Total cordial labeling of corona product of paths and second power of fan graph

Atef Abd El-hay[a], Ashraf Elrokh[b]

[a,b] Dept. of Math., Faculty of Science, Menoufia University, Shebeen Elkom, Egypt.
E mail address: [a]atef _1992@yahoo.com and [b]ashraf.hefnawy68@yahoo.com

Abstract: A graph is called total cordial if it has a $0-1$ labeling such that the total number of vertices and edges labelled with ones and zeros differ by at most one. In this paper, we contribute some new results on Total Product cordial labeling and investigate necessary and sufficient conditions of the corona product between paths and second power of fan graphs to be total cordial.
Keywords: Corona operation, Second power, Total cordial, Path graph, Fan graph.

1. Introduction

Graph labeling is one of graph theory's oldest issues. It is the assignment of numbers to vertices, edges, or both under specific condition. Many authors have been interested in graph labeling [1, 3, 4]. Gallian [2] has published an effective survey on whole graph labeling and its applications. The notion of product cordial labeling was introduced in 2004 [6], and it's proved that trees, unicyclic graphs of odd order, triangular snakes, helms, and unions of two path graphs are product cordial.

M. Sundaram, R. Ponraj, and S. Somasundaram introduced a new type of graph labeling known as total product cordial labeling and investigated the total product cordial behavior of some standard graphs [7]. Suppose that $G=(V,E)$ is a graph, where V is the set of its vertices and E is the set of its edges. A mapping $f: V\to\{0,1\}$ is called binary vertex labeling of G and $f(u)$ denotes the label of vertex u of G under f. A binary vertex labeling of a graph G with the induced edge labeling $f^*: E\to\{0,1\}$ defined by $f^*(uv)=\big(f(u)+f(v)\big)\text{mod } 2$ is called Total cordial labeling if $|(v_0+e_0)-(v_1+e_1)|\le 1$, where v_0 and v_1 are the numbers of vertices labeled by 0 and 1, respectively, and e_0 and e_1 are the corresponding numbers of edges. A graph with a Total cordial labeling defined on it is called Total cordial.

Definition 1. A second power of a fan F_n^2 is the graph obtained from the join of the second power of a path P_n^2 and a null graph N_1, i.e.$F_n^2=P_n^2+N_1$. So the order of F_n^2 is $n+1$ and its size is $3n-3$, in particular $F_1^2\equiv P_2$,$F_2^2\equiv C_3$ and $F_3^2\equiv K_4$.

Definition 2. The corona $G\odot H$ of two graphs G (with n_1 vertices and m_1 edges) and H (with n_2 vertices and m_2 edges) is the graph denoted by $G\odot H$ and is obtained by taking one copy of G and n_1 copies of H, and then joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H [17]. It follows from the definition of the corona that $G\odot H$ has $n_1+n_1.n_2$ vertices and $m_1+n_1.m_2+n_1.n_2$ edges.

It is easy to see that $G\odot H$ is not in general isomorphic to $H\odot G$.

In this paper we proposed the corona $P_m\odot F_n^2$ and show that is Total cordial for all $m\ge 1$ and $n\ge 4$.

2. Terminologies and Notations

A path with m vertices and $m-1$ edges is denoted by P_m , and second power of fan graph has $n+1$ vertices and $3n-3$ edges is denoted by F_n^2. We let $M_r$denote the labeling $0101\ldots01$, zero-one repeated r-times if r is even and $0101\ldots010$ if r is odd; for example, $M_6=010101$ and $M_5=01010$. We let M'_{2r} denote the labeling $1010\ldots10$. Sometimes, we modify the labeling M_r or M'_r by adding symbols at one end or the other (or both). We let S_{4r} denote the labeling $0_211\ 0_211\ldots0_211$ (repeated r-times), Let S'_{4r} denote the labeling $01_20\ 01_20\ldots01_20$ (repeated r-times).

The labeling $1_20_21_20_2\ldots1_20_2$ (repeated r-times) and labeling $10_2110_21\ldots10_21$ (repeated r-times) are written L_{4r} and L'_{4r}. Let M_r denote the labeling $01\ 01\ldots01$, zero-one repeated $r$times if r is even and $01\ 01\ldots010$ if r is odd; for example, $M_6 = 010101$ and $M_5 = 01010$. We let M'_r denote the labeling $1010\ldots10$. Sometimes, we modify the labeling M_r or M'_r by adding symbols at one end or the other (or both). Also, L_{4r} (or L'_{4r}) with extra labeling from right or left (or both sides) [9-12]. For specific labeling L and M of $G \odot H$ where G is path and H is a second power of fan graph, we let $[L; M]$ denote the corona labeling. Additional notation that we use is the following. For a given labeling of the corona $G \odot H$, we let v_i and e_i (for $i = 0,1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let $y_i$and b_i be those for H, which are connected to the vertices labeled 0 of G. Likewise, let y'_i and b'_i be those for H, which are connected to the vertices labeled 1 of G. In case it increases by one more vertex, so y''_i and b''_i will be those for H, which are connected to the vertex labeled 1 or 0 of G. It is easy to verify that $v_0 = x_0 + x_0y_0 + x_1y'_0$,$v_1 = x_1 + x_0y_1 + x_1y'_1$, $e_0 = a_0 + x_0b_0 + x_1b'_0 + x_0y_0 + x_1y'_1$ and $e_1 = a_1 + x_0b_1 + x_1b'_1 + x_0(x_0y_1) + x_1y'_0$. Thus $|(v_0 + e_0) - (v_1 + e_1)| = |(x_0 - x_1) + (a_0 - a_1) + x_0(b_0 - b_1) + x_1(b'_0 - b'_1) + 2x_0(y_0 - y_1)|$. Finally, for particular labeling A and B that are used for P_m and F_n^2.

3. The Total cordial of corona Product between paths and second power of Fan graphs

In this section, we explain that the corona between paths and the second power of Fan graphs $P_m \odot F_n^2$ are Total cordial for all $m \geq 1$, and $n \geq 4$. This goal will be achieved after the following lemmas.

Lemma 3.1 $P_m \odot F_n^2$ is Total cordial for all $m \geq 1$ and $n \equiv 0(mod4)$.

Proof. We should investigate the following cases :

Case (1). $\boldsymbol{m \equiv 0(mod4)}$.

Let $m = 4r$,$r \geq 1$. Then, one can select the labeling $[S_{4r}: 0M'_{4s}, 0\ M'_{4s}, 1M_{4s}, 1M_{4s}, \ldots (r - times)]$ for $P_{4r} \odot F_{4s}^2$. Therefore $x_0 = x_1 = 2r, a_1 = 2r - 1, a_0 = 2r, y_0 = 2s + 1, y_1 = 2s, b_1 = 6s - 1, b_0 = 6s - 2, y'_0 = 2s, y'_1 = 2s + 1, b'_1 = 6s - 1$ *and* $b'_0 = 6s - 2$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 1$. Thus $P_{4r} \odot F_{4s}^2$, $s \geq 1$ is Total cordial. As an example, Figure (4.1) illustrates this case $P_4 \odot F_4^2$.

Case (2). $m \equiv 1(mod4)$.

Let $m = 4r + 1$, $r > 0$. Then, one can select the labelling $[S_{4r}1: 0M_{4s}, 0M_{4s}, 1M_{4s}, 1M_{4s}, \ldots (r - times), 0M_{4s}]$ for $P_{4r+1} \odot F_{4s}^2$. Therefore $x_0 = 2r, x_1 = a_1 = 2r - 1, a_0 = 2r + 1, y_0 = 2s + 1, y_1 = 2s, b_1 = 6s - 2, b_0 = 6s - 1, y'_0 = 2s, y'_1 = 2s + 1, b'_1 = 6s - 2, b'_0 = 6s - 1, y''_0 = 2s + 1, y''_1 = 2s, b''_1 = 6s - 2$ *and* $b''_0 = 6s - 1$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 0$. Thus $P_{4r+1} \odot F_{4s}^2$, $s \geq 1$ is Total cordial.

Case (3). $m \equiv 2(mod4)$.

Let $m = 4r + 2$, $r > 0$. Then, one can select the labelling $[S_{4r}10: 0M'_{4s}, 0M'_{4s}, 1M_{4s}, 1M_{4s}, \ldots (r - times), 1M_{4s}, 0M'_{4s}]$ for $P_{4r+2} \odot F_{4s}^2$. Therefore $x_0 = x_1 = 2r + 1, a_1 = 2r, a_0 = 2r + 1, y_0 = 2s + 1, y_1 = 2s, b_1 = 6s - 1, b_0 = 6s - 2, y'_0 = 2s, y'_1 = 2s + 1, b'_1 = 6s - 1$ and $b'_0 = 6s - 2$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 1$. Thus $P_{4r+2} \odot F_{4s}^2$, $s \geq 1$ is Total cordial.

Case (4). $m \equiv 3(mod4)$.

Let $m = 4r + 3, r > 0$. Then, one can select the labelling $[S_{4r}110: 0M'_{4s}, 0M'_{4s}, 1M'_{4s}, 1M'_{4s} \ldots (r - times), 1M'_{4s}, 1M_{4s}, 0M'_{4s}]$ for $P_{4r+3} \odot F_{4s}^2$. Therefore $x_0 = 2r + 1, x_1 = a_1 = 2r, a_0 = 2r + 2, y_0 = 2s + 1, y_1 = 2s, b_1 = 6s - 1, b_0 = 6s - 2, y'_0 = 2s, y'_1 = 2s + 1, b'_1 = 6s - 1, b'_0 = 6s - 2, y''_0 = 2s + 1, y''_1 = 2s, b''_1 = 6s - 1$ and $b''_0 = 6s - 2$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 0$. Thus $P_{4r+3} \odot F_{4s}^2, s \geq 1$ is Total cordial.

Lemma 3.2 $P_m \odot F_n^2$ is Total cordial for all $m \geq 1$ and $n \equiv 1(mod4)$.

Proof. We should investigate the following cases:

Case (1). $\boldsymbol{m \equiv 0(mod4)}$.

Let $m = 4r$, $r \geq 1$. Then, one can select the labeling $[S_{4r}: 11_3 0_2 M_{4s-4}, 11_3\ 0_2 M_{4s-4}, 0_4 1_2 M'_{4s-4},$

$0_4 1_2 M'_{4s-4}, \ldots (r - times)]$ for $P_{4r} \odot F_{4s+1}^2$. Therefore $x_0 = x_1 = a_1 = 2r - 1, a_0 = 2r, y_0 = 2s, y_1 = 2s + 2, b_1 = 6s - 1, b_0 = 6s + 1, y'_0 = 2s + 2, y'_1 = 2s$ and $b'_1 = 6s - 1, b'_0 = 6s + 1$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 1$.Thus $P_{4r} \odot F_{4s+1}^2, s \geq 1$ is Total cordial. As an example, Figure (4.2) illustrates this case $P_4 \odot F_5^2$.

Case (2). $\boldsymbol{m \equiv 1(mod4)}$.

Let $m = 4r + 1, r > 0$. Then, one can select the labeling $[S_{4r}0{:}\, 11_3 0_2 M_{4s-4}, 11_3 0_2 M_{4s-4}, 0_4 1_2 M'_{4s-4}, 0_4 1_2 M'_{4s-4}, \ldots (r - times), 10_3 1_2 M_{4s-4}]$ for $P_{4r+1} \odot F^2_{4s+1}$. Therefore $x_0 = 2r + 1, x_1 = a_0 = a_1 = 2r, y_0 = 2s + 2, y_1 = 2s, b_1 = 6s + 1, b_0 = 6s - 1, y'_0 = 2s, y'_1 = 2s + 2, b'_1 = 6s + 1, b'_0 = 6s - 1, y''_0 = y''_1 = 2s + 1$, and $b''_0 = b''_1 = 6s$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 1$. Thus $P_{4r+1} \odot F^2_{4s+1}, s \geq 1$, is Total cordial.

Case (3). $\boldsymbol{m \equiv 2(mod4)}$.

Let $m = 4r + 2, r \geq 0$. Then, one can select the labeling $[S_{4r}101_4 0_2 M_{4s-4}, 11_3 0_2 M_{4s-4}, 0_4 1_2 M'_{4s-4}, 0_4 1_2 M'_{4s-4} \ldots (r - times), 0_4\, 1_2 M'_{4s-4}, 11_3 0_2 M_{4s-4}]$ for $P_{4r+2} \odot F^2_{4s+1}$. Therefore $x_0 = x_1 = a_1 = 2r, a_0 = 2r + 1, y_0 = 2s + 2, y_1 = 2s, b_1 = 6s + 1, b_0 = 6s - 1, y'_0 = 2s, y'_1 = 2s + 2$ and $b'_1 = 6s + 1, b'_0 = 6s - 1$. Hence It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 1$. Thus $P_{4r+2} \odot F^2_{4s+1}, s \geq 1$, is Total cordial.

Case (4). $\boldsymbol{m \equiv 3(mod4)}$.

Let $m = 4r + 3, r \geq 0$. Then, one can select the labeling $[S_{4r}101{:}\, 1_4 0_2 M_{4s-4}, 1_4 0_2 M_{4s-4}, 0_4 1_2 M'_{4s-4}, 0_4 1_2 M'_{4s-4}, \ldots (r - times), 0_4 1_2 M'_{4s-4}, 1_4 0_2 M_{4s-4}, 0_3 1_3 M'_{4s-4}]$ for $P_{4r+3} \odot F^2_{4s+1}$. Therefore $x_0 = a_0 = a_1 = 2r + 1, x_1 = 2r + 2, y_0 = 2s, y_1 = 2s + 2, b_1 = 6s - 1, b_0 = 6s + 1, y'_0 = 2s + 2, y'_1 = 2s, b'_1 = 6s - 1, b'_0 = 6s + 1, y''_0 = y''_1 = 2s + 1$, and $b''_0 = b''_1 = 6s$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 1$. Thus $P_{4r+3} \odot F^2_{4s+1}, s \geq 1$, is Total cordial.

Lemma 3.3 $P_m \odot F^2_n$ is Total cordial for all $m \geq 1$ and $n \equiv 2(mod4)$.

Proof. We need to study the following cases:

Case (1). $\boldsymbol{m \equiv 0(mod4)}$.

Let $m = 4r, r \geq 1$. Then, one can select the labeling $[S_{4r}{:}\, 0M'_{4s+2}, 0\, M'_{4s+2}, 1M_{4s+2}, 1M_{4s+2}, \ldots (r - times)]$ for $P_{4r} \odot F^2_{4s+2}$. Therefore $x_0 = x_1 = 2r, a_1 = 2r - 1, a_0 = 2r, y_0 = 2s + 2, y_1 = 2s + 1, b_1 = 6s + 1, b_0 = 6s + 2, y'_0 = 2s + 1, y'_1 = 2s + 2, b'_1 = 6s + 1$ and $b'_0 = 6s + 2$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 1$.Thus$P_{4r} \odot F^2_{4s+2}, r \geq 1$ is Total cordial. As an example, Figure (4.3) illustrates this case $P_4 \odot F^2_6$.

Case (2). $\boldsymbol{m \equiv 1(mod4)}$.

Let $m = 4r + 1, r \geq 0$. Then, one can select the labeling $[S_{4r}1{:}\, 0M'_{4s+2}\ , 0M'_{4s+2}, 1M'_{4s+2}, 1M'_{4s+2}, \ldots, (r - times), 0M_{4s+2}]$ for $P_{4r+1} \odot F^2_{4s+2}$. Therefore $x_0 = 2r, x_1 = 2r + 1, a_1 = 2r - 1, a_0 = 2r + 1, y_0 = 2s + 2, y_1 = 2s + 1, b_1 = 6s + 2, b_2 = 6s + 1, y'_0 = 2s + 1, y'_1 = 2s + 2, b'_1 = 6s + 2, b'_0 = 6s + 1, y"_0 = 2s + 2, y''_1 = 2s + 1, b''_1 = 6s + 2$ and $b"_0 = 6s + 1$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 0$. Thus$P_{4r+1} \odot F^2_{4s+2}, s \geq 1$ is Total cordial.

Case (3). $\boldsymbol{m \equiv 2(mod4)}$.

Let $m = 4r + 2, r \geq 0$. Then, one can select the labeling $[S_{4r}10{:}\, 0\ M'_{4s+2}, 0M'_{4s+2}, 1M_{4s+2}, 1M_{4s+2}, \ldots, (r - times), 1M_{4s+2}, 0M'_{4s+2}]$ for $P_{4r+2} \odot F^2_{4s+2}$. Therefore $x_0 = x_1 = a_0 = 2r + 1, a_1 = 2r, y_0 = 2s + 2, y_1 = 2s + 1, b_1 = 6s + 2, b_0 = 6s + 1, y'_0 = 2s + 1, y'_1 = 2s + 2, b'_1 = 6s + 2$ and $b'_0 = 6s + 1$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 1$. Thus $P_{4r+2} \odot F^2_{4s+2}, s \geq 1$ is Total cordial.

Case (4). $\boldsymbol{m \equiv 3(mod4)}$.

Let $m = 4r + 3, r \geq 0$. Then, one can select the labeling $[S_{4r}1_2 0{:}\, 0M'_{4s+2}, 0M'_{4s+2}, 1M_{4s+2}, 1M_{4s+2}, \ldots, (r - times), 1M_{4s+2}, 0_3 1_2\ \ M'_{4s-2}, 0M'_{4s+2}]$ for $P_{4r+3} \odot F^2_{4s+2}$. Therefore $x_0 = 2r + 1, x_1 = a_0 = 2r + 2, a_1 = 2r, y_0 = 2s + 2, y_1 = 2s + 1, b_1 = 6s + 2, b_0 = 6s + 1, y'_0 = 2s + 1, y'_1 = 2s + 2, b'_1 = 6s + 2, b'_0 = 6s + 1, y''_0 = 2s + 2, y''_1 = 2s + 1, b''_1 = 6s + 2$ and $b''_0 = 6s + 1$. Hence, It is simple to prove that $|(v_0 + e_0) - (v_1 + e_1)| = 0$. Thus $P_{4r+3} \odot F^2_{4s+2}, s \geq 1$ is Total cordial.

Lemma 3.4 $P_m \odot F^2_n$ is Total cordial for all $m \geq 1$ and $n \equiv 3(mod4)$.

Proof: We should examine the following cases:

Case (1). $\boldsymbol{m \equiv 0(mod4)}$.

Let $m = 4r, r \geq 1$. Then, one can select the labeling $[S_{4r}{:}\, 10_3 M'_{4s}, 10_3\ M'_{4s}, 101_2 M_{4s}, 101_2 M_{4s}, \ldots, (r - times)]$ for $P_{4r} \odot F^2_{4s+3}$. Therefore $x_0 = x_1 = a_0 = 2r, a_1 = 2r - 1, y_0 = 2s + 3, y_1 = 2s + 1, b_1 = 6s + 4, b_0 = 6s +$

$2, y_0' = 2s+1, y_1' = 2s+3$ and $b_1' = 6s+4, b_0' = 6s+2$. Hence, It is simple to prove that $|(v_0+e_0)-(v_1+e_1)| = 1$. Thus $P_{4r} \odot F_{4s+3}^2, s \geq 1$ is Total cordial. As an example, Figure (4.4) illustrates this case $P_4 \odot F_7^2$.

Case (2). $\boldsymbol{m \equiv 1(mod4)}$.

Let $m = 4r+1, r \geq 0$. Then, one can select the labeling $[S_{4r}1{:}\,10_3M_{4s}', 1\ 0_3M_{4s}', 101_2M_{4s}, 101_2M_{4s}, \ldots, (r-times), 01_20M_{4s}']$ for $P_{4r+1} \odot F_{4s+3}^2$. Therefore $x_0 = 2r, x_1 = a_1 = 2r-1, a_0 = 2r+1, y_0 = 2s+3, y_1 = 2s+1, b_1 = 6s+2, b_0 = 6s+4, y_0' = 2s+1, y_1' = 2s+3, b_1' = 6s+2, b_0' = 6s+4, y_0'' = y_1'' = 2s+2, b_1'' = 6s+2$ and $b_0'' = 6s+4$. Hence It is simple to prove that $|(v_0+e_0)-(v_1+e_1)| = 1$. Thus $P_{4r+1} \odot F_{4s+3}^2, s \geq 1$ is Total cordial.

Case (3). $\boldsymbol{m \equiv 2(mod4)}$.

Let $m = 4r+2, r \geq 0$. Then, one can select the labeling $[S_{4r}10{:}\,10_3M_{4s}', 10_3M_{4s}', 101_2M_{4s}, 101_2M_{4s}, \ldots, (r-times), 101_2\ M_{4s}, 10_3M_{4s}']$ for $P_{4r+2} \odot F_{4s+3}^2$. Therefore $x_0 = x_1 = a_1 = 2r+1, a_0 = 2r, y_0 = 2s+3, y_1 = 2s+1, b_1 = 6s+4, b_0 = 6s+2, y_0' = 2s+1, y_1' = 2s+3, b_1' = 6s+4$ and $b_0' = 6s+2$. Hence, It is simple to prove that $|(v_0+e_0)-(v_1+e_1)| = 1$. Thus $P_{4r+2} \odot F_{4s+3}^2, s \geq 1$ is Total cordial.

Case (4). $\boldsymbol{m \equiv 3(mod4)}$.

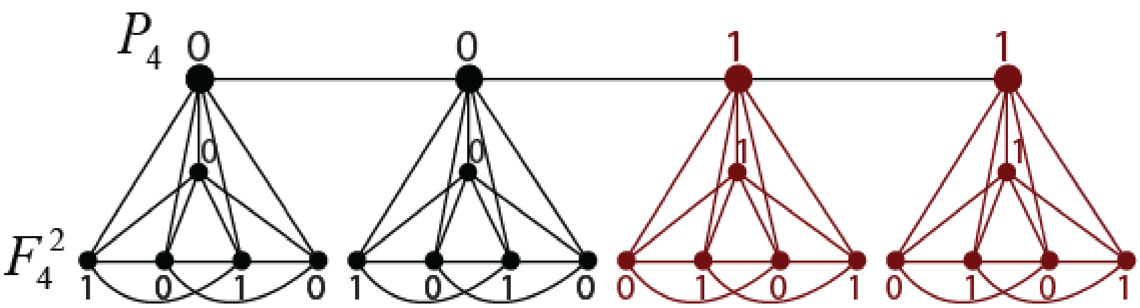
Let $m = 4r+3, r \geq 0$. Then, one can select the labeling $[S_{4r}10_2{:}\,10_3M_{4s}', 10_3M_{4s}', 101_2M_{4s}, 101_2M_{4s}, \ldots, (r-times), 101_2\ M_{4s}, 10_3M_{4s}', 01_20M_{4s}']$ for $P_{4r+3} \odot F_{4s+3}^2$. Therefore $x_0 = a_0 = 2r+2, x_1 = 2r+1, a_1 = 2r, y_0 = 2s+3, y_1 = 2s+1, b_1 = 6s+4, b_0 = 6s+2, y_0' = 2s+1, y_1' = 2s+3, b_1' = 6s+4, b_0' = 6s+2, y_0'' = y_1'' = 2s+2, b_1'' = 6s+4$ and $b_0'' = 6s+2$. Hence It is simple to prove that $|(v_0+e_0)-(v_1+e_1)| = 1$. Thus $P_{4r+3} \odot F_{4s+3}^2, s \geq 1$ is Total cordial.

The following theorem may be established as a result of all preceding lemmas.

Theorem 3.1. The corona between path and fourth power of fan graphs $P_m \odot F_n^2$ is Total cordial for all $m \geq 1$ and $n \geq 4$.

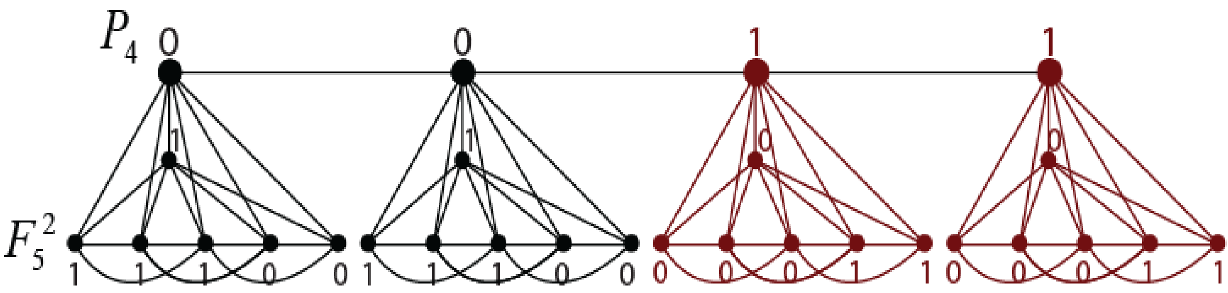
4. Example

The Total cordial graph of $P_4 \odot F_4^2, P_4 \odot F_5^2, P_4 \odot F_6^2$ and $P_4 \odot F_7^2$ are illustrated in Figures (4.1, ..., 4.4).



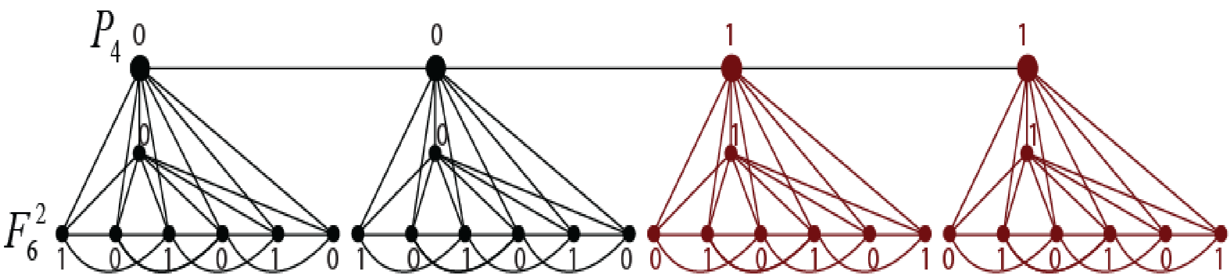
$|(v_0+e_0)-(v_1+e_1)| = 1$

Figure (4.1). $P_4 \odot F_4^2$ is Total cordial



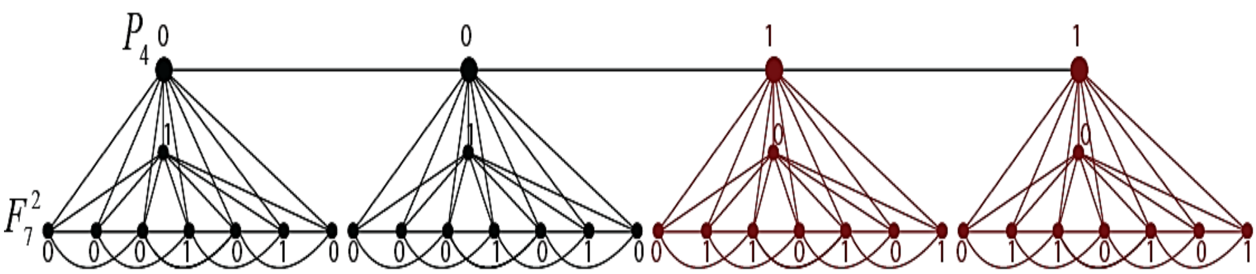
$$|(v_0 + e_0) - (v_1 + e_1)| = 1$$

Figure (4.2). $P_4 \odot F_5^2$ is Total cordial.



$$|(v_0 + e_0) - (v_1 + e_1)| = 1$$

Figure (4.3). $P_4 \odot F_6^2$ is Total cordial



$$|(v_0 + e_0) - (v_1 + e_1)| = 1$$

Figure (4.4). $P_4 \odot F_7^2$ is Total cordial.

5. Conclusion

In this work, we proved that the corona between paths and second power of Fan graphs $P_m \odot F_n^2$ is Total cordial for all $m \geq 1$, and $n \geq 4$. An example is introduced in section 4.

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