Elzaki Substitution Method with Adomian Polynomials for Solving Initial Value Problems of n^th Order Non-linear Mixed Type Partial Differential Equations.

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Abstract: A method entitled the Elzaki Substitution Method (ESM) will be generalized in this article , an overview of the ESM for n^{th}-order non-linear partial differential equations (NPDEs) involving the following three types of mixed derivatives: $\frac{\partial^n u}{\partial x^{n-1}\partial y}, \frac{\partial^n u}{\partial x \partial y^{n-1}}, \frac{\partial^n u}{\partial x^p \partial y^q}$ (where p and q are positive integers such that $p+q=n$) will be included. ESM technique often includes Adomian polynomial for non-linear terms to find exact solutions of non-linear homogeneous and non-homogeneous higher-order initial value problems containing mixed partial derivatives. Illustrative examples are presented in order to demonstrate the utility and applicability of the proposed procedure.

Keywords: Elzaki Transforms, Mixed Partial Derivatives, n^{th}-Order Partial Derivatives, Elzaki Substitution Method, Adomian Polynomial.

1. Introduction

By NLPDEs many topics of applied mathematics, physics, and engineering-related problems with physical interest and fundamental significance can be explained. NLPDEs which involve mixed partial derivatives with initial conditions naturally occur in various fields of science, physics and engineering. The comprehensive applicability of these equations has gained so much attention from many mathematicians. However, in physics and applied mathematics it is still a major problem to obtain the exact solutions of these NLPDEs, some new methods applied to discover new exact solutions of these NLPDEs. Many researchers have paid special attention in recent years to seeking a solution of NLPDEs by using various methodologies. Some of these are the Adomian decomposition method (Evans & Raslan, 2005), homotopy perturbation method (Sweilam & Khader,2009; Sharma & Methi, 2011), Laplace Substitution Method(LSM) (Handibag & Karande, 2012 ; Handibag & Karande,2015), Laplace Variation Iteration Method (Al-Fayadh & Faraj,2019), Sumudu Homotopy Perturbation Method (Singh & Kumar,2011), Elzaki Variation Iteration Method (Elzaki & Kim, 2015), Elzaki Projected Differential Transform Method (Elzaki, Hilal, Arabia, & Arabia, 2012b), Elzaki Homotopy Perturbation Method (Elzaki, Hilal, Arabia, & Arabia, 2012a), and Elzaki Decomposition Method (Ziane & Cherif, 2015; Nuruddeen , 2017). Motivated and inspired by this ongoing research fields, a new method entitled ESM will be determined which is based on Elzaki Transform (Elzaki, 2011; Elzaki & Ezaki, 2011), introduced by Tarig Elzaki in 2011. Furthermore Adomian Polynomials can be applied to manage the non-linear term in this proposed method. The ESM was introduced in (Datta, Habiba, & Hossain, 2020; Hossain & Datta, 2018) to solve the second-order linear and non-linear partial differential equations (PDEs) with mixed partial derivatives. By ESM, the solution of n^{th}-order NLPDEs involving mixed partial derivatives with initial conditions will be explained. Also, the basic concepts and fundamental properties of Elzaki transform will be discussed in the section (2). This proposed approach is elaborated in the next section (3). Three subsections are provided in the same section that are classified on the basis of mixed derivatives forms i.e $\frac{\partial^n u}{\partial x^{n-1}\partial y}, \frac{\partial^n u}{\partial x \partial y^{n-1}}, \frac{\partial^n u}{\partial x^p \partial y^q}$ (where p and q are positive integers such that $p+q=n$). In section (4), the strength and efficiency of the proposed method is verified with illustrative examples. Finally the conclusion of this article will be presented in the last section.

2. Preliminaries

2.1. Elzaki transform

A new integral transform named Elzaki transform operator is denoted by E [.] and is defined by the integral equation as;

$E[f(t)] = T(v) = v\int_0^{\infty} f(t)\ e^{-\frac{t}{v}}dt, \quad t > 0$

Where the variable v is used in function f reasoning to factor the variable *t*.

2.2. Fundamental properties of Elzaki transformation

(1) $E[1] = v^2$

(2) $E[t] = v^3$

(3) $E[t^n] = n!\, v^{n+2},\ n \geq 0$

(4) $E[e^{at}] = \frac{v^2}{1-av}$

(5) $E[sin\ a\ t] = \frac{av^3}{1+a^2v^2}$

(6) $E[cos\ a\ t] = \frac{v^3}{1+a^2v^2}$

Theorem: Let $u(x,t)$ be a function of two independent variables u and t, then

(1) $E[u(x,t)] = T(x,v)$

(2) $E\left[\frac{\partial u(x,t)}{\partial t}\right] = \frac{1}{v}T(x,v) - vu(x,0)$

(3) $E\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = \frac{1}{v^2}T(x,v) - u(x,0) - v\frac{\partial u(x,0)}{\partial t}$

(4) $E\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] = \frac{E[u(x,t)]}{v^n} - \frac{u(x,0)}{v^{n-2}} - \frac{1}{v^{n-3}}\frac{\partial u(x,0)}{\partial t} - \quad \cdots \quad \cdots \quad \cdots \quad - \frac{\partial^{n-2}u(x,0)}{\partial t^{n-2}} - v\frac{\partial^{n-1}u(x,0)}{\partial t^{n-1}}.$

3. Elzaki Substitution Method

In this section, ESM for n^{th} non-linear PDEs involving mixed partial derivatives will be described. In this description, the nonlinear term can be decomposed with the help of Adomian polynomials. In the same section, three subsections are also provided which based on separated types of mixed derivatives that embedded in equation.

3.1. ESM for n^{th} order non-linear PDEs involving mixed partial derivatives of type $\frac{\partial^n u}{\partial x^{n-1}\partial y}$

Consider a n^{th} order non-linear partial differential equation,

$Lu(x,y) + Nu(x,y) + Ru(x,y) = \square(x,y)$
(3.1.1)

with initial condition

$u(x,0) = g(x), u_y(0,y) = f_0(y), u_{xy}(0,y) = f_1(y), u_{x^2y}(0,y) = f_2(y), u_{x^3y}(0,y) = f_3(y), \ldots \quad \ldots \quad \ldots,$
$u_{x^{n-2}y}(0,y) = f_{n-2}(y)$ (3.1.2)

where $L = \frac{\partial^n}{\partial x^{n-1}\partial y}$, $Nu(x,y)$ is a nonlinear term, $Ru(x,y)$ is a remaining linear term and $\square(x,y)$ is a source term.

Re-write the equation (3.1.1) as,

$\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial u}{\partial y}\right) + Nu(x,y) + Ru(x,y) = h(x,y)$

Consider the substitution $U = \frac{\partial u}{\partial y}$, the above equation is converted to,

$\frac{\partial^{n-1}U}{\partial x^{n-1}} + Nu(x,y) + Ru(x,y) = h(x,y)$ (3.1.3)

Applying Elzaki transform with respect to x equation (3.1.3) is reduced to

$$\frac{1}{v^{n-1}}E_x[U(x,y)] - \frac{1}{v^{n-3}}U(0,y) - \frac{1}{v^{n-4}}U_x(0,y) - \frac{1}{v^{n-5}}U_{x^2}(0,y) - \cdots - vU_{x^{n-2}}(0,y)$$
$$= E_x[h(x,y) - Nu(x,y) - Ru(x,y)]$$

Using initial conditions from equation (3.1.2)-

$$E_x[U(x,y)] - v^2 f_0(y) - v^3 f_1(y) - v^4 f_2(y) - \cdots - v^n f_{n-2}(y) = v^{n-1}E_x[h(x,y) - Nu(x,y) - Ru(x,y)]$$

Applying inverse Elzaki transform with respect to x, the above equation is converted to

$$U(x,y) = f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \cdots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y) + E_x{}^{-1}\left[v^{n-1}E_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Re-substitute the value of $U = \frac{\partial u}{\partial y}$ on the above equation,

$$\frac{\partial u(x,y)}{\partial y} = f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \cdots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y) + E_x{}^{-1}\left[v^{n-1}E_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Taking Elzaki transform with regard to y and initial conditions on the above equation,

$$E_y[u(x,y)] = v^2 g(x) + vE_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \cdots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]$$
$$+vE_y\left[E_x{}^{-1}\left[v^{n-1}E_x[\square(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]$$

Using inverse Elzaki transform with respect to y on the above equation implies that

$$u(x,y) = g(x) + E_y{}^{-1}\left[vE_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \ \dots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]\right]$$
$$+E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}\left[v^{n-1}E_x[\square(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]\right] \quad (3.1.4)$$

For solving n^{th} order NLPDEs involving mixed derivatives by ESM, consider the solution in an infinite series form.

Consider that,

$$u(x,y) = \sum_{n=\infty}^{\infty} u_n(x,y) \quad (3.1.5)$$

be the required series solution of equation (3.1.1). The non-linear term can be decomposed by using Adomian polynomial which is defined by the following formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\gamma^n}\left[N[\textstyle\sum_{i=0}^{\infty}\gamma^i u_i]\right]\right]_{\gamma=0} \quad (3.1.6)$$

And therefore $Nu(x,y) = \sum_{n=0}^{\infty} A_n$ (3.1.7)

Where, A_n is the Adomian polynomial with components $u_0(x,y), u_1(x,y), \dots \ \dots \ \dots, u_n(x,y)$, $n \geq 0$ of series (3.1.5).

Substitute the values of $u(x,y)$ and $Nu(x,y)$ in equation (3.1.4), then new one below is obtained

$$\sum_{n=0}^{\infty} u_n(x,y) = g(x) + E_y{}^{-1}\left[vE_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \ \dots\ \dots\ \dots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]\right]$$
$$+E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}\left[v^{n-1}E_x\left[h(x,y) - \sum_{n=0}^{\infty} A_n - R\left[\sum_{n=0}^{\infty} u_n(x,y)\right]\right]\right]\right]\right]$$

Comparing them on both sides, the following results is obtained

$$u_0(x,y) = g(x) + E_y{}^{-1}\left[vE_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \ \dots\ \dots\ \dots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]\right]$$
$$+E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}\left[v^{n-1}E_x[\square(x,y)]\right]\right]\right]$$
$$u_1(x,y) = -E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}\left[v^{n-1}E_x[A_0 + Ru_0(x,y)]\right]\right]\right]$$
$$u_2(x,y) = -E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}\left[v^{n-1}E_x[A_1 + Ru_1(x,y)]\right]\right]\right]$$
$$\dots\ \dots\ \dots$$
$$u_n(x,y) = -E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}\left[v^{n-1}E_x[A_{n-1} + Ru_{n-1}(x,y)]\right]\right]\right] \quad ,n \geq 1$$

The following recursive relation is found from the above equations

$$\left.\begin{aligned} &u_0(x,y) = k(x,y)\\ &u_{n+1}(x,y) = -E_y{}^{-1}[vE_y[E_x{}^{-1}[v^{n-1}E_x[A_n + Ru_n(x,y)]]]], \quad n \geq 0 \end{aligned}\right\} \tag{3.1.8}$$

Where, $K(x,y) =$
$g(x) + E_y{}^{-1}\left[vE_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \ \dots\ \dots\ \dots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]\right] +$
$E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}[v^{n-1}E_x[\square(x,y)]]\right]\right]$
From recursive relation (3.1.8), the components $u_0(x,y), u_1(x,y),\ \dots\ \dots\ \dots, u_n(x,y), n \geq 0$ of series (3.1.5) can be calculated.
Substituting the values of $u_n(x,y), n \geq 0$ in equation (3.1.5), the series solution of equation (3.1.1) is obtained.
The initial conditions of equation (3.1.1) are converted to

$$\left.\begin{aligned} &u_{x^{n-1}}(x,0) = g(x), u(0,y) = f_0(y), u_x(0,y) = f_1(y), u_{x^2}(0,y) = f_2(y),\\ &u_{x^3}(0,y) = f_3(y),\ \dots\ \dots\ \dots, u_{x^{n-2}}(0,y) = f_{n-2}(y) \end{aligned}\right\} \tag{3.1.9}$$

Now the substitution $U = \frac{\partial u}{\partial y}$ is not applicable for equation (3.1.1). In this case, the above-explained method can't be applied. Thus when such a situation appears, the ESM needs to be described again. Using Young's theorem, the mixed derivative of n^{th} order $\frac{\partial^n u}{\partial x^{n-1}\partial y}$ appeared in equation (3.1.1) which can be written in the form $\frac{\partial^n u}{\partial y \partial x^{n-1}}$. Therefore the n^{th} order non-linear PDEs containing mixed derivatives (3.1.1) with initial conditions (3.1.9) becomes,

$$\frac{\partial^n u}{\partial y \partial x^{n-1}} + Nu(x,y) + Ru(x,y) = h(x,y) \tag{3.1.10}$$

$u_{x^{n-1}}(x,0) = g(x), u(0,y) = f_0(y), u_x(0,y) = f_1(y), u_{x^2}(0,y) = f_2(y), u_{x^3}(0,y) = f_3(y),\ \dots\ \dots\ \dots,$
$u_{x^{n-2}}(0,y) = f_{n-2}(y)$
Substituting $\frac{\partial^{n-1}u}{\partial x^{n-1}} = U$ in equation (3.1.10), then the equation is converted to,

$$\frac{\partial U}{\partial y} + Nu(x,y) + Ru(x,y) = h(x,y)$$

Applying Elzaki transform with respect to y subject to initial conditions in the above equation and then taking inverse Elzaki transform, then new one below is found

$$U(x,y) = g(x) + E_y{}^{-1}\left[vE_y[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

In the above equation, re-substituting the value of $U(x,y)$ means that

$$\frac{\partial^{n-1}u(x,y)}{\partial x^{n-1}} = g(x) + E_y{}^{-1}\left[vE_y[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Regarding to x taking Elzaki transform in the above equation with initial conditions and then using inverse Elzaki transform, the above equation is transformed to

$$\begin{aligned} &u(x,y) = f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) - \cdots\ \dots\ \dots - \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\\ &+E_x{}^{-1}[v^{n-1}E_x[g(x + E_y{}^{-1}[vE_y[h(x,y) - Nu(x,y) - Ru(x,y)]]]]] \end{aligned} \tag{3.1.11}$$

Suppose that,

$$u(x,y) = \sum_{n=\infty}^{\infty} u_n(x,y) \tag{3.1.12}$$

be the required solution of equation (3.1.10).As in the above description ,the nonlinear term $Nu(x,y)$ can be decomposed by using Adomian polynomial same as the equation (3.1.7).
Now the equation (3.1.11) is converted to

$$\begin{aligned} &\sum_{n=0}^{\infty} u_n(x,y) = f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) -\ \dots\ \dots\ \dots - \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\\ &+E_x{}^{-1}\left[v^{n-1}E_x\left[g(x) + E_y{}^{-1}\left[vE_y[\square(x,y) - \sum_{n=0}^{\infty} A_n - R\sum_{n=0}^{\infty} u_n(x,y)]\right]\right]\right] \end{aligned} \tag{3.1.13}$$

The following recursive relations are obtained by comparing both sides of the above equation

$$\left.\begin{aligned} &u_0(x,y) = k(x,y)\\ &u_{n+1}(x,y) = -E_x{}^{-1}[v^{n-1}E_x[E_y{}^{-1}[vE_y[A_n + Ru_n(x,y)]]]]] \end{aligned}\right\} \tag{3.1.14}$$

Where

$$k(x,y) = f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) -\ \dots\ \dots\ \dots - \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)$$

$$+E_x{}^{-1}\left[v^{n-1}E_x[g(x)]\right] + E_x{}^{-1}\left[v^{n-1}E_x\left[E_y{}^{-1}\left[vE_y[h(x,y)]\right]\right]\right]$$

is a source term. Calculating and substituting the components of $u_n(x,y), n \geq 0$ in equation (3.1.12), the series solution of the given equation is obtained.

3.2. ESM for nth order non-linear PDEs with mixed partial derivatives of type $\frac{\partial^n u}{\partial x \partial y^{n-1}}$

Considering a n^{th} order non-linear PDE –

$$Lu(x,y) + Nu(x,y) + Ru(x,y) = \square(x,y) \tag{3.2.1}$$

with initial conditions,

$$\left.\begin{array}{l} u_{y^{n-1}}(0,y) = g(y), u(x,0) = f_0(x), u_y(x,0) = f_1(x), u_{y^2}(x,0) = f_2(x), \\ u_{y^3}(x,0) = f_3(x), \;\ldots\;\ldots\;\ldots, u_{y^{n-2}}(x,0) = f_{n-2}(x) \end{array}\right\} \tag{3.2.2}$$

Where $L = \frac{\partial^n}{\partial x \partial y^{n-1}}$, $Nu(x,y)$ is the nonlinear term, $Ru(x,y)$ is the remaining linear term and $\square(x,y)$ is the source term.

Substituting $= \frac{\partial^{n-1} u}{\partial y^{n-1}}$, the given equation is reduced to

$$\frac{\partial U}{\partial x} + Nu(x,y) + Ru(x,y) = h(x,y) \tag{3.2.3}$$

Regarding x, applying Elzaki transform with initial conditions in the above equation and then inverse Elzaki transform, finally the following result is obtained-

$$U(x,y) = g(y) + E_x{}^{-1}\left[vE_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

$$\frac{\partial^{n-1} u}{\partial y^{n-1}}(x,y) = g(y) + E_x{}^{-1}\left[vE_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Again, using Elzaki transform subject to initial conditions and inverse Elzaki transform with regard to y the equation is transformed to-

$$u(x,y)] = f_0(x) + yf_1(x) + \frac{y^2}{2!}f_2(y) + \;\ldots\;\ldots\;\ldots + \frac{y^{n-2}}{(n-2)!}f_{n-2}(x)$$

$$+E_y{}^{-1}\left[v^{n-1}E_y\left[g(y) + E_x{}^{-1}\left[vE_x[\square(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]\right] \tag{3.2.4}$$

For solving n^{th} order non-linear PDE by ESM let us suppose that,

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \tag{3.2.5}$$

be the series solution of the given equation. The nonlinear term $Nu(x,y)$ can be decomposed with the help of Adomian polynomial which is defined by the equation (3.1.6)

Therefore the non-linear term $Nu(x,y) = \sum_{n=0}^{\infty} A_n$ (3.2.6)

Putting the values of $u(x,y)$ and $Nu(x,y)$ in equation (3.2.4), the equation is converted to-

$$\sum_{n=0}^{\infty} u_n(x,y) = f_0(x) + yf_1(x) + \frac{y^2}{2!}f_2(x) - \;\ldots\;\ldots\;\ldots - \frac{y^{n-2}}{(n-2)!}f_{n-2}(x) +$$

$$+E_y{}^{-1}\left[v^{n-1}E_y\left[g(y) + E_x{}^{-1}\left[vE_x\left[h(x,y) - \sum_{n=0}^{\infty} A_n - R\left[\sum_{n=0}^{\infty} u_n(x,y)\right]\right]\right]\right]\right] \tag{3.2.7}$$

The following recursive relation is obtained by comparing both sides of the equation (3.2.7)

$$\left.\begin{array}{l} u_0(x,y) = k(x,y) \\ u_{n+1}(x,y) = -E_y{}^{-1}[v^{n-1}E_y[E_x{}^{-1}[vE_x[A_n + Ru_n(x,y)]]]] \end{array}\right\} \tag{3.2.8}$$

Where $k(x,y) = f_0(x) + yf_1(x) + \frac{y^2}{2!}f_2(x) - \;\ldots\;\ldots\;\ldots - \frac{y^{n-2}}{(n-2)!}f_{n-2}(x) + E_y{}^{-1}\left[v^{n-1}E_y[g(y)]\right]$

$$+E_y{}^{-1}\left[v^{n-1}E_y\left[E_x{}^{-1}\left[vE_x[h(x,y)]\right]\right]\right]$$

By using the recursive relation of (3.2.8), evaluating and substituting the components $u_n(x,y), n \geq 0$ in equation (3.2.5), the series solution of considered equation is obtained.

The initial conditions in the equation (3.2.1) are becomes:

$$\left.\begin{array}{l} u(0,y) = g(y), u_x(x,0) = f_0(x), u_{yx}(x,0) = f_1(x), u_{y^2x}(x,0) = f_2(x), \\ u_{y^3x}(x,0) = f_3(x), \;\ldots\;\ldots\;\ldots, u_{y^{n-2}x}(x,0) = f_{n-2}(x) \end{array}\right\} \tag{3.2.9}$$

Then in this situation, use the substitution $U = \frac{\partial u}{\partial x}$ in equation (3.2.1). In subsection (3.1), ESM for non-linear PDEs with mixed partial derivatives of type $\frac{\partial^n u}{\partial x^{n-1} \partial y}$ have been explained. In this case, using the same process the solution of non-linear PDEs with mixed partial derivatives of type $\frac{\partial^n u}{\partial x \partial y^{n-1}}$ by the proposed ESM can be found.

3.3. ESM for n[th] order non-linear PDEs involving mixed partial derivatives of type $\frac{\partial^n u}{\partial x^p \partial y^q}$, (where p and q are positive integers with $p+q=n$)

Considering a n^{th} order non-linear PDE-

$$Lu(x,y)+Nu(x,y)+Ru(x,y)=\square(x,y) \tag{3.3.1}$$

With initial conditions,

$$\left.\begin{array}{l} u_{y^q}(0,y)=f_0(y), u_{xy^q}(0,y)=f_1(y), u_{x^2y^q}(0,y)=f_2(y), \ \ldots \ \ldots \ \ldots, u_{x^{p-2}y^q}(0,y) \\ \quad =f_{p-2}(y), u_{x^{p-1}y^q}(0,y)=f_{p-1}(y), u(x,0)=g_0(x), u_y(x,0) \\ \quad =g_1(x), u_{y^2}(x,0)=g_2(x), \ \ldots \ \ldots \ \ldots, u_{y^{q-1}}(x,0)=g_{q-1}(x) \end{array}\right\} \tag{3.3.2}$$

Where $L=\frac{\partial^n}{\partial x^p \partial y^q}$, $Nu(x,y)$ be nonlinear term, $Ru(x,y)$ be the remaining linear term and $\square(x,y)$ be the source term.

The equation (3.3.1) can be written as

$$\frac{\partial^p}{\partial x^p}\left(\frac{\partial^q u}{\partial y^q}\right)+Nu(x,y)+Ru(x,y)=\square(x,y)$$

Using the substitution $U=\frac{\partial^q u}{\partial y^q}$, and then the above equation becomes

$$\frac{\partial^p U}{\partial x^p}+Nu(x,y)+Ru(x,y)=h(x,y)$$

Applying Elzaki transform with regard to x and initials conditions of the above equation, then new one below is found

$$E_x[U(x,y)]-v^2f_0(y)-v^3f_1(y)-v^4f_2(y)- \ \ldots \ \ldots \ \ldots -v^pf_{p-2}(y)-v^{p+1}f_{p-1}(y)=v^pE_x[\square(x,y) -Nu(x,y)-Ru(x,y)]$$

Again, with regard to x taking inverse Elzaki transform, the above equation is converted to

$$U(x,y)=f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)+ \ \ldots \ \ldots \ \ldots +\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}(y) +E_x^{-1}\left[v^pE_x[h(x,y)-Nu(x,y)-Ru(x,y)]\right]$$

By re-substituting $U=\frac{\partial^q u}{\partial y^q}$, the above equation becomes

$$\frac{\partial^q u}{\partial y^q}=f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)+ \ \ldots \ \ldots \ \ldots +\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}(y) +E_x^{-1}\left[v^pE_x[\square(x,y)-Nu(x,y)-Ru(x,y)]\right]$$

Now, taking Elzaki transform with regard to y subject to initial conditions, the equation implies that

$$E_y[u(x,y)]-v^2g_0(x)-v^3g_1(x)-v^4g_2(x)- \ \ldots \ \ldots \ \ldots -v^{q+1}g_{q-1}(x)=v^qE_y[f_0(y) +xf_1(y)+\frac{x^2}{2!}f_2(y)+ \ \ldots \ \ldots \ \ldots +\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}(y)]+v^qE_y[E_x^{-1} [v^pE_x[h(x,y)-Nu(x,y)-Ru(x,y)]]]$$

Again with respect to y applying inverse Elzaki transform,

$$u(x,y)=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)+ \ \ldots \ \ldots \ \ldots +\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)+E_y^{-1}[v^qE_y [f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)+ \ \ldots \ \ldots \ \ldots +\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}(y)]] +E_y^{-1}\left[v^qE_y\left[E_x^{-1}\left[v^pE_x[\square(x,y)-Nu(x,y)-Ru(x,y)]\right]\right]\right] \tag{3.3.3}$$

For solving n[th] order non-linear PDE of this type by ESM, let's consider that

$$u(x,y)=\sum_{n=\infty}^{\infty}u_n(x,y) \tag{3.3.4}$$

be the required series solution of equation (3.3.1). With the help of Adomian polynomial the nonlinear term $Nu(x,y)$ can be decomposed and is defined by the equation (3.1.6)

$$Nu(x,y)=\sum_{n=0}^{\infty}A_n \tag{3.3.5}$$

Substituting the values of $u(x,y)$ and $Nu(x,y)$ values in the equation (3.3.3) which is transformed to-

$$\sum_{n=0}^{\infty}u_n(x,y)=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)+ \ \ldots \ \ldots \ \ldots +\frac{y^{q-1}}{(q-1)!}g_{q-1}(x) +E_y^{-1}\left[v^qE_y\left[f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)+ \ \ldots \ \ldots \ \ldots +\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}(y)\right]\right]$$

$$+E_y{}^{-1}\left[v^q E_y\left[E_x{}^{-1}\left[v^p E_x\left[\square(x,y)-\sum_{n=0}^{\infty}A_n-R\left[\sum_{n=0}^{\infty}u_n\,(x,y)\right]\right]\right]\right]\right]$$

The following recursive relation is obtained by comparing both sides of the above equation.

$$\left.\begin{aligned} &u_0(x,y)=K(x,y)\\ &u_{n+1}(x,y)=-E_y{}^{-1}[v^q E_y[E_x{}^{-1}[v^p E_x[A_n+Ru_n(x,y)]]]],\quad n\geq 0\end{aligned}\right\} \tag{3.3.6}$$

Where

$$K(x,y)=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)+\ \dots\ \dots\ \dots+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)$$
$$+E_y{}^{-1}\left[v^q E_y\left[f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)+\ \dots\ \dots\ \dots+\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}(y)\right]\right]$$
$$+E_y{}^{-1}[v^q E_y[E_x{}^{-1}[v^p E_x[\square(x,y)]]]]]$$

From recursive relation (3.3.6), the components $u_0(x,y),u_1(x,y),\ \dots\ \dots\ \dots,u_n(x,y),n\geq 0$ of equation (3.3.4) are found.

Therefore the series solution of the given equation is obtained by substituting all these values in equation (3.3.4), Let's solve the nonlinear equation (3.3.1) with the following conditions

$$\left.\begin{aligned} &u_{x^p}(x,0)=g_0(x),u_{yx^p}(x,0)=g_1(y),u_{y^2x^p}(x,0)=g_2(x),\ \dots\ \dots\ \dots,\\ &u_{y^{q-2}x^p}x,0)=g_{q-2}(x),u_{y^{q-1}x^p}x,0)=g_{q-1}(x),u(0,y)=f_0(y),\\ &u_x(0,y)=f_1(y),u_{x^2}(0,y)=f_2(y),\ \dots\ \dots\ \dots,u_{x^{p-1}}(0,y)=f_{p-1}(y)\end{aligned}\right\} \tag{3.3.7}$$

Then by Young's theorem, the equation (3.3.1) can be written in the form:

$$\frac{\partial^q}{\partial y^q}\left(\frac{\partial^p u}{\partial x^p}\right)+Nu(x,y)+Ru(x,y)=\square(x,y)$$

By substitution $U=\frac{\partial^p u}{\partial x^p}$, the above equation becomes

$$\frac{\partial^q U}{\partial y^q}+Nu(x,y)+Ru(x,y)=h(x,y)$$

Applying Elzaki transform with respect to y subject to initial conditions and then taking inverse Elzaki transform, which is converted to the new one below,

$$U(x,y)=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)+\ \dots\ \dots\ \dots+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)+E_y{}^{-1}[v^q E_y[h(x,y)$$
$$-Nu(x,y)-Ru(x,y)]]$$
$$\frac{\partial^p u}{\partial x^p}=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)+\ \dots\ \dots\ \dots+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)+E_y{}^{-1}[v^q E_y[\square(x,y)$$
$$-Nu(x,y)-Ru(x,y)]]$$

Again with respect to x applying Elzaki transform subject to initial conditions and inverse Elzaki transform, then the equation implies that

$$u(x,y)=f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)+\ \dots\ \dots\ \dots+\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}(y)$$
$$+E_x{}^{-1}\left[v^p E_x\left[g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2\,(x)+\ \dots\ \dots\ \dots+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)\right]\right]$$
$$+E_x{}^{-1}\left[v^p E_x\left[E_y{}^{-1}\left[v^q E_y[\square(x,y)-Nu(x,y)-Ru(x,y)]\right]\right]\right] \tag{3.3.8}$$

Now assume that,

$$u(x,y)=\textstyle\sum_{n=\infty}^{\infty}u_n\,(x,y) \tag{3.3.9}$$

be the solution of the given equation. Here $Nu(x,y)$ can be decomposed by using Adomian polynomial A_n which is identified up in the equation (3.1.6).

And nonlinear term $Nu(x,y)=\sum_{n=0}^{\infty}A_n$ (3.3.10)

Therefore, the equation (3.3.8) is transformed to

$$\sum_{n=0}^{\infty} u_n(x,y) = f_0(y) + xf_1(y) + \frac{x^2}{2!} f_2(y) + \dots \quad \dots \quad \dots + \frac{x^{p-2}}{(p-2)!} f_{p-2}(y) + \frac{x^{p-1}}{(p-1)!} f_{p-1}(y)$$
$$+ E_x^{-1}[v^p E_x[g_0(x) + yg_1(x) + \frac{y^2}{2!} g_2(x) + \dots \quad \dots \quad \dots + \frac{y^{q-1}}{(q-1)!} g_{q-1}(x)]]$$
$$+ E_x^{-1}[v^p E_x[E_y^{-1}[v^q E_y[h(x,y) - \sum_{n=0}^{\infty} A_n - R[\sum_{n=0}^{\infty} u_n(x,y)]]]]]$$

The following recursive relation is obtained by comparing both sides of the above equation

$$\left.\begin{array}{l} u_0(x,y) = K(x,y) \\ u_{n+1}(x,y) = -E_x^{-1}[v^p E_x[E_y^{-1}[v^q E_y[A_n + Ru_n(x,y)]]]], \quad n \geq 0 \end{array}\right\} \tag{3.3.11}$$

Where

$$k(x,y) = f_0(y) + xf_1(y) + \frac{x^2}{2!} f_2(y) + \dots \quad \dots \quad \dots + \frac{x^{p-2}}{(p-2)!} f_{p-2}(y) + \frac{x^{p-1}}{(p-1)!} f_{p-1}(y)$$
$$+ E_x^{-1}\left[v^p E_x\left[g_0(x) + yg_1(x) + \frac{y^2}{2!} g_2(x) + \dots \quad \dots \quad \dots + \frac{y^{q-1}}{(q-1)!} g_{q-1}(x)\right]\right]$$
$$+ E_x^{-1}\left[v^p E_x\left[E_y^{-1}\left[v^q E_y[\square(x,y)]\right]\right]\right]$$

is a source term. From the recursive relative (3.3.11), the components $u_n(x,y), n \geq 0$ can be found. Substituting the values of all these components in equation (3.3.9), then the required solution of the given equation is obtained in series form.

4. Applications

In this section, to illustrate ESM for non-linear PDEs of the n^{th} order with three kinds of mixed partial derivative, such as $\frac{\partial^n u}{\partial x^{n-1} \partial y}$, $\frac{\partial^n u}{\partial x \partial y^{n-1}}$ and $\frac{\partial^n u}{\partial x^p \partial y^q}$ (Where p and q are positive integers with $p + q = n$), four examples of homogeneous and non-homogeneous PDEs with non-linear term and involving mixed partial derivatives with initial conditions are provided.

Example 1: Considering a 3rd order nonlinear homogenous PDE is-

$$\frac{\partial^3 f}{\partial x^2 \partial y} - 2\left(\frac{\partial f}{\partial y}\right)^3 = 0 \tag{4.1}$$

With initial conditions

$$f_y(0,y) = 1, f_{xy}(0,y) = 1, f(x,0) = 0 \tag{4.2}$$

Substituting $U = \frac{\partial f}{\partial y}$ in the given equation

$$\frac{\partial^2 U}{\partial x^2} - 2U^3 = 0$$

Applying Elzaki transform with regard to x and initial conditions in the given equation, which is converted to the new one below,

$$E_x[U(x,y)] = v^2 + v^3 + 2v^2 E_x[U^3]$$

Taking the inverse Elzaki transformation with respect to x in the above equation is transformed to

$$U(x,y) = 1 + x + 2E_x^{-1}[v^2 E_x[U^3]]$$
$$\frac{\partial f}{\partial y}(x,y) = 1 + x + 2E_x^{-1}\left[v^2 E_x\left[\left(\frac{\partial f}{\partial y}\right)^3\right]\right]$$

Using Elzaki transform with regard to y and initial condition on the above equation, we obtain

$$E_y[f(x,y)] = v^3 + xv^3 + 2vE_y\left[E_x^{-1}\left[v^2 E_x\left[\left(\frac{\partial f}{\partial y}\right)^3\right]\right]\right]$$

With respect to y the inverse Elzaki Transformation is applied on the above equation,

$$f(x,y)] = y + xy + 2E_y^{-1}\left[vE_y\left[E_x^{-1}\left[v^2 E_x\left[\left(\frac{\partial f}{\partial y}\right)^3\right]\right]\right]\right] \tag{4.3}$$

In ESM, solution is represented in infinite series form as below,

$f(x,y) = \sum_{n=\infty}^{\infty} f_n(x,y)$ (4.4)

which is the required solution of the given equation .The nonlinear term $Nu(x,y) = \left(\frac{\partial f}{\partial y}\right)^3$ can be decomposed by Adomian polynomial A_n defined by the equation (3.1.6).

Consequently, $\left(\frac{\partial f}{\partial y}\right)^3 = \sum_{n=0}^{\infty} A_n$ (4.5)

Where $A_{n,}, n \geq 0$ Adomian polynomial with components $f_0(x,y), f_1(x,y), \ldots \quad \ldots \quad \ldots, f_n(x,y)$ of series (4.4).
Some values of the Adomian polynomials A_n are found, as below
$A_0 = f_{0y}^3, A_1 = 3f_{0y}^2 f_{1y}, A_2 = 3f_{0y}f_{1y}^2 + 3f_{0y}^2 f_{2y}$ and so on
Then the equation (4.3) becomes

$$\sum_{n=0}^{\infty} f_n(x,y) = y + xy + 2E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}\left[v^2E_x\left[\sum_{n=0}^{\infty} A_n\right]\right]\right]\right]$$

The recursive relationship is obtained by comparing both sides of the above equation.

$$\left.\begin{aligned} &f_0(x,y) = y + xy \\ &f_{n+1}(x,y) = 2E_y{}^{-1}[vE_y[E_x{}^{-1}[v^2E_x[A_n]]]] \quad , n \geq 0 \end{aligned}\right\}$$

The following new components of $f_n(x,y)$, $n \geq 0$ have been found from the above relationship.

$$f_0(x,y) = y + xy$$

$$f_1(x,y) = 2E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}[v^2E_x[A_0]]\right]\right] = 2E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}\left[v^2E_x[f_{0y}^3]\right]\right]\right] = x^2y + x^3y + \frac{x^4}{2}y + \frac{x^5}{10}y$$

$$f_2(x,y) = 2E_y{}^{-1}\left[vE_y\left[E_x{}^{-1}[v^2E_x[A_1]]\right]\right] = \frac{1}{2}x^4y + \frac{9}{10}x^5y + \frac{7}{10}x^6y + \frac{3}{10}x^7y + \frac{3}{56}x^8y + \frac{1}{120}x^9y + \cdots$$

Likewise, find the values of $f_3(x,y),\ f_4(x,y), \cdots \ \cdots \cdots, f_n(x,y)$

Substituting the values of $f_n(x,y), n \geq 0$ in equation (4.4), finally the result is found as,
$f(x,y) = y(1 + x + x^2 + x^3 + x^4 + x^5 + \ \ldots \quad \ldots \quad \ldots) = y\sum_{n=0}^{\infty} x^n$
Where $\sum_{n=0}^{\infty} x^n$ is a geometric series and converges to $\frac{1}{1-x}$, for $|x| < 1$
Therefore, $f(x,y) = \frac{y}{1-x},\ |x| < 1$
Which is the required solution with the initial conditions defined and checked by the substitution. The solution is same as the solution is obtained by LSM.
Figure 1 below, shows the solution of the equation (4.1)

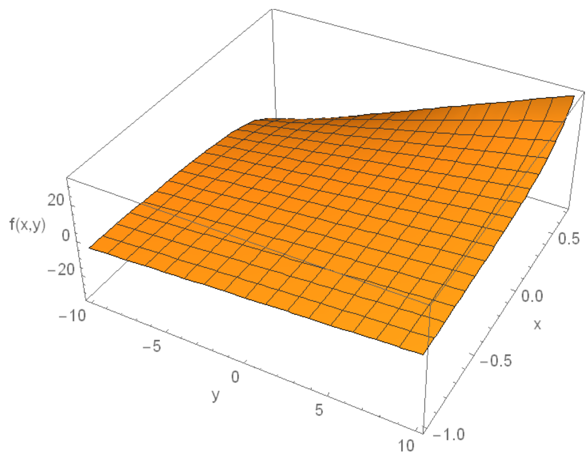

Fig.1. The graph of solution of example (4.1).

Example 2: Consider a 3rd order non-linear non-homogeneous PDE,

$$\frac{\partial^3 u}{\partial x \partial y^2} - \left(\frac{\partial u}{\partial y}\right)^2 + u^2 = e^y \cos x \quad (4.6)$$

With initial conditions

$$u_{yy}(0,y) = 0, u(x,0) = \sin x\,, u_y(x,0) = \sin x \quad (4.7)$$

Substituting $U = \frac{\partial^2 u}{\partial y^2}$ in the above equation, the equation is converted to-

$$\frac{\partial U}{\partial x} - \left(\frac{\partial u}{\partial y}\right)^2 + u^2 = e^y \cos x$$

Applying Elzaki transform with respect to x in the above equation subject to initial condition and then taking inverse Elzaki transform-

$$U(x,y) = e^y \sin x + E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2 - u^2]]$$

Re-substitution the value of $U = \frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial^2 u(x,y)}{\partial y^2} = e^y \sin x + E_x^{-1}[vE_x[\left(\frac{\partial u}{\partial y}\right)^2 - u^2]]$$

Again, with respect to y taking Elzaki transform subject to initial conditions and then applying inverse Elzaki transform the above equation becomes-

$$u(x,y) = e^y \sin x + E_y^{-1}[v^2 E_y[E_x^{-1}[vE_x[\left(\frac{\partial u}{\partial y}\right)^2 - u^2]]]] \tag{4.8}$$

For solving this problem by ESM let's consider that,

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \tag{4.9}$$

be the solution of the given equation. The non-linear term $Nu(x,y) = \left(\frac{\partial u}{\partial y}\right)^2 - u^2$ can be decomposed by using Adomian polynomials A_n and B_n which is defined by the equation (3.1.6).

Then $\left(\frac{\partial u}{\partial y}\right)^2 = \sum_{n=0}^{\infty} A_n$ and $u^2 = \sum_{n=0}^{\infty} B_n$ (4.10)

Adomian polynomials of A_n and B_n are as follows:

$A_0 = u_0 u_{0y}$, $B_0 = u_0^2$

$A_1 = u_0 u_{1y} + u_1 u_{0y}$, $B_1 = 2u_0 u_1$

$\dots \quad \dots \quad \dots$ $\dots \quad \dots \quad \dots$

Finally, the equation (4.8) is converted to

$$\sum_{n=0}^{\infty} u_n(x,y) = e^y \sin x + E_y^{-1}\left[v^2 E_y\left[E_x^{-1}\left[vE_x\left[\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n\right]\right]\right]\right]$$

The following recursive relation is found by comparing the above equation on both sides,

$$\left.\begin{array}{l} u_0(x,y) = e^y \sin x \\ u_{n+1} = E_y^{-1}[v^2 E_y[E_x^{-1}[vE_x[\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n]]]] \quad , n > 1 \end{array}\right\}$$

Few components $u_n(x,y)$, $n \geq 0$ are found from the above recursive relation,

$u_0(x,y) = e^y \sin x$

$$u_1(x,y) = E_y^{-1}\left[v^2 E_y\left[E_x^{-1}[vE_x[A_0 - B_0]]\right]\right] = 0$$

Consequently, $u_n(x,y) = 0, n \geq 1$

In equation (4.9), substituting all the values of $u_n(x,y), n \geq 0$, the following result is obtained

$u(x,y) = e^y \sin x$

Which is the required solution with respect to initial conditions and checked by the substitution.The solution is same as the solution obtained by LSM.

Figure 2 below, shows the solution of the equation (4.6)

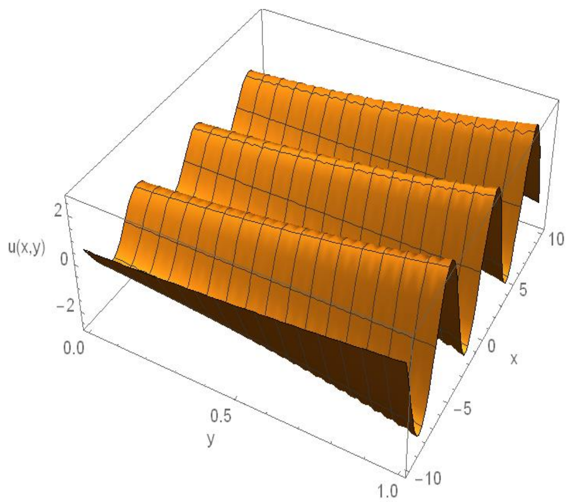

Fig.2. The graph of solution of example (4.6).

Example 3: Consider a 4th order non-linear homogeneous PDE,

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} + u\frac{\partial u}{\partial y} - u^2 = 0 \tag{4.11}$$

Subject to initial conditions

$$u(x,0) = x\ , u_y(x,0) = x\ , u_{y^2}(0,y) = 0, u_{y^2x}(0,y) = e^y \tag{4.12}$$

Substituting $U = \frac{\partial^2 u}{\partial y^2}$ in the given equation,

$$\frac{\partial^2 U}{\partial x^2} + u\frac{\partial u}{\partial y} - u^2 = 0$$

Taking Elzaki transform with regard to x subject to initial conditions and then using inverse Elzaki transform, the equation implies that

$$U(x,y) = xe^y + E_x{}^{-1}\left[v^2 E_x\left[u^2 - u\frac{\partial u}{\partial y}\right]\right]$$

Re-substituting $U = \frac{\partial^2 u}{\partial y^2}$, the above equation is converted to

$$\frac{\partial^2 u(x,y)}{\partial y^2} = xe^y + E_x{}^{-1}\left[v^2 E_x\left[u^2 - u\frac{\partial u}{\partial y}\right]\right]$$

Again, with respect to y, applying Elzaki transform and inverse Elzaki transform in the same process which is transformed to-

$$u(x,y) = xe^y + E_y{}^{-1}\left[v^2 E_y\left[E_x{}^{-1}\left[v^2 E_x\left[u^2 - u\frac{\partial u}{\partial y}\right]\right]\right]\right] \tag{4.13}$$

Suppose that,

$$u(x,y) = \sum_{n=\infty}^{\infty} u_n(x,y) \tag{4.14}$$

be the solution of equation (4.11).The nonlinear term $Nu(x,y) = u^2 - u\frac{\partial u}{\partial y}$ can be decomposed by using Adomian polynomials.

$$u^2 = \sum_{n=0}^{\infty} A_n \text{and } u\frac{\partial u}{\partial y} = \sum_{n=0}^{\infty} B_n \tag{4.15}$$

Where $A_n, B_n, n \geq 0$ are Adomian polynomials with components $u_0(x,y), u_1(x,y), \dots \quad \dots \quad \dots, u_n(x,y), n \geq 0$.
Adomian polynomials of A_n and B_n are found that:

$A_0 = u_0^2$, $B_0 = u_0 u_{0y}$

$A_1 = 2u_0 u_1$, $B_1 = u_0 u_{1y} + u_1 u_{0y}$

… … … … … …

Now the equation (4.13) is converted to

$$\sum_{n=0}^{\infty} u_n(x,y) = xe^y + E_y{}^{-1}\left[v^2 E_y\left[E_x{}^{-1}\left[v^2 E_x\left[\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n\right]\right]\right]\right]$$

The following recursive relation is found by comparing the above equation on both sides,

$$\left.\begin{array}{l} u_0(x,y) = xe^y \\ u_n(x,y) = E_y{}^{-1}[v^2 E_y[E_{yx}{}^{-1}[v^2 E_x[A_{n-1} - B_{n-1}]]]] \quad , n > 0 \end{array}\right\} \tag{4.16}$$

The following few components of $u_n(x,y)\ , n \geq 0$ are determined from the above relation

$$u_0(x,y) = xe^y$$

$$u_1(x,y) = E_y{}^{-1}\left[v^2 E_y\left[E_x{}^{-1}[v^2 E_x[A_0 - B_0]]\right]\right] = 0$$

Correspondingly, $u_n(x,y) = 0\ , n \geq 1$

Substituting all the values of $u_n(x,y), n \geq 0$ in equation (4.14), then the following solution is obtained

$$u(x,y) = xe^y$$

Which is the required solution with given initial conditions and checked by substitution. The solution is same as the solution is obtained by LSM.

Figure 3 below, shows the solution of the equation (4.11)

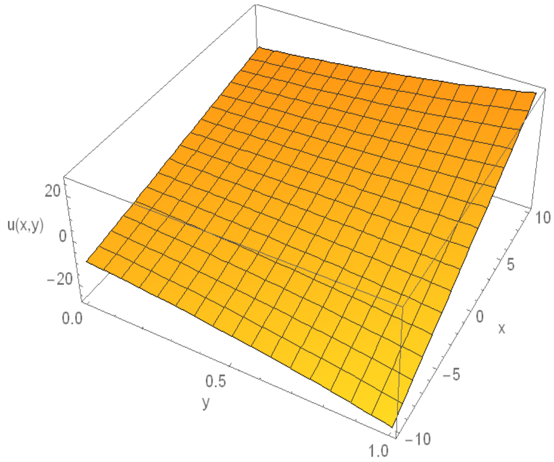

Fig.3. The graph of solution of example (4.11).

Example 4: Consider a 5th order non-linear non-homogeneous PDE,

$$\frac{\partial^5 u}{\partial x^3 \partial y^2} - u\frac{\partial u}{\partial x} + u^2 = e^x \cos y \tag{4.17}$$

With initial conditions

$$u(0,y) = -\cos y \ , u_x(0,y) = -\cos y \ , u_{x^2}(0,y) = -\cos y \ ,$$
$$u_{x^3}(x,0) = -e^x, u_{x^3 y}(x,0) = 0 \tag{4.18}$$

By using the same method in example 3, finally the new one below is obtained

$$u(x,y) = -e^x \cos y + E_x^{-1}\left[v^3 E_x\left[E_y^{-1}\left[v^2 E_y\left[u\frac{\partial u}{\partial x} - u^2\right]\right]\right]\right] \tag{4.19}$$

Suppose that,

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \tag{4.20}$$

be the required solution of equation (4.17).

Now the nonlinear terms $u\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} A_n$ and $u^2 = \sum_{n=0}^{\infty} B_n$ (4.21)

Where $A_n, B_n, n \geq 0$ are Adomian polynomials with components $u_0(x,y), u_1(x,y), \dots \quad \dots \quad \dots, u_n(x,y), n \geq 0$ from series (4.20).

Adomian polynomials of A_n and B_n are calculated as:

$$A_0 = u_0 u_{0x} \qquad , \quad B_0 = u_0^2$$
$$A_1 = u_0 u_{1x} + u_1 u_{0x} \qquad , \quad B_1 = 2u_0 u_1$$
$$\dots \ \dots \ \dots \qquad \qquad \dots \ \dots \ \dots$$

Then the equation (4.19) is transformed to-

$$\sum_{n=0}^{\infty} u_n(x,y) = -e^x \cos y + E_x^{-1}\left[v^3 E_x\left[E_y^{-1}\left[v^2 E_y\left[\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n\right]\right]\right]\right]$$

The following recursive relation is found by comparing the above equation on both sides

$$\left.\begin{aligned} &u_0(x,y) = -e^x \cos y \\ &u_n(x,y) = E_x^{-1}[v^3 E_x[E_y^{-1}[v^2 E_y[A_{n-1} - B_{n-1}]]]] \quad , n > 0 \end{aligned}\right\} \tag{4.22}$$

Few components of $u_n(x,y)$, $n \geq 0$ are calculated from the above relation

$$u_0(x,y) = -e^x \cos y$$
$$u_1(x,y) = E_x^{-1}\left[v^3 E_x\left[E_y^{-1}\left[v^2 E_y[A_0 - B_0]\right]\right]\right] = 0$$

Correspondingly, $u_n(x,y) = 0$, $n \geq 1$

Substituting the values of $u_n(x,y), n \geq 0$ in equation (4.20), the solution is obtained as,

$u(x,y) = -e^x \cos y.$

which is exact with regarding initial conditions and checked by the substitution.The solution is same as the solution obtained by LSM.

Figure 4 below, shows the solution of the equation (4.17)

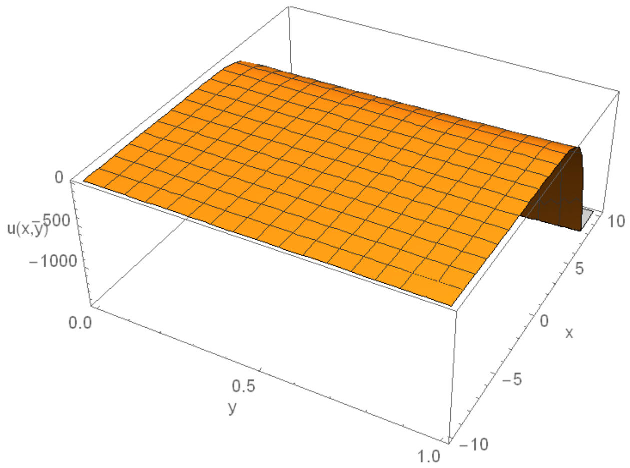

Fig.4. The graph of solution of example (4.17).

5. Conclusion

In the article, the proposed ESM was successfully used without using linearization, disruption, or restrictive assumption to find a solution of n^{th}-order non-linear initial value problems containing mixed partial derivatives with the help of Adomian polynomials. Implementation of this technique is simple, economically beneficial, time saving and exquisite. The capability of this method has been demonstrated by solving some nonlinear PDEs of different order. An important recurrence relationship are found by using this method to solve the n^{th} non-linear mixed order PDE which concludes that ESM with Adomian polynomial provides highly reliable numerical solutions compared to other methods for non-linear mixed order problems.

Conflicts of interest

The authors don't have any conflict of interest.

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