On singed product cordial of cone graph and its second power

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Abstract: A graph $G=(V,E)$ is called singed cordial if it is possible to label the vertex by the function $f: V\to\{-1,1\}$ and label the edges by $f^*: E\to\{-1,1\}$, where $f^*(uv)=f(u)f(v)$, $u,v\in V$ so that $|v_{-1}-v_1|\le 1$ and $|e_{-1}-e_1|\le 1$. In our work we present necessary and sufficient conditions for which cone graphs and their second power are singed product cordial.
Keywords: Cone, Second power, Singed product cordial graph, Sum graph

1. Introduction

Labeling graphs are used widely in different subjects including astronomy and communication networks. The concept of graph labeling was introduced during the sixties' of the last century by Rosa [1]. Many researches have been working with different types of labeling graphs [2][3]. In 1954 Harray introduced S-cordiality [4]. An excellent reference for this purpose is the survey written by Gallian [5]. All graphs considered in this theme are finite, simple and undirected. The original concept of cordial graphs is due to Chait [3]. He showed that each tree is cordial; an Euerlian graph is not cordial if its size is congruent to $2(mod\ 4)$. Let $G=(V,E)$ be a graph and let $f: V\to\{-1,1\}$ be a labeling of its vertices, and let the induced edge labeling $f^*: E\to\{-1,1\}$ be given by $f^*(e)=f(u)f(v)$, where $e=uv$ and $u,v\in V$. Let v_{-1} and v_1 be the numbers of vertices that are labeled by (-1) and 1, respectively, and let e_{-1} and e_1 be the corresponding numbers of edges. Such a labeling is called *signed-cordial* if both $|v_{-1}-v_1|\le 1$ and $|e_{-1}-e_1|\le 1$ hold. A graph is called *signed-cordial* if it has a signed-cordial labeling. In [8] J.Devaraj and P.Delphy defined signed graphs, and started by labeling edges and then induced the labeling of vertices. In [9] Jayapal Baskar Babujee and Shobana Loganathan proved that path graph, cycle graphs, star-$K_{1,n}$, Bistar-$B_{n,n}$,$P_n^+, n\ge 3$ and $C_n^+, n\ge 3$ are signed product cordial. The *sum* or *join* of the two graphs G_1 and G_2, denoted by G_1+G_2, is the graph with vertex set and edge set given by $V(G_1+G_2)=V(G_1)\cup V(G_2)$ and $E(G_1+G_2)=E(G_1)\cup E(G_2)\cup J$, respectively, where J consists of edges join each vertex of G_1 to every vertex of G_2. A cone graph is $\mathcal{C}_{n,m}=N_n+C_m$ where N_n is the null graph with n vertices and no edges, and C_m is the cycle graph of m vertex and edges. It is clear that any cone graph $n+m$ vertices and $m+nm+1$ edges. In this paper, we show that the cone graph and it's second power are signed product-cordial

2. Terminologies and Notations

The second power of a cone graph is the sum of null graph and the second power of a cycle graph. Given one cycle of the cone graph with $4r$ vertices, we let L_{4r} denote the labeling $(-1)_2(1)_2...(-1)_2(1)_2$ (repeated r-times), let L'_{4r} denote the labeling $(1)_2(-1)_2...(1)_2(-1)_2$ (repeated r times). We denote the labeling $1(-1)_21$ $1(-1)_21...1(-1)_21$ (repeated r times) and $(-1)(1)_2(-1)...(-1)(1)_2(-1)$ (repeated r times) by S_{4r} and S'_{4r}, respectively. Sometimes, we modify this by adding symbols at one end or the other (or both), thus $L_{4r}1(-1)1$ denotes the labeling $(-1)_2(1)_2$ ...$(-1)_2(1)_2$ $1(-1)1$ when $r\ge 1$ and $1(-1)1$ when $r=0$. Similarly, $1L'_{4r}$ is the labeling 1 $(1)_2(-1)_2...(1)_2(-1)_2$ when $r\ge 1$ and 1 when $r=0$. The labeling $(-1)(1)\dots(-1)(1)$ (repeated r times) denoted by M_r if r is even and $(-1)(1)\dots(-1)(1)(-1)$ if r is odd. Likewise $(1)(-1)\dots(1)(-1)$, is denoted by M'_r if r is even and $(1)(-1)\dots(1)(-1)(1)$, if r is odd. For a given labeling of the join $G+H$, we let v_i and e_i (for $i=(-1),1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H. It follows that $v_{-1}=x_{-1}+y_{-1}; v_1=x_1+y_1; e_{-1}=a_{-1}+b_{-1}+x_{-1}y_1+x_1y_{-1}$ and $e_1=a_1+b_1+x_{-1}y_{-1}+x_1y_1$, thus, $v_{-1}-v_1=(x_{-1}-x_1)+(y_{-1}-y_1)$ and $e_{-1}-e_1=(a_{-1}-a_1)+(b_{-1}-b_1)-(x_{-1}-x_1)(y_{-1}-y_1)$. The labeling of N_n+C_m and its second power are denoted by $[A;B]$; where the labeling A is given to N_n and the labeling B is given to C_m or C_m^2.

3. The signed product-cordial of cone graphs

In this section, we show that the cone graphs $\mathcal{C}_{n,m}$ is signed product-cordial for all $n\ge 1, m\ge 3$. A theorem given by Seoud and Abdel Maqsoud [7] can be applied for signed product-cordial and therefore we have:

Lemma 3.1 If the graph G has the sum of it is order and it is size is $2(mod4)$ and the degree of each vertex is odd, then the graph is not signed product-cordial.

Our target will be achieved after the following series of lemmas.

Lemma 3.2 The cone graph $\mathcal{C}_{n,3}$ is signed product-cordial if and only if n even.

Proof. Suppose that n is odd. Then the degree of each vertex of $\mathcal{C}_{n,3}$ is odd. Hence is also that the sum of order and size of $\mathcal{C}_{n,3}$ is congruent to $2(mod4)$. This is true since the degree is $n+3$ and it is size is $3n+3$. It follows that $\mathcal{C}_{n,3}$ is not product signed cordial as indicate in the Lemma 3.1. If n is even i.e. $n=2r$ where $r \geq 1$, then we may label the vertices of C_3 by $(-1)_2(1)$, and we label the vertices of N_n by M_{2r}. Therefore, the number of vertices of $\mathcal{C}_{n,3}$ increases by r for each label. So the difference $v_{-1}-v_1$ remains the same. Also the number of edge of $\mathcal{C}_{n,3}$ of each label increases by $3r$. So the difference $e_{-1}-e_1$ remains the same. Therefore, $\mathcal{C}_{n,3}$ is signed product-cordial. Thus the lemma follows. As an example, Figure (1) illustrates $\mathcal{C}_{2,3}$. □

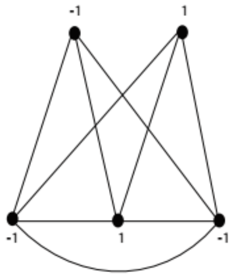

Figure.1

Lemma 3.3 If $m \equiv 0(mod4)$, then the cone graph $\mathcal{C}_{n,m}$ is signed product-cordial for all $n \geq 1$.

Proof. Suppose that $m=4s$, where $s \geq 1$. Then we label the vertices of C_{4s} by $B_0=L_{4s}$, i.e. $y_{-1}=y_1=2s$ and $b_{-1}=b_1=2s$. Now Let us study the different cases for n; either n is even or odd. If n is even i.e. $n=2r$ where $r \geq 1$, then we may label the vertices of N_n by $A_0=M_{2r}$, i.e. $x_{-1}=x_1=r$ and $a_{-1}=a_1=0$. Thus we have $v_{-1}-v_1=(x_{-1}-x_1)+(y_{-1}-y_1)=0$ and $e_{-1}-e_1=(a_{-1}-a_1)+(b_{-1}-b_1)-(x_{-1}-x_1)(y_{-1}-y_1)=0$. If n is odd i.e. $n=2r+1$ where $r \geq 0$, then if we may label the vertices of N_n by $A_1=M_{2r+1}$, i.e. $x_{-1}=r+1$, $x_1=r$ and $a_{-1}=a_1=0$. Thus we have $v_{-1}-v_1=1$ and $e_{-1}-e_1=0$. Therefore $\mathcal{C}_{n,m}$ is signed product-cordial. Thus the lemma follows. As an examples, Figure (2) and Figure (3) illustrate $\mathcal{C}_{1,4}$ and $\mathcal{C}_{2,4}$ respectively.□

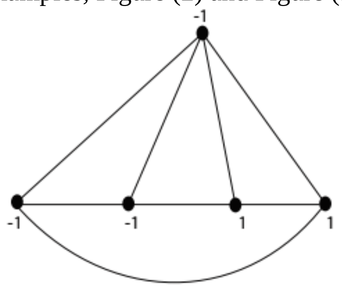

Figure.2

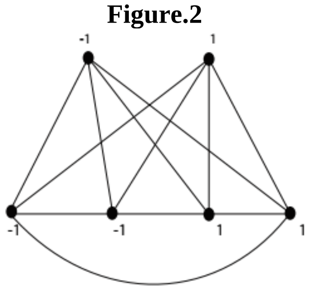

Figure.3

Lemma 3.4 If $m \equiv 1(mod4)$, then the cone graph $\mathcal{C}_{n,m}$ is signed product-cordial for all $n \geq 1$.

Proof. Suppose that $m=4s+1$, where $s \geq 1$, and we label the vertices of C_{4s+1} as $B_1=L_{4s}1$, i.e. $y_{-1}=2s$, $y_1=2s+1$, $b_{-1}=2s$ and $b_1=2s+1$. Let us study two cases for n; either n is even or odd. If n is even i.e. $n=2r$ where $r \geq 1$, then we may label the vertices of N_n by $A_0=M_{2r}$, i.e. $x_{-1}=x_1=r$ and $a_{-1}=a_1=0$. Thus we have $v_{-1}-v_1=-1$ and $e_{-1}-e_1=-1$. Conversely, If n is odd, i.e. $n=2r+1$ where $r \geq 0$, then we may label the vertices of N_n by $A_1=M_{2r+1}$, i.e. $x_{-1}=r+1, x_1=r$ and $a_{-1}=a_1=0$. Thus we have $v_{-1}-v_1=0$ and $e_{-1}-e_1=0$. Therefore $\mathcal{C}_{n,m}$ is signed product-cordial. Thus the lemma follows. As an examples, Figure (4) and Figure (5) illustrate $\mathcal{C}_{1,5}$ and $\mathcal{C}_{2,5}$ respectively.□

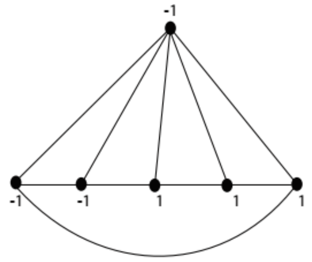

Figure.4

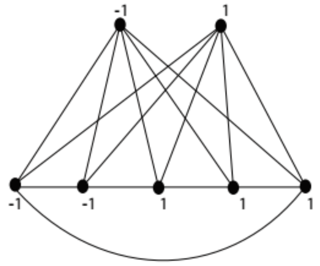

Figure.5

Lemma 3.5 If $m \equiv 2(mod4)$, then the cone graph $\mathcal{C}_{n,m}$ is signed product-cordial for all n odd, $n \geq 3$.

Proof. Suppose that $m = 4s + 2$, where $s \geq 1$, then we may label the vertices of C_{4s+2} by $B_2 = (-1)L_{4s}(-1)$, i.e. $y_{-1} = 2s + 2,\ y_1 = 2s,\ b_{-1} = 2s$ and $b_1 = 2s + 2$. Let $n = 2r + 1$ where $r \geq 1$, then we may label the vertices of N_n by $A_1 = M'_{2r+1}$, i.e. $x_{-1} = r,\ x_1 = r + 1$ and $a_{-1} = a_1 = 0$. Thus we have $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Therefore $\mathcal{C}_{n,m}$ is signed product-cordial. Thus the lemma follows. As an example, Figure (6) illustrates $\mathcal{C}_{1,6}$.□

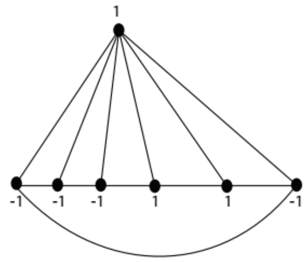

Figure.6

Lemma 3.6 If $m \equiv 3(mod4)$, then the cone graph $\mathcal{C}_{n,m}$ is signed product-cordial for all n even, $n \geq 1$.

Proof. Suppose that $m = 4s + 3$, where $s \geq 1$, then we label the vertices of C_{4s+3} by $B_3 = L_{4s}1(-1)1$, i.e. $y_{-1} = 2s + 1,\ y_1 = 2s + 1,\ b_{-1} = 2s + 2$ and $b_1 = 2s + 1$. let $n = 2r$ where $r \geq 1$, then we may label the vertices of N_n by $A_0 = M_{2r}$, i.e. $x_{-1} = x_1 = r$ and $a_{-1} = a_1 = 0$. Thus we have $v_{-1} - v_1 = -1$ and $e_{-1} - e_1 = 1$. Therefore $\mathcal{C}_{n,m}$ is signed product-cordial. Thus the lemma follows. As an example, Figure (7) illustrates $\mathcal{C}_{1,7}$.□

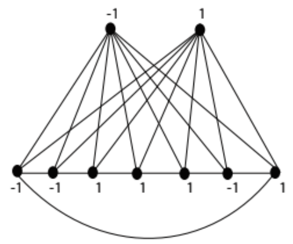

Figure.7

An Important result given by Cahit [3] can be applied for signed product-cordial and therefore we have: If a graph G is an Eulerian graph with size congruent to $2(mod4)$, then the graph G is not signed product-cordial.

Lemma 3.7 If $m \equiv 2(mod4)$, then the cone graph $\mathcal{C}_{n,m}$ is not signed product-cordial for all n even.

Proof. It is easy to verify that the graph $\mathcal{C}_{n,m}$ is an Eulerian graph with size congruent to $2(mod4)$, and consequently $C_{n,m}$ is not signed product-cordial.

Lemma 3.8 If $m \equiv 3(mod4)$,then the cone graph $C_{n,m}$ is not signed product-cordial for all odd n.
Proof. It is easy to verify that the degree of all vertices of $C_{n,m}$ is odd, and the sum of order and the size of $C_{n,m}$ is congruent to $2(mod4)$, and consequently by Lemma 3.1 $C_{n,m}$ is not signed product-cordial.

As a consequence of all lemmas mentioned above we conclude the following theorem.

Theorem 3.1 The $C_{n,m}$ is signed product-cordial for all n and all m if and only if m is not congruent to $3(mod4)$ and n odd, or when m is not congruent to $2(mod4)$ and n even.

Algorithm . Algorithm for the signed product-cordiality of cone graphs
Cone graph(n,m): joint between cyclic graph and null graph
e: the number of edge
r: random value of -1,1
V: the number of vertex

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input : $n \longrightarrow$ cyclic graph
$m \longrightarrow$ null graph
output : identify our cone graph is signed product-cordial or not
START
using n,m to draw the joint between cyclic graph and null graph denoted by cone graph
$R \longleftarrow (-1,1)$
FOR all vertices $\in n$ **do**
define $r \longleftarrow R$
labeling n graph using r
END FOR
FOR all vertices $\in m$ **do**
define $r \longleftarrow R$
labeling n graph using r
END FOR
$e_{-1} \longleftarrow 0$
$e_1 \longleftarrow 0$
FOR vertex$(i) i = 1$ to n step vertex$++$ **do**
IF $i < n$
  IF vertex(i) = vertex$(i+1)$
   $e_1 ++$ **ELSE** $e_{-1} ++$
  END IF
ELSE
  IF vertex(i) = vertex(0)
   $e_1 ++$ **ELSE** $e_{-1} ++$
  END IF
END IF
END FOR
FOR vertex m$(i) i = 1$ to m step $i ++$ **do**
  FOR vertex n$(j) j = 1$ to n step $j ++$ **do**
   IF vertex $m(i)$ = vertex $n(j)$
    $e_1 ++$ **ELSE** $e_{-1} ++$
   END IF
  END FOR
END FOR
$v_{-1} \longleftarrow 0$
$v_1 \longleftarrow 0$
FOR vertex $n(i) i = 1$ to n step $i ++$ **do**
  IF vertex $n(i) = -1$
  $v_{-1} ++$ **ELSE** $v_1 ++$
  END IF
END FOR
FOR vertex $m(i) i = 1$ to m step $i ++$ **do**
  IF vertex $m(i) = -1$

$v_{-1}++$ **ELSE** v_1++
END IF
END FOR
IF$(|v_{-1}-v_1| <= 1 \ and \ |e_{-1}-e_1| <= 1)$
THEN the cone graph(n, m) is signed product-cordial
ELSE IF ((cone graph(n, m) is eulerian graph AND cone graph (n, m) size congruent to $2(mod4)$)
OR (cone graph(n, m) with size congruent to $2(mod4)$ AND degree of each vertex of cone graph(n, m) is odd))
THEN the cone graph(n, m) is not signed product-cordial
ELSE go to **START**
END IF.

4. Signed product-cordial of the second power of cone graphs

In this section, we study the signed product-cordial of the second power of cone graphs and show that all second power of cone graphs $C_{n,m}^2$ are signed product-cordial except at m is congruent to $3(mod4)$, where n is odd, or m is congruent to $2(mod4)$ and n is even.

Lemma 4.1. The second power of cone graph $C_{n,3}^2$ is signed product-cordial if and only if n even.

Proof. Since $C_3^2 = C_3$, from Lemma 3.2 the lemma follows.□

Lemma 4.2. If $m = 4$, then the second power of cone graph $C_{n,4}^2$ is not signed product-cordial for all $n \geq 1$.

Proof. We need to study the following two cases for n, either n is odd or even.

__Case 1__. n odd.

In this case $C_{n,4}^2$ is not signed product-cordial because it is easy to see that $C_4^2 = K_4$ is a complete graph and the degree of all vertices of $C_{n,4}^2$ are even so that, it is Eulerian graph with size congruent to $2(mod4)$. Hence $C_{n,4}^2$ is not signed product-cordial.

__Case 2__. n even.

Let $n = 2r, r \geq 1$. Then the graph $C_{2r,4}^2$ has an even order. If this graph was signed product-cordial, it would have an equal number of vertices that are labeled ones and that are labeled negative ones. Otherwise $|v_{-1} - v_1| > 1$. Consequently, $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) - (x_{-1} - x_1)^2$, because $v_{-1} - v_1 = 0$. The set $(-1)_4, 1(-1)_3, 1_2(-1)_2$ contains all different possibilities of labeling of the vertices of C_4^2. It is obvious that any other labeling for C_4^2 will be equivalent to one of the above mentioned possibilities. Now, we turn our attention to examine these three different possibilities through the following subcases.

__Sub Case 2.1__.

If we label the vertices of C_4^2 by $(-1)_4$. Then, the vertices of N_{2r} should have four vertices that are labeled ones and the rest of vertices must be divided equality by negative ones and ones. Thus, the number of vertices that are labeled ones in N_{2r} is $\frac{2r-4}{2} + 4 = r + 2$ and the number of vertices that are labeled negative one is $\frac{2r-4}{2} = r - 2$. Since $a_{-1} = a_1 = 0$, $e_{-1} - e_1 = 0 - 6 + 8 = 2$; contradiction.

__Sub Case 2.2__.

Label the vertices of C_4^2 by $1(-1)_3$. Then, the number of vertices labeled ones in N_{2r} is $\frac{2r-4}{2} + 3 = r + 1$ and the number of vertices labeled negative ones in N_{2r} is $(\frac{2r-4}{2}) + 1 = r - 1$. It follows that $e_{-1} - e_1 = 0 + 0 + 4 = 4$; contradiction.

__Sub Case 2.3 .__

Label the vertices of C_4^2 by $(-1)_2 1_2$. In this situation, the number of vertices labeled ones in N_{2r} is $\frac{2r-4}{2} + 2 = r$ and the number of vertices labeled negative ones in N_{2r} is also r. It follows that $e_{-1} - e_1 = 2 + 0 + 0 = 2$. Contradiction.

Hence $C_{2r,4}^2$ is not signed product-cordial as we wanted to show.□

For $C_{n,m}^2$, $m > 4$ we have:

Lemma 4.3. If $m \equiv 0(mod4)$, then the second power of cone graph $C_{n,m}^2$ is signed product-cordial for all $n \geq 1$.

Proof. Suppose that $m = 4s$, where $s > 1$. The following two cases will be examined.

Case 1. n even.

Suppose that $n = 2r$, where $r \geq 1$. Then we label the vertices of C_{4s}^2 by $B_0 = (-1)_2 1_2 M'_{4s-4}$. Therefore $y_{-1} = y_1 = 2s$ and $b_{-1} = b_1 = 4s - 1$. Label the vertices of N_{2r} by $A_0 = M_{2r}$. Therefore, $x_{-1} = x_1 = r$ and $a_{-1} = a_1 = 0$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) - (x_{-1} - x_1)(y_{-1} - y_1) = 0$. Hence, $C_{2r,4s}^2$ is signed product-cordial. As an example, Figure (8) illustrates $C_{2,8}^2$.

Case 2. n odd.

Suppose that $n = 2r + 1$ where $r \geq 0$. We label the vertices of C_{4s}^2 by B_0 as in case 1 and the vertices of N_{2r+1} by $A_1 = M_{2r+1}$. Therefore, $x_{-1} = r + 1, x_1 = r$ and $a_{-1} = a_1 = 0$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{2r+1,4s}^2$ is signed product-cordial. Thus the lemma follows. As an example, Figure (9) illustrates $C_{3,8}^2$.□

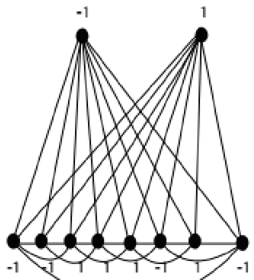

Figure.8

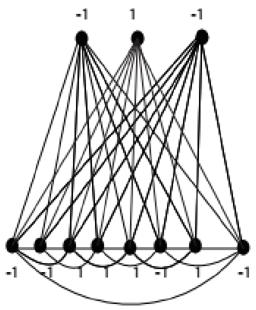

Figure.9

Lemma 4.4. If $m \equiv 1(mod4)$, then the cone graph $C_{n,m}^2$ is signed product-cordial for all $n \geq 1$.

Proof. Let $m = 4s + 1$, $s > 1$. Then, we can choose the label $B_1 = (-1)_3 1_3 M_{4s-6} 1$ for C_{4s+1}^2. Consequently $y_{-1} = 2s, y_1 = 2s + 1$ and $b_{-1} = b_1 = 4s$. It is reasonable to divide our proof into two cases.

Case 1. n even.

Let $n = 2r, r \geq 1$. Then we can take the labeling $A_0 = M_{2r}$ for N_{2r}. Therefore, $x_{-1} = x_1 = r$, and $a_{-1} = a_1 = 0$. It follows that $v_{-1} - v_1 = -1$ and $e_{-1} - e_1 = 0$. Hence, $C_{2r,4s+1}^2$ is signed product-cordial if $s > 1$. As an example, Figure (10) illustrates $C_{2,9}^2$.

Case 2. n odd.

Let $n = 2r + 1, r \geq 0$. Then we can choose the labeling $A_1 = M_{2r+1}$ for N_{2r+1}. Therefore, $x_{-1} = r + 1, x_1 = r$, and $a_{-1} = a_1 = 0$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $C_{2r+1,4s+1}^2$ is signed product-cordial if $s > 1$. As an example, Figure (11) illustrates $C_{3,9}^2$.

It remains to study $C_{n,5}^2$. If n is even, i.e. $n = 2r$, then we choose $A_0 = M_{2r}$ for N_{2r} and we choose the label $(-1)_2 1_3$ for C_5^2. Therefore, $v_{-1} - v_1 = -1$ and $e_{-1} - e_1 = 0$. If n is odd, i.e. $n = 2r + 1$, then we choose the label $A_1 = M_{2r+1}$ for N_{2r+1} and we labeling $(-1)_2 1_3$ for C_5^2. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$.

Thus $C^2_{n,4s+1}$ is signed product-cordial for all $s \geq 1$ as we wanted to prove. Figure (12) and Figure (13) illustrate $C^2_{1,5}$ and $C^2_{3,5}$, respectively.

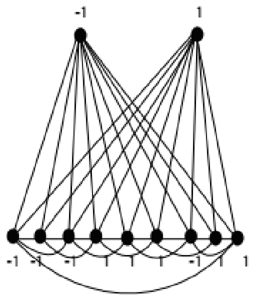

Figure.10

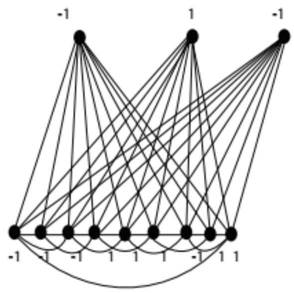

Figure.11

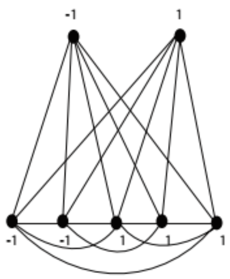

Figure.12

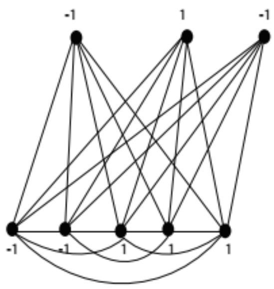

Figure.13

Lemma 4.5. If $m \equiv 2(mod4)$, then the second power of a cone graph $C^2_{n,m}$ is signed product-cordial for all $n \geq 1$.

Proof. Suppose that $m = 4s + 2$, where $s > 1$. we need to study the following cases.

Case 1. n even.

Let $n = 2r$, where $r \geq 1$. Then we choose the label $B_2 = (-1)_3 1_3 M_{4s-4}$ for the vertices of C^2_{4s+2}. Therefore, $y_{-1} = y_1 = 2s + 1$ and $b_{-1} = b_1 = 4s + 1$. Choose the labeling $A_0 = M_{2r}$ for the vertices of N_{2r}. It follows that, $x_{-1} = x_1 = r$, and $a_{-1} = a_1 = 0$. Hence $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Thus, $C^2_{2r,4s+2}$ is signed product-cordial. As an example, Figure (14) illustrates $C^2_{2,10}$.

Case 2. n odd.

Let $n = 2r + 1$, where $r \geq 0$. Then we select the same label B_2 for the vertices of C_{4s+2}^2 and we choose the label $A_1 = M_{2r+1}$ for the vertices of N_{2r+1}. Therefore, $x_{-1} = r + 1, x_1 = r$ and $a_{-1} = a_1 = 0$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $\mathcal{C}_{2r+1,4s+2}^2$ is signed product-cordial. As an example, Figure (15) illustrates $\mathcal{C}_{3,10}^2$.

It remains to examine the case $\mathcal{C}_{n,6}^2$. If n is even i.e. $n = 2r$, then we choose the labeling $(-1)_2 1_3(-1)$ for the vertices of C_6^2. Therefore, $y_{-1} = y_1 = 3$ and $b_{-1} = b_1 = 5$. Choose the labeling $A_0 = M_{2r}$ for N_{2r}. Therefore, $x_{-1} = x_1 = r$ and $a_{-1} = a_1 = 0, r \geq 1$. It follows that $v_{-1} - v_1 = -1$ and $e_{-1} - e_1 = 0$. Hence, $\mathcal{C}_{2r,6}^2$ is signed product-cordial. If n is odd i.e. $n = 2r + 1, r \geq 0$. Then we choose the labeling $(-1)_2 1_3(-1)$ for the vertices of C_6^2 and the labeling $A_1 = M_{2r+1}$ for the vertices of N_{2r+1}. Therefore, $x_{-1} = r + 1, x_1 = r$ and $a_{-1} = a_1 = 0$, where $r \geq 0$. It follows that $v_{-1} - v_1 = -1$ and $e_{-1} - e_1 = 0$. Hence, $\mathcal{C}_{2r+1,6}^2$ is signed product-cordial. Thus the lemma follows.As an examples, Figure (16) and Figure (17) illustrate $\mathcal{C}_{1,6}^2$ and $\mathcal{C}_{3,6}^2$, respectively.□

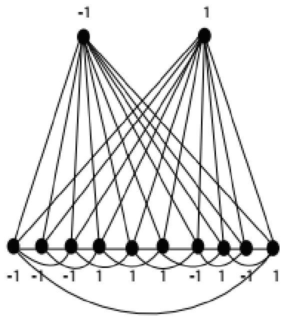

Figure.14

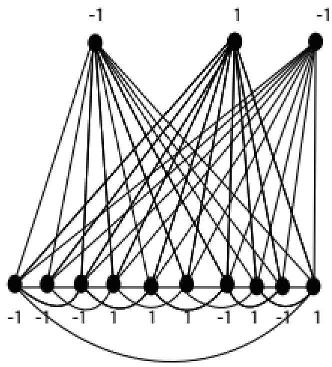

Figure.15

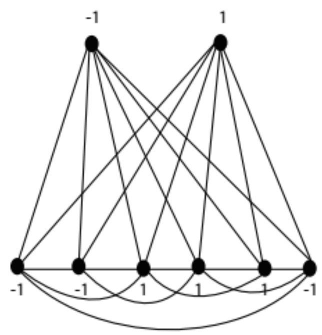

Figure.16

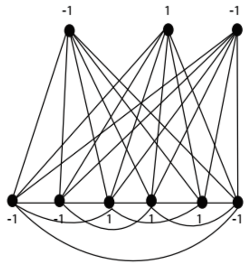

Figure.17

Lemma 4.6. If $m \equiv 3(mod4)$, then the second power of the cone graph $N_n + C_m^2$ is signed product-cordial for all $n, m \geq 1$.

Proof. Suppose that $m = 4s + 3$ where $s \geq 1$. we need to study the following two cases for n.

<u>Case 1</u>. n even.

Suppose that $n = 2r, r \geq 1$. Then we label the vertices of C_{4s+1}^2 by $B_3 = M_{4s+3}$. Therefore, $y_{-1} = 2s + 2,\ y_1 = 2s + 1,\ b_{-1} = b_1 = 4s + 2$. The label of the vertices of N_{2r} may be chosen as $A_0 = M_{2r}$. Therefore $x_{-1} = x_1 = r$ and $a_{-1} = a_1 = 0$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{2r,4s+3}^2$ is signed product-cordial. As an example, Figure (18) illustrates $C_{2,7}^2$.

<u>Case 2</u>. n odd.

Suppose that $n = 2r + 1$, $r \geq 0$. Then we label the vertices of N_{2r+1} by $A_1 = M'_{2r+1}$, $r \geq 0$. Therefore, $x_{-1} = r + 1,\ x_1 = r$ and $a_{-1} = a_1 = 0$. We also shall use the label B_3 for C_{4s+3}^2. It follows that $v_{-1} - v_1 == 0$ and $e_{-1} - e_1 = 1$. Hence, $C_{2r+1,4s+3}^2$ is signed product-cordial, and the lemma follows. As an example, Figure (19) illustrates $C_{3,7}^2$.□

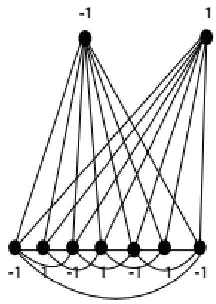

Figure.18

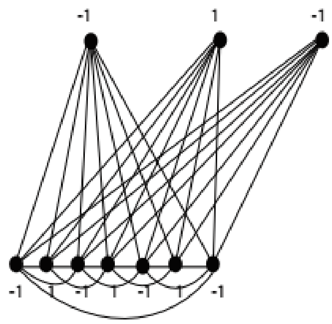

Figure.19

As a consequence of all lemmas mentioned above we conclude that the second power of the cone graph $C_{n,m}^2$ is signed product-cordial for all n and all m if and only if m is not equal to 3 and n odd, or when m is not equal to 4 and all n .

Theorem 4.1 The sum of the cycles C_m^2 and an isolated vertices N_n is signed product-cordial for all n and all m if and only if m is not equal to 3 and n odd, or when m is not equal to 4 and all n

5. Conclusion

In this work we have discussed and established necessary and sufficient conditions for which cone graphs and their second power are singed product cordial labeling concept on other families of graphs and finding the application of this labelling will be our future.

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