Comparison between Touchard and Bernstein polynomials solutions of integral equations

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Abstract: The idea of this work is to find the approximate solutions to the Fredholm and Volterra integral equations of the second kind using the Touchard and Bernstein polynomials. The approximate solutions were compared with the exact solutions to see the accuracy of the method and how effective it is, and to compare the solutions obtained using many terms used among them and in the last they can be compared also with the previously studied works.
Keywords: linear integral equations, Touchard polynomials, Bernstein polynomials, collocation method

1. Introduction

Integral equations play an important role in many theoretical and applied researches due to the possibility of expressing the integral equation as a continuous or discontinuous integral operator. This work deals with the two types of integral equations Volterra and Fredholm. Here we will show how to find their solutions using polynomials of two different types, namely, Touchard and Bernstein with the collocation method and the comparison of the approximate solutions obtained with the exact solutions, in addition to comparing them also with some of the works carried out by (Yalcinbas, Aynigul, 2011; Yalcinbas, Aynigul, and Akkaya, 2010; Peter Alpha, 2021; Shoukralla, Ahmed, 2020). Which were previously accomplished in different methods.

2. Problem Formulation

Now consider the following linear integral equations of the second kind.

$$u(x) = f(x) + \lambda \int_{\Omega} k(x,t)u(t)dt \tag{1}$$

Where the functions $f(x)$, and $k(x,t)$ are given and continuous functions in Ω and $\Omega \times \Omega$ respectively, the function $u(t)$ is to be determined as continuous function in Ω. Depending on the domain $\Omega = [a,t]$ or $[a,b]$ the equation (1) describes the Volterra integral equation or Fredholm integral equation, respectively.

The equation (1) can be put in the form of a linear functional equation

$$u(x) - Au(x) = f(x),\ x \in \Omega$$

With the linear mapping A given by

$$Au(x) = \int_{\Omega} k(x,t)\, u(t)dt.$$

For the solution of the equation (1) in the complete function spaces, usually take it $C(\Omega)$, we choose a sequence of finite dimensional subspaces $V_n, n \geq 1$, having n basis functions $\{P_1, P_2, \dots, P_n\}$ with dimension of $V_n = n$.

Seeking the approximate function $u_n \in V_n$ of the function u given by

$$u_n(x) = \sum_{k=1}^{n} \alpha_k P_k(x) \tag{2}$$

Where the expression (2) describes the truncated Touchard series or Bernstein series of the solution of the equation (1), with the functions $\{P_k\}_{0\leq k\leq n}$ represent the Touchard or Bernstein polynomials and $\{\alpha_k\}_{0\leq k\leq n}$ the coefficients to be determined. In other words, we can write

$$r_n(x) = u_n(x) - A_n u(x) - f(x)$$

$$r_n(x) = u_n(x) - \int_{\Omega} k(x,t)\, u_n(t)dt - f(x)$$

$$= \sum_{k=1}^{n} \alpha_k P_k(x) - \sum_{k=1}^{n} \alpha_k \int_{\Omega} k(x,t) P_k(t)dt - f(x)$$
$$= \sum_{k=1}^{n} \alpha_k \left[P_k(x) - \int_{\Omega} k(x,t) P_k(t)dt \right] - f(x), x\epsilon\Omega$$

And α_k, are coefficients to be determined.

3. Solution with collocation-Polynomials methods

Choose a selection of distinct points $x_1, x_2, \ldots\ldots x_n \in \Omega$ and require
$r_n(x_j) = 0, \; j = 1,2,\ldots,n.$ (3)

The condition (3) leads us to determine the coefficients $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ solution of the linear system
$\sum_{k=1}^{n} \alpha_k \left[P_k(x_j) - \int_{\Omega} k(x_j,t) P_k(t)dt \right] = f(x_j), j = 1,2,\ldots,n$ (4)

Define the matrices

$$P = (P_{kj}) = P_k(x_j)$$

And

$$K = (K_{kj}) = \int_{\Omega} k(x_j,t) P_k(t)dt$$

If the $det(P-K) \neq 0$, we can ensure that, there exists a solution of the linear system (4) and consequently the approximate solution $u_n(x)$ as a linear combination

$$u_n(x) = \sum_{k=1}^{n} \alpha_k P_k(x)$$

For which

$$u_n(x_j) - \int_{\Omega} k(x_j,t) u_n(t)dt = f(x_j), j = 1,2,\ldots,n$$

In fact, the linear system may be written in matrix
$(P-K) \propto = F$ (5)

Where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{tr}$ and $F = (f(x_1), f(x_2), \ldots, f(x_n))^{tr}$. For the determinant of the system (5) is different from zero $det(P-K) \neq 0$, then it has a unique solution

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{tr} = (P-K)^{-1}F$$

The corresponding approximate solution

$$u_n(x) = \sum_{k=1}^{n} \alpha_k P_k(x)$$

Has the property that its residual $r_n(x)$ vanishes at the selected nodes x_j.

4. Touchard polynomials

The Touchard polynomials $T_n(x)$ is defined by $T_0(x) = 1$ and the following recursion

$$T_n(x) = \sum_{k=0}^{n} C_k^n (x)^k$$

The Touchard polynomial $T_n(x)$ is polynomials with rational coefficients

n	T_n
0	1
1	$1+x$
2	$1+2x+x^2$
3	$1+3x+3x^2+x^3$
4	$1+4x+6x^2+4x^3+x^4$
5	$1+5x+10x^2+10x^3+5x^4+x^5$
6	$1+6x+15x^2+20x^3+15x^4+6x^5+x^6$
7	$1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7$
8	$1+8x+28x^2+56x^3+70x^4+56x^5+28x^6+8x^7+x^8$
9	$1+9x+36x^2+84x^3+126x^4+126x^5+84x^6+36x^7+9x^8+x^9$
10	$1+10x+45x^2+120x^3+210x^4+252x^5+210x^6+120x^7+45x^8+10x^9+x^{10}$

5. Bernstein polynomials

The Bernstein polynomials of degree n over the interval $[a,b]$ are defined by

$$B_{i,n}(x)=\binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^{\mathrm{n}}},\qquad a\le x\le b,\qquad i=0,1,2,..,n$$

The Bernstein polynomials having degree n satisfies the following properties

$$\begin{cases}1)\ B_{i,n}(x)=0, & if\ \ i<0\ or\ \ i>n\\ 2)\ \sum_{i=0}^{n}B_{i,n}(x)=1 & \\ 3)\ B_{i,n}(a)=B_{i,n}(b)=0, 1\le i\le n-1 & \end{cases}$$

Noting that, the Bernstein polynomial $B_{i,n}(x)$ is polynomials with rational coefficients

i	$B_{i,10}$
0	$(b-x)^{10}/(b-a)^{10}$
1	$10(b-x)^9(x-a)/(b-a)^{10}$
2	$45(b-x)^8(x-a)^2/(b-a)^{10}$
3	$120(b-x)^7(x-a)^3/(b-a)^{10}$
4	$210(b-x)^6(x-a)^4/(b-a)^{10}$
5	$252(b-x)^5(x-a)^5/(b-a)^{10}$
6	$210(b-x)^4(x-a)^6/(b-a)^{10}$
7	$120(b-x)^3(x-a)^7/(b-a)^{10}$
8	$45(b-x)^2(x-a)^8/(b-a)^{10}$
9	$10(b-x)(x-a)^9/(b-a)^{10}$
10	$(x-a)^{10}/(b-a)^{10}$

5. Existence and uniqueness theorems

Here we present the following two theorems which confirm the existence and the uniqueness of the solution for the work we are going to study.

Consider the linear integral equation (1).

Where the conditions

1- $f\in C(\Omega), u\in C(\Omega), k\in C(\Omega\times\Omega)$

2- $M=max_{(x,t)\in\Omega\times\Omega}|k(x,t)|$

5.1 Theorem

Let $A: X \to X$ be compact and the equation

$$(I - A)u = f \tag{6}$$

Admit a unique solution. That the projections $R_n : X \to X_n$ satisfy to $\|R_n A - A\| \to 0, n \to \infty$. Then, for sufficiently large n, the approximate equation

$$u_n - R_n A u_n = R_n f \tag{7}$$

Has a unique solution for all $f \in X$ and there holds an error estimate

$$\|u - u_n\| \leq M\|u - R_n u\| \tag{8}$$

With some positive constant M depending on A.

Proof

As it is known for all sufficiently large n the inverse operators $(I - R_n A)^{-1}$ exist and are uniformly bounded. To verify the error bound, we apply the projection operator R_n to the equation (6) and get

$$R_n u - R_n A u = R_n f$$

Or again

$$u - R_n A u = R_n f + u - R_n u$$

Subtracting this from (7) we find

$$(I - R_n A)(u - u_n) = (I - R_n)u$$

Hence the estimate (8) follows.

5.2 Theorem

Under the above continuity conditions (1) and (2), suppose there is a constant $c > 0$ such that

$$\frac{1}{c} M exp(c(b-a)) \prec 1$$

Then equation (1) has a unique solution $u \in C(\Omega)$

Proof. Let u and v be two solutions of equation (1)
Let the integral operator

$$Au(x) = f(x) + \int_{\Omega} k(x,t)\, u(t) dt$$

We have

$$|Au(x) - Av(x)| \leq \left| \int_{\Omega} k(x,t)\,(u(t) - v(t)) dt \right|$$

$$\leq \int_{\Omega} |k(x,t)|\, |(u(t) - v(t))| dt$$

$$\leq M \int_{\Omega} |(u(t) - v(t))|\, exp(-c(t-a)) exp(c(t-a)) dt$$

$$\leq \|u - v\| \left(\frac{M}{c}\right) exp\big(c(b-a)\big), because\ (t-a) \leq b - a$$

$$\leq \|u - v\| (\frac{M}{c}) exp(c(x - a + b - x))$$

$$= \|u - v\| (\frac{M}{c}) (exp(c(x-a). exp\ c(b-x))$$

$$\leq (\frac{M}{c}) (exp(c(x-a)) \|u - v\| exp\ c(b-x)$$

$$\le \left(\frac{M}{c}\right)(exp(c(x-a))\|u-v\|exp\ c(b-a)$$

It follows that for all $x \in \Omega$

$$|Au(x)-Av(x)|exp(-c(x-a) \le \left(\frac{M}{c}\right)\|u-v\|exp\ c(b-a)$$

So

$$\|Au-Av\| \le \left(\frac{M}{c}\right)exp\ c(b-a)\|u-v\|$$

We deduce that the operator A is Lipschitzian with constant $k=\left(\frac{M}{c}\right)exp\ c(b-a) \prec 1$. Then A is a contraction, and A has a unique solution in $C(\Omega)$.

The absolute error for this formulation is defined by $er=|u(x)-u_n(x)|$.

6. Numerical Examples

For numerical verification of the above method we consider the following examples

Example 1 We consider the following Fredholm integral equation

$$u(x)=\sin(\pi x^2)-\frac{x^2}{\pi}+\int_0^1 (x^2 t)u(t)dt$$

Where the function $f(x)$ is chosen so that the exact solution is given by $u(x)=\sin(\pi x^2)$

Table.1. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for $n=10$.

x	***Sol ext***	***Sol app*** T_n	***Sol app*** B_n	***er,*** T_n	***er,*** B_n
0	0.000E+00	0.000E+00	0.000E+00	1.876E-12	0.000E+00
0.1	3.141E-02	3.141E-02	3.141E-02	1.059E-08	1.059E-08
0.2	1.253E-01	1.253E-01	1.253E-01	4.235E-08	4.235E-08
0.3	2.790E-01	2.790E-01	2.790E-01	9.529E-08	9.529E-08
0.4	4.818E-01	4.818E-01	4.818E-01	1.694E-07	1.694E-07
0.5	7.071E-01	7.071E-01	7.071E-01	2.647E-07	2.647E-07
0.6	9.048E-01	9.048E-01	9.048E-01	3.812E-07	3.812E-07
0.7	9.995E-01	9.995E-01	9.995E-01	5.190E-07	5.188E-07
0.8	9.048E-01	9.048E-01	9.048E-01	6.777E-07	6.776E-07
0.9	5.621E-01	5.621E-01	5.621E-01	8.576E-07	8.576E-07
1	1.225E-16	1.059E-06	1.059E-06	1.059E-06	1.059E-06

Example 2 Consider the following Fredholm integral equation

$$u(x)=-x+e^{2x}+\int_0^1 xe^{-2t}u(t)dt$$

Where the function $f(x)$ is chosen so that the exact solution is given by $u(x)=e^{2x}$

Table.2. We present the exact and the approximate solutions of the equation in the example 2 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in (Yalcinbas, Aynigul, and Akkaya, 2010).

x	***Sol ext***	***Sol app*** $\boldsymbol{T_n}$	***Sol app*** $\boldsymbol{B_n}$	***er,*** $\boldsymbol{T_n}$	***er,*** $\boldsymbol{B_n}$	***er(Yalcinbas, Aynigul, and Akkaya, 2010)***
0	1.000	1.000	1.000	0.000E+00	0.000E+00	0.000
0.1	1.221	1.221	1.221	2.501E-12	2.501E-12	2.68139265e-6
0.2	1.492	1.492	1.492	5.002E-12	5.002E-12	5.362784215e-6
0.3	1.822	1.822	1.822	7.504E-12	7.505E-12	8.044082301e-6
0.4	2.226	2.226	2.226	1.001E-11	1.001E-11	1.072326582e-5
0.5	2.718	2.718	2.718	1.251E-11	1.251E-11	1.337965059e-5
0.6	3.320	3.320	3.320	1.501E-11	1.501E-11	1.58817245e-5
0.7	4.055	4.055	4.055	1.751E-11	1.751E-11	1.762273736e-5
0.8	4.953	4.953	4.953	2.002E-11	2.001E-11	1.637473125e-5
0.9	6.050	6.050	6.050	2.252E-11	2.251E-11	5.233654626e-6
1	7.389	7.389	7.389	2.503E-11	2.501E-11	3.457600943e-5

Example 3 We consider the following Fredholm integral equation

$$u(x) = e^{(x+2)} - 2\int_0^1 e^{(x+t)}u(t)dt$$

Where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = e^x$

Table.3. We present the exact and the approximate solutions in (Ahmet, 2016; Peter Alpha, 2021) of the equation in the example 3 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in (Ahmet, 2016; Peter Alpha, 2021).

x	***Sol ext***	***Sol app*** $\boldsymbol{T_n}$	***Sol app*** $\boldsymbol{B_n}$	***Sol ap (Ahmet, 2016; Peter Alpha, 2021).***	***er,*** $\boldsymbol{T_n}$	***er,*** $\boldsymbol{B_n}$
0	1.000	1.000	1.000	0.999999998	3.664E-15	0.000E+00
0.1	1.105	1.105	1.105	1.105170916	4.885E-15	2.220E-15
0.2	1.221	1.221	1.221	1.221402756	4.663E-15	2.887E-15
0.3	1.350	1.350	1.350	1.349858803	7.550E-15	8.882E-15
0.4	1.492	1.492	1.492	1.491824693	5.995E-15	5.995E-15
0.5	1.649	1.649	1.649	1.648721264	6.217E-15	7.105E-15
0.6	1.822	1.822	1.822	1.822118793	6.439E-15	8.438E-15
0.7	2.014	2.014	2.014	2.013752696	5.773E-15	7.105E-15
0.8	2.226	2.226	2.226	2.225540911	1.021E-14	1.199E-14
0.9	2.460	2.460	2.460	2.459603079	1.155E-14	1.155E-14
1	2.718	2.718	2.718	2.718281765	1.377E-14	1.066E-14

Example 4 We consider the following Fredholm integral equation

$$u(x) = e^x - 1 + \int_0^1 e^{(-t)} u(t)dt$$

Where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = e^x$

Table.4. We present the exact and the approximate solutions of the equation in the example 4 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in (Mustafa, Yalçın, 2010).

x	***Sol ext***	***Sol app T_n***	***Sol app B_n***	***er, T_n***	***er, B_n***	***er, (Mustafa,Yalçın, 2010)***
0	1.000	1.000	1.000	1.255E-14	0.000E+00	0.210E-5
0.1	1.105	1.105	1.105	1.243E-14	7.772E-15	0.200E-5
0.2	1.221	1.221	1.221	1.266E-14	1.021E-14	0.200E-5
0.3	1.350	1.350	1.350	1.288E-14	1.243E-14	0.300E-5
0.4	1.492	1.492	1.492	1.266E-14	1.199E-14	0.300E-5
0.5	1.649	1.649	1.649	1.288E-14	1.177E-14	0.200E-5
0.6	1.822	1.822	1.822	1.266E-14	1.221E-14	0.200E-5
0.7	2.014	2.014	2.014	1.288E-14	1.199E-14	0.200E-5
0.8	2.226	2.226	2.226	1.243E-14	1.199E-14	0.100E-5
0.9	2.460	2.460	2.460	1.243E-14	1.155E-14	0.100E-5
1	2.718	2.718	2.718	1.155E-14	1.243E-14	0.00000

Example 5 Consider the linear integral equation of Volterra

$$u(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x - t)u(t)dt$$

Where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = 1 - \sinh(x)$

Table.5. We present the exact and the approximate solutions of the equation in the example 5 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in (Babolian, Davari 2005).

x	***Sol ext***	***Sol app T_n***	***Sol app B_n***	***er, T_n***	***er, B_n***	***er, (Babolian, Davari 2005)***
0	1.000	1.000	1.000	0.000E+00	0.000E+00	0
0.1	8.998E-01	8.998E-01	8.998E-01	4.441E-16	4.441E-16	0.563890E-5
0.2	7.987E-01	7.987E-01	7.987E-01	1.332E-15	1.332E-15	0.00002202
0.3	6.955E-01	6.955E-01	6.955E-01	1.887E-15	9.992E-16	0.00004821
0.4	5.892E-01	5.892E-01	5.892E-01	2.220E-15	2.220E-15	0.00008333
0.5	4.789E-01	4.789E-01	4.789E-01	4.552E-15	2.887E-15	0.00012656

0.6	3.633E-01	3.633E-01	3.633E-01	1.665E-15	3.497E-15	0.00017715
0.7	2.414E-01	2.414E-01	2.414E-01	2.776E-15	4.191E-15	0.00023436
0.8	1.119E-01	1.119E-01	1.119E-01	1.443E-15	4.802E-15	0.00029745
0.9	-2.652E-02	-2.652E-02	-2.652E-02	8.882E-16	5.697E-15	0.00036566
1	-1.752E-01	-1.752E-01	-1.752E-01	2.887E-15	6.273E-15	0.00043821

Example 6 Consider the linear integral equation of Volterra

$$u(x) = \cos(x) - e^x \sin(x) + \int_0^x e^x u(t)dt$$

Where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = \cos(x)$

Table.6. We present the exact and the approximate solutions of the equation in the example 4 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in (Shoukralla, Ahmed 2020).

x	***Sol ext***	***Sol app T_n***	***Sol app B_n***	***er, T_n***	***er, B_n***	***er, (Shoukralla, Ahmed 2020)***
0	1.000	1.000	1.000	0.000E+00	0.000E+00	0
0.1	9.950E-01	9.950E-01	9.950E-01	3.442E-15	2.998E-15	1.54654067330284e-13
0.2	9.801E-01	9.801E-01	9.801E-01	3.664E-15	3.109E-15	1.50213175231784e-13
0.3	9.553E-01	9.553E-01	9.553E-01	4.885E-15	4.996E-15	1.26765264951700e-12
0.4	9.211E-01	9.211E-01	9.211E-01	5.773E-15	5.662E-15	3.47766260233584e-11
0.5	8.776E-01	8.776E-01	8.776E-01	7.772E-15	7.438E-15	4.06040867595436e-10
0.6	8.253E-01	8.253E-01	8.253E-01	9.881E-15	9.881E-15	3.00572522426990e-09
0.7	7.648E-01	7.648E-01	7.648E-01	1.321E-14	1.310E-14	1.63047718659826e-08
0.8	6.967E-01	6.967E-01	6.967E-01	1.721E-14	1.799E-14	7.04649298910454e-08
0.9	6.216E-01	6.216E-01	6.216E-01	2.665E-14	2.587E-14	2.55996352449550e-07
1	5.403E-01	5.403E-01	5.403E-01	2.376E-14	2.909E-14	8.10970208320327e-07

Example 7 Consider the linear integral equation of Volterra

$$u(x) = \exp(\mathrm{x}) + \int_0^x u(t)dt$$

Where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = (1 + x)\exp(x)$

Table.7. We present the exact and the approximate solutions of the equation in the example 7 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in (Mamadu, Njoseh, 2016).

x	***Sol ext***	***Sol app T_n***	***Sol app B_n***	***er, T_n***	***er, B_n***	***er, (Mamadu, Njoseh, 2016)***
0	1.000E+00	1.000E+00	1.000E+00	0.000E+00	0.000E+00	5.4670E-03
0.1	1.216E+00	1.216E+00	1.216E+00	1.306E-13	1.292E-13	1.1050E-03
0.2	1.466E+00	1.466E+00	1.466E+00	1.252E-13	1.237E-13	2.2853E-03
0.3	1.755E+00	1.755E+00	1.755E+00	1.439E-13	1.437E-13	8.4845E-04

0.4	2.089E+00	2.089E+00	2.089E+00	1.572E-13	1.550E-13	1.0669E-03
0.5	2.473E+00	2.473E+00	2.473E+00	1.754E-13	1.736E-13	2.0398E-03
0.6	2.915E+00	2.915E+00	2.915E+00	1.923E-13	1.905E-13	1.4560E-03
0.7	3.423E+00	3.423E+00	3.423E+00	2.212E-13	2.136E-13	3.8930E-04
0.8	4.006E+00	4.006E+00	4.006E+00	2.425E-13	2.309E-13	2.1784E-03
0.9	4.673E+00	4.673E+00	4.673E+00	3.091E-13	2.718E-13	1.4434E-03
1	5.437E+00	5.437E+00	5.437E+00	2.363E-13	1.723E-13	5.5777E-03

4.Conclusion

In this paper, we study comparing the solutions of the integral equations of the type Volterra and Fredholm of the second type, using the Touchard and Barnstein polynomials with the collocation method, and comparing the solutions between them and with the exact solutions as well. (Yalcinbas, Aynigul, 2011; Yalcinbas, Aynigul, and Akkaya, 2010; Peter Alpha, 2021; Mustafa, Yalçın, 2010). We also note that the higher the degree of the polynomial, the better the results. However, we noticed that the results obtained using Barenstein's polynomial were somewhat better than the results obtained using Touchard's polynomial. Finally, we dealt with many examples that showed the effectiveness of the method for both.

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