

we get

$d(Tu, u) \leq \phi(\max\{d(Tu, u), 0, 0, k_3 d(Tu,u)\})$

$= \phi(d(Tu,u))$.

Hence, $d(Tu, u) \leq \phi(d(Tu, u)) < d(T u, u)$, which is a contradiction. So, $T u = u$. simarlary since U is contionous we obtain that

that $\lim_{n\to\infty} U^2x_{2n+1} = Uu$ and $\lim_{n\to\infty} UBx_{2n+1} = Uu$.

then $\lim_{n\to\infty} d(\,Bx_{2n+1}\,, BUx_{2n+1}) = 0$

So we have $\lim_{n\to\infty} Bx_{2n+1} = u$.

by condition (2), we obtain

$d(\,Ax_{2n}, Bx_{2n+1})$

$\leq \phi(\max\{d(Tx_{2n}, U^2x_{2n+1})k_1d(Ax_{2n}, Tx_{2n}), k_2d(BUx_{2n+1}, U^2x_{2n+1}), \quad k_3d(Ax_{2n}, BUx_{2n+1},)\})$.
(3.3) Taking the upper limit as $n \to \infty$ in (4), we get

$d(u, Ru) \leq \phi(\max\{d(u, Uu), 0, 0, k_3d(u, Uu)\}) = \phi(d(u, Uu))$.

Consequently, $d(u, Uu) \leq \phi(d(u, Uu)) < d(u, Uu)$ which is a contradiction. Uu=u Also, we can apply condition (3.1) to obtain we have Tu=Uu=u

$d(Au,Bx_{2n+1}) \leq \phi(\max\{d(Tu, Ux_{2n+1}), k_1d(Au, Tu), k_2d(Bx_{2n+1}, Ux_{2n+1}), k_3d(Au, Bx_{2n+1})\})$. (3.4)

Taking the upper limit as $n \to \infty$ in (3.4),

we have

$d(Au, u) \leq \phi(\max\{d(Tu, u), k_1d(Au, Tu), k_2d(u,u), k_3d(Au,u)\}) \leq \max\{k_1, k_3\}d(Au,u)$. if $Au \neq u$, then this implies that $\max\{k_1, k_3\} \geq 1$, which is a contradiction. Hence, from $\phi(t) < t$ for all $t > 0$, we have $Au = u$.

Finally, by using of condition (3.1), we get

$d(u,Bu) = S(Au,Bu)$

$\leq \phi(\max\{S(Tu, Uu), k_1d(Au, Tu), k_2d(Bu, Uu), k_3S(Au,Bu)\})$

$\leq \max\{k_2, k_3\}d(u,Bu)$.

if $Bu \neq u$, then this implies that $\max\{k_2, k_3\} \geq 1$, which is a contradiction. Hence, from $\phi(t) < t$ for all $t > 0$, we have $Bu = u$.

Thus, we have Tu=Uu=Au = Bu = u, that is, u is a common fixed point of A, B. U and T Suppose that p is another common fixed point of A, B,U and T that is, p = Ap = Bp=Up=Tp. If $u \neq p$, then by condition (3.1), we have that

$d(u, p) = d(Au, Bp) \leq \phi(\max\{d(Tu, Up), k_1S(Au, Tu), k_2S(Bp, Up), k_3S(Au, Bp)\})$

$= \phi(\max\{d(u,p), k_1d(u,u), k_2d(p,p), k_3d(u,p)\})$

$\leq \phi(d(u,p))$.

Hence, S(u,p) ≤ $\alpha$ (d(u, p)) which is a contradiction. Hence, u = p. Therefore, u is a unique common fixed point of A, B, U and T. This completes the proof.

**Corollary 3.1.** Let (X, d) be a complex valued metric space and let A, U: X → X be mappings satisfying the following conditions S(Ax,Bz) ≤ $\alpha$(max{d(Ux,Uz), $k_1$d(Ax,T x), $k_2$d(Az, Uz), $k_3$S(Ay, Bz)}).for all x,y in X where $\alpha \in (0,1)$

(1) A(x)⊂ $U(x)$
(2) the pair (A, U) and (A, T) are complete and weaklycompatible and also satisfies property (E.A)

Then the maps A and U have a unique common fixed point in X

**Corollary 3.2.** Let (X, d) be a complex valued metric space and let A, B,: X → X be mappings satisfying the following conditions

S(Ax,Bz) ≤ ϕ(max{d(x,Uz), $k_1$d(Ax, x), $k_2$d(Bz, z), $k_3$S(Ay,Bz)}).then A and B have a unique common fixed point in X.

(1)A(x)⊂ $B(x)$

(2) the pair (A, B) are complete and weaklycompatible and also satisfies property (E.A)

Then the maps B and A have a unique common fixed point in X

**Theorem 3.3.**

Let (X, d) be a complex valued metric space and let A, B, U, T : X → X be mappings satisfying the following conditions:

(i) A(X) ⊆ U(X), B(X) ⊆ T(X);
(ii) for all x, y, z ∈ X, there exists a function ϕ ∈ Φ and 0 < $k_1$, $k_2$, $k_3$ < 1 and p, q ∈ N such that d($A^p$x,$B^q$ z) ≤ ϕ(max{d(Tx,Uz), $k_1$d($A^p$x,T x), $k_2$d($B^q$z, Uz), $k_3$d($A^p$y,$B^q$z)}). the pair (B, U) and (A, T) are weakly compatible
(i) one of the pairs (B, U) and (A, T) satisfies property (E.A)

if the range of one of the mapping U(X), T(X) is complete subspace of X

Then the maps A, B, U and T have a unique common fixed point.

Proof. (i) When p = q = 1, we have Au = Bu = Uu = T u = u, u is a unique common fixed point of A, B, U and T.

(ii). If one of p and q is not equal to 1. Similar to Theorem 3.1, we can prove that $A^p$ , $B^q$ , U and T have a unique common fixed point u, that is, $A^p$u = $B^q$u = Uu = T u = u. Now, we should prove u is unique common fixed point of A, B, U and T. Indeed, $A^p(Au) = A^{p+1}u = A(A^pu) = Au = A(T u) = T(Au)$. So, Au is a common fixed point of $A^p$ and T.

Suppose that $Au \neq u$, and

$S(Au, u) = S(Au, B^qu)$

$= S(A^p(Au), B^qu)$

$\leq \phi(\max\{S(T(Au), Uu), k_1S(A^p(Au), T(Au)), k_2S(B^qu, Uu), k_3S(A^p(Au), B^qu)\})$

$= \phi(\max\{S(Au, u), k_1S(Au,Au), k_2S(u,u), k_3S(Au,u)\})$

$\leq \phi(S(Au,u))$. Hence, $S(Au,u)$

$\leq \phi(S(Au,u)) < S(Au,u)$, which is a contradiction. It means that Au = u. And, $B^q(Bu) = B^{q+1}u = B(B^qu) = Bu = B(Uu) = U(Bu)$.

Thus, Bu is a common fixed point of $B^q$ and U.

Suppose that $Bu \neq u$, and

$S(u,Bu) = S(A^pu, B^q(Bu))$

$\leq \phi(\max\{S(T u,U(Bu)),k_1S(A^pu, T u), k_2S(B^q(Bu),U(Bu)), k_3S(A^pu, B^q(Bu))\})$

$= \phi(\max\{S(u, u, Bu), k_1S(u, u), k_2S(Bu,Bu), k_3S(u,Bu)\}) \leq \phi(S(u,Bu))$.

Hence, $S(u,Bu) \leq \phi(S(u,Bu)) < S(u,Bu)$, which is a contradiction. It means that Bu = u. Therefore, Au = Bu = Uu = T u = u. Thus, u is a unique common fixed point of A, B, U and T. This completes the proof.

**Corollary 3.4.**

Let (X, d) be a complex valued metric space and let $A, B, U, T : X \to X$ be mappings satisfying the following conditions

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;

(ii) for all $x, y, z \in X$, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2, k_3 < 1$ and $p \in N$such that $S(A^px, B^p z) \leq \phi(\max\{S(T x, Uz), k_1S(A^px, , T x), k_2S(B^pz, Uz), k_3S(A^py, B^p z)\})$. the pair (B, U) and (A, T) are complete and weaklycompatible

(iii) one of the pairs (B, U) and (A, T) satisfies property (E.A)

if the range of one of the mapping U(X), T(X) is complete of X .Then the maps A, B, U and T have a unique common fixed point.

Proof. Let p = q and the process of proof is similar to the proof of Theorem 2.6.

**Theorem 3.5.**

Let (X, d) be a complex valued metric space and let A, B, U, T : X → X be mappings satisfying the following conditions

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) for all x, y, z ∈ X, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2 < 1$ such that
$d(Ax, Bz) \leq \phi(\max\{d(T x,Uz), k_1 d(Ax,T x), k_2 S(Bz,Uz)\})$
(iii) the pair (B, U) and (A, T) are weaklycompatible
(iv) one of the pairs (B, U) and (A, T) satisfies property (E.A)

if the range of one of the mapping U(X), T(X) is complete of X

Then the maps A, B, U and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary point of X. From condition (i) we can construct a sequence $\{y_n\}$ in X as follows:

$y_{2n} = Ax_{2n} = Ux_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$, $n \geq 0$.

Now, we show that $\{y_n\}$ is a Cauchy sequence.

Let $d_{n+1} = S(y_n,y_{n+1})$. Then we have

$d_{2n+1} = S(y_{2n},y_{2n+1}) = S(Ax_{2n}, Bx_{2n+1})$

$\leq \phi(\max\{S(T x_{2n}, Ux_{2n+1}), k_1 S(Ax_{2n},Tx_{2n}), k_2 S(Bx_{2n+1}, Ux_{2n+1})\})$

$= k_1\ d(y_{2n,}y_{2n-1})\ k_2\ d(y_{2n+1,}y_{2n})\ k_3 d(y_{2n,}y_{2n+1})\}$

$= \phi(\max\{d_{2n}, k_1 d_{2n}, k_2 d_{2n+1}, k_3 d_{2n+1}\})$.

Thus $d_{2n+1} \leq \phi(d_{2n})$. By similar arguments we have,

$d_{2n} = d(y_{2n-1}, y_{2n}) = d(y_{2n}, y_{2n-1}) = d(Ax_{2n}, Bx_{2n-1})$

$\leq \phi(\max\{d(x_{2n}, x_{2n-1}), k_1 d(Ax_{2n}, x_{2n}), k_2 d(Bx_{2n-1}, x_{2n-1}), k_3 d(Ax_{2n},Bx_{2n-1})\})$

$= \phi(\max\{\{\ d(y_{2n,}y_{2n})\ k_1\ d(y_{2n,}y_{2n-1})\ k_2\ d(y_{2n+1,}y_{2n})\ k_3 d(y_{2n,}y_{2n+1})\}$

$= \phi(\max\{d_{2n-1}, k_1 d_{2n}, k_2 d_{2n-1}, k_3 d_{2n}\})$.

Thus $d_{2n+1} \leq \phi(d_{2n})$. By similar arguments we have,

$d_{2n} = d(y_{2n-1}, y_{2n}) = d(y_{2n}, y_{2n-1}) = d(Ax_{2n}, Bx_{2n-1})$

$\leq \phi(\max\{d(x_{2n}, x_{2n-1}), k_1 d(Ax_{2n}, x_{2n}), k_2 d(Bx_{2n-1}, x_{2n-1}), k_3 d(Ax_{2n},Bx_{2n-1})\})$

$= \phi(\max\{\{ d(y_{2n,}y_{2n})\ k_1\ d(y_{2n,}y_{2n-1})\ k_2\ d(y_{2n+1,}y_{2n})\ k_3 d(y_{2n,}y_{2n+1})\}$

$= \phi(\max\{d_{2n-1}, k_1 d_{2n}, k_2 d_{2n-1}, k_3 d_{2n}\})$.

Thus $d_{2n} \leq \phi(d_{2n-1})$. The process of next proof is similar to the proof of Theorem 2.1.

**Theorem 3.6.**

Let (X, d) be a complex valued metric space and let A, B, U, T : X → X be mappings satisfying the following conditions

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;
(ii) for all x, y, z ∈ X, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2 < 1$ and p, q ∈ N such that

$S(A^p x, B^q z) \leq \phi(\max\{S(T x,Uz), k_1 S(A^p x,T x), k_2 S(B^q z,Uz)\})$

(iii).the pair (B, U) and (A, T) are weaklycompatible

(iv).one of the pairs (B, U) and (A, T) satisfies property (E.A)

if the range of one of the mapping U(X), T(X) is complete subspace of xX Then the maps A, B, U and T have a unique common fixed point.

**Corollary 3.7.**

Let (X, d) be a complex valued metric space and let A, B, U, T : X → X be mappings satisfying the following conditions

(i) $A(X) \subseteq U(X)$, $B(X) \subseteq T(X)$;

(ii). for all x, y, z ∈ X, there exists a function $\phi \in \Phi$ and $0 < k_1, k_2 < 1$ and p ∈ N such that $S(A^p x,B^p z) \leq \phi(\max\{S(T x, Uz), k_1 S(A^p x, T x), k_2 S(B^p z, Uz)\})$

(iii). the pair (B, U) and (A, T) are weaklycompatible

(iv)one of the pairs (B, U) and (A, T) satisfies property (E.A)

if the range of one of the mapping U(X), T(X) is complete subspace of xX Then the maps A, B, U and T have a unique common fixed point.

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