Random Differential Equations With Nonlinear Boundary Conditions

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Abstract: We discuss the first order random functional differential equation subject to nonlinear boundary conditions with discontinuity under the existence of extremal random solution. The methodology used is generalized iterative technique.

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1.Statement of the Problem

The general form of a delay functional differential equation is

$$p'(\mathrm{t},\omega) = u(\mathrm{p}, p_{\mathrm{t}},\omega) + v(\mathrm{p}, p_{\mathrm{t}},\omega) \text{ , for all } \mathrm{t} \in \mathrm{I} = [0,\mathrm{T}],\ \omega \in \Omega, \tag{1.1}$$

For fixed $r > 0$ and $T > 0$, where for every $\mathrm{t} \in \mathrm{I}$ and every $\xi \in \mathrm{C}([\text{-r},\mathrm{T}]), \xi_{\mathrm{t}})$, ξ_{t} denotes the element of $\mathrm{C}([\text{-r},0],\Omega)$ defined by

$$\xi_{\mathrm{t}}(\theta,\omega) = \xi(t+\theta,\omega) \text{ for all } \theta \in [-r,0] , \quad \omega \in \Omega . \tag{1.2}$$

The function ξ_{t} can be regarded as the history of ξ from time $t-r$ up to time t

and the function u , v are assumed to be continuous on the set $I \times \mathrm{C}([\text{-r},0]) \times \Omega \to R$

The delay differential equation (1.1) can be studied in a more general frame of functional differential equations. Indeed, it suffices to define the function $u{:}\mathrm{I} \times \mathrm{C}([\text{-r, T}]) \times \Omega \to R$ as

$$u(t,\ \xi,\omega) = v(t,\ \xi_t,\omega)$$

in order to rewrite equation ([1.1] as

$$p'(t,\omega) = u(t,\ p,\omega) + v(t,\ p,\omega) \text{ for all } t \in I = [0,\ T] ,\ \omega \in \Omega . \tag{1.3}$$

Also,other types of well-known functional differential equations can be reduced to an equation of the form of (1.3).

Another example is furnished by the equations with maxima, which are of the type

$$p'(t,\omega) = w\left(t, p(t,\omega), \omega \max_{s\in S(t)} p(s)\right) \text{ for all } t \in I = [0,\ T] , \quad \omega \in \Omega . \tag{1.4}$$

When $S(t)=[t-r,\ t],$ (1.4) can be regarded as a particular case of (1.1), but formulation (1.3) permits us to consider more general forms of $S(t)\subset\Re$.

Equations with maxima have attracted much attention recently due to their importance in mathematical models such as the automatic control of technical systems.

Monotone iterative techniques have been extensively applied to study the existence of solutions for delay differential equations and equations with maxima. S, where equation (1.1) is studied with periodic boundary conditions of type

$$p(0,\omega)=p(T,\omega) \tag{1.5}$$

and a constant initial condition $p(0,\omega)=p(t,\omega),\ t\in[-r,0].$

For (1.4), Stepanov [15] considered a nonlinear two-point boundary condition of the form $w(p(0,\omega),\ p(T,\omega))=0$ and a Carathéodory function u.

We shall consider here a functional differential equation

$$p'(t,\omega)=u(t,p(t,\omega),\ p,\omega)+v(t,p(t,\omega),\ p,\omega)\ \text{ for a.e. } t\in I=[0,T], \tag{1.6}$$

with less restrictive conditions on u, in the spirit of [6].

In relation to the boundary conditions, we would like to emphasize the fact that the usual pointwise boundary conditions which naturally arise in ordinary differential equations, do not seem to be so natural for functional differential equations. Indeed, if we regard the problem of finding a periodic solution as the problem of finding a fixed point for the Poincaré map, then the natural condition that appears in connection with equation (1.1) or (1.4) is

$$p_0(\theta,\omega)=p_T(\theta,\omega)\ \text{ for all } \theta\in[-r,0],\ \omega\in\Omega \tag{1.7}$$

instead of (1.5). Condition (1.7) was considered in [3], where the monotone iterative method was developed for (1.1) with u continuous.

We also remark that this periodic condition can be seen as a particular case of the more general functional boundary condition

$$\mathrm{H}(\mathrm{p}_0,\ p_T)=0, \tag{1.8}$$

for a given mapping $H:(C([-r,0]))^2\to C([-r,0]),$ proposed by Halanay [4]. The linear case of Halanay's boundary conditions $A_1p_0+A_2p_T=\psi,$

where $A_1,\ A_2 : (C([-r,0]))^2 \to C([-r,0])$ are given linear operators, was considered in [9, Chapter 11] and becomes (1.7) for $A_1 = -A_2 = Id,\ \ \psi = 0.$

As far as the authors are aware, there is no previous work on existence of solutions for equation (1.1) or (I.4) together with condition (1.8) in the frame of upper and lower solutions. In this paper we consider the general functional boundary condition

$$p(\theta,\omega) = B\,p(\theta,\omega) \text{ for all } \theta \in [-r,0],\ \omega \in \Omega. \tag{1.9}$$

where B is defined over a suitable set of functions $p:[-r,\ T]\times \Omega \to R$ .More precisely, for $r > 0$ and $T > 0$ fixed, we will define $B : S_r^T \times \Omega \to L^\infty([-r,0])$

where $L^\infty([-r,0])$ denotes the space of functions which are Lebesgue-measurable and bounded on $[-r,0]$,

$$S_r^T = \{\xi : [-r,T] \to R : \xi_{|[-r,\,0]} \in L^\infty([-r,0]) \text{ and } \xi_{|[0,\,T]} \in AC([0,T])\},$$

and $AC([0,T])$ is the space of absolutely continuous functions on $[0,T]$. In S_r^T we consider the supremum norm

$$\|\xi\|_\infty = \sup_{t\in[-r,T]} |\xi(t)|.$$

We shall consider the following boundary value problem

$$\begin{aligned} &p'(t,\omega) = u(t, p_t, p, \omega) + v(t, p_t, p, \omega),\ \ for\ a.e. t \in I = [0,\ T], \\ &p(\theta,\omega) = B\,p(\theta,\omega)\ \ for\ all\ \theta \in [-r,\ 0], \omega \in \Omega \end{aligned} \tag{P}$$

subject to the following assumptions on the functions $u, v : I \times R \times S_r^T \times \Omega \to R$ and $B : S_r^T \times \Omega \to L^\infty([-r,0])$.

(F) (f1) For all $(u,\xi) \in R \times S_r^T$, $(v,\xi) \in R \times S_r^T$ the function $u(., p, \xi, \omega)$, $v(., p, \xi, \omega)$ are measurables on I .

(f2) For a.e. $t \in I$ and all $(u,\xi) \in R \times S_r^T$, $(v,\xi) \in R \times S_r^T$ we have

$$\limsup_{q\to p^-} u(t,q,\ \xi,\omega) \le u(t,\ p,\ \xi,\omega) \le \liminf_{q\to p^+} u(t,\ q,\ \xi,\omega),$$

$$\limsup_{q\to p^-} v(t,q,\ \xi,\omega) \le v(t,\ p,\ \xi,\omega) \le \liminf_{q\to p^+} v(t,\ q,\ \xi,\omega).$$

(f3) For a.e. $t \in I$ and all $u, v \in R$ the function $u(t, p, ., \omega)$, $v(t, p, ., \omega)$ are nondecreasing on S_r^T.

(f4) For all $R > 0$ there exists $\psi_R \in L^1(I)$ such that for a.e. $t \in I$ and for all

$$(p, \xi) \in R \times S_r^T \text{ with } |p| \le R \text{ and } \|\xi\|_\infty \le R \text{ we have}$$

$$|u(t, p, \xi, \omega| \le \psi_R(t, \omega)$$

$$|v(t, p, \xi, \omega| \le \psi_R(t, \omega).$$

(B) $B : S_r^T \times \Omega \to L^\infty([-r, 0])$ is nondecreasing.

The aim of this paper is to prove a general existence result for problem (P) under conditions (F) and (B) assuming the existence of lower and upper solutions. The proof of such a result is based upon an existence result for the Cauchy problem for ordinary differential equations and the generalized iterative techniques from [1.7].

Finally, several types of periodic conditions can be included in the formulation of problem (P). For instance, the functional periodic condition (1.7) corresponds with $\mathrm{B}\xi = \xi_{\mathrm{T}}$ *for* $\xi \in S_r^T$.

The ordinary periodic conditions $p(0, \omega) = p(T, \omega) = p(t, \omega), \mathrm{t} \in [\text{-r}, 0], \omega \in \Omega$ correspond with $\mathrm{B}\xi(\theta, \omega) = \xi(T, \omega)$ *for all* $\theta \in [-r, 0].$

2. Initial Value Problem

Let $T > 0$ be fixed and consider the initial value problem

$$p'(t, \omega) = u(t, p(t, \omega), \omega) + v(t, p(t, \omega), \omega) \text{ for a.e. } t \in I - [0, T],\ p(0, \omega) = a \tag{2.1}$$

The following result is Theorem 3.1 in [6], and we include it here for the convenience of the reader.

Theorem 2.1. Assume that $u : I \times R \to R$ satisfies the following conditions:

1. for all $p \in R$ the function $u(., p)$ is measurable on I,
2. for a.e. $t \in I$ and all p, we have

$$\lim_{p \to q^-} \sup u(t, q) \le u(t, p) \le \lim_{p \to q^+} u(t, q)$$

3. there exists $\psi \in L^1(I)$ such that for a.e. $t \in I$ *and all* $p \in R$

$$|u(t, p)| \le \psi(t).$$

Then the initial value problem (2.1) has extremal solutions for every $a \in R$.

This last theorem can be extended by using lower and upper solutions.

Definition 2.3. We say that α is a lower random solution of (2.1) if $\alpha \in AC(I)$,

$$\alpha'(t,\omega) \leq (u+v)p(t,\alpha(t,\omega),\omega) \ \ for\ a.e.\ \ t \in I \ \ and \ \ \alpha(0,\omega) \leq a.$$

An upper solution β is defined analogously reversing the inequalities.

Let α and β be lower and upper solutions of (2.1) and $\alpha \leq \beta$. We say that a solution of (2.1), x_m, is the maximal solution of (2.1) in $[\alpha,\bar{\beta}]$ if $x_m \in [\alpha,\beta]$ and $x_m \geq x$ for every solution x of (2.1) in $[\alpha,\beta]$. In an analogous way we define the concept of minimal solution in $[\alpha,\beta]$ and when the minimal and maximal solutions in $[\alpha,\beta]$ exist, we call them extremal solutions in $[\alpha,\beta]$.

Theorem 2.4. Assume that $u: I \times R \to R$ satisfies the following conditions:

1. for all $p \in R$ the function $u(.,p)$ is measurable on I;
2. for a.e. $t \in I$ and all p, we have

$$\lim_{q \to p^-} \sup(u+v)(t,q) \leq (u+v)(t,p) \leq \lim_{q \to p^+} \inf(u+v)(t,q)$$

3. for every $S > 0$ there exists $\psi_R \in L^1(I)$ such that for a.e. $t \in I$ and all $p \in R$ with $|P| \leq S$ we have

$$|(u+v)(t,p)| \leq \ \psi_R(t)$$

If α and β are, respectively, a lower and an upper solution of problem (2.1) and $\alpha \leq \beta$, then the problem (2.1) has extremal solutions in $[\alpha,\beta]$.

3. Bondary Value Problem

Main Result

To prove our main result, we need the following lemma, which is Theorem 1.2.2 in [7].

Lemma 3.1. Let Y be a subset of an ordered metric space X, $[a,b]$ a nonempty interval in Y and $G:[a,b] \to [a,b]$ a nondecreasing mapping.

If $\{G\,p_n\}$ converges in Y whenever $p_n \subset [a,b]$ is a monotone sequence, then G has a minimal fixed point $p_* \in [a,b]$ and a maximal one $p^* \in [a,b]$.

Moreover, these extremal fixed points satisfy the relations

$$p_* = \min\{p \in [a,b] : G\,p \le p\} \text{ and } p^* = \max\{p \in [a,b] : G\,p \ge p\}.$$

Now, to make a proper use of this lemma, we give some preliminaries about the space S_r^T defined in Section 1. We can define a partial ordering on S_r^T as follows:

for $\xi_1, \xi_2 \in S_r^T$ we write $\xi_1 \le \xi_2$ if and only if $\xi_1(t) \le \xi_2(t)$ for all $t \in [-r, T]$. If $\xi_1, \xi_2 \in S_r^T$ and $\xi_1 \le \xi_2$, we define the functional interval

$$[\xi_1, \xi_2] = \{\xi \in S_r^T : \xi_1 \le \xi \le \xi_2\}.$$

It is easy to see that the intervals $[\xi_1) = \{\xi \in S_r^T : \xi_1 \le \xi\}$ and $(\xi_1] = \{\xi \in S_r^T : \xi_1 \ge \xi\}$ are closed in S_r^T, considering the supremum norm. Therefore $(S_r^T, \|.\|_\infty)$ is an ordered metric space (see [7]).

Next we introduce the concepts of lower and upper random solution for problem (P).

Definition 3.2. We say that $\alpha : [-r, T] \times \Omega \to R$ is a lower random solution of (P) if $\alpha \in S_r^T$ and

$$\begin{aligned} &\alpha'(t,\omega) \le f(t, \alpha(t,\omega), \alpha, \omega) \quad \textit{for a.e. } t \in [0,T], \\ &\alpha(\theta,\omega) \le B\alpha(\theta,\omega) \qquad \textit{for all } \theta \in [-r, 0]. \end{aligned}$$

Analogously, we say that $\beta : [-r, T] \times \Omega \to R$ is an upper solution of (P) if $\beta \in S_r^T$ and

$$\begin{aligned} &\beta'(t,\omega) \ge f(t, \beta(t,\omega), \beta, \omega) \quad \textit{for a.e. } t \in [0, T], \\ &\beta(\theta,\omega) \ge B\beta(\theta,\omega) \qquad \textit{for all } \theta \in [-r, 0]. \end{aligned}$$

Finally, we say that p is a random solution of (P) if it is both a lower and an upper random solutions.

The following theorem is our main result.

Theorem 3.3. Assume that conditions (F) and (B) are satisfied.

If there exist α and β, lower and upper random solutions of (P), and $\alpha \le \beta$, then (P) has extremal random solutions in $[\alpha, \beta]$.

Proof. We shall only prove the existence of a minimal random solution, since the arguments to show that there is a maximal one are analogous.

By condition (f4) there exists $\psi \in L^1(I)$ such that for a.e. $t \in I$, all $p \in [\alpha(t,\omega), \beta(t,\omega)]$

and all $\xi \in [\alpha, \beta]$ we have

$$\left|(u+v)(t,p,\xi,\omega) \le \psi(t,\omega)\right|. \quad (3.1)$$

Let us consider the mapping $G:[\alpha,\beta] \to [\alpha,\beta]$, where for a given $\xi \in [\alpha,\beta]$ the function $G\xi \in [\alpha,\beta]$ is defined as follows:

Definition of $G\xi$ on $[-r,0]$. We define

$$G\xi = B\xi(\theta,\omega) \text{ for all } \theta \in [-r,0] \quad (3.2)$$

Notice that $\alpha \le B\alpha \le B\xi \le B\beta \le \beta$ on $[-r,0]$ and thus $\alpha \le G\xi \le \beta$ on $[-r,0]$.

Definition of $G\xi$ on $[0,T]$. By condition (f3), and since $\alpha(0,\omega) \le G\xi(0,\omega) \le \beta(0,\omega)$, the restrictions α_1 and β_1 are, respectively, a lower and an upper random solution (in the sense of Definition 2.3) of the initial value problem

$$p'(t,\omega) = (u+v)(t,p(t,\omega),\xi,\omega) \text{ for a.e. } t \in I, x(0,\omega) = B\xi(0,\omega). \quad (3.3)$$

Hence we can apply Theorem 2.4 to ensure that (3.3) has extremal random solutions between α_1 and β_1. We define $G\xi$ on $[0,T]$ as the minimal random solution of (3.3) between α_1 and β_1.

Since $G\xi_{|[-r,0]} \in L^\infty([-r,0])$ and $G\xi_{|I} \in AC(I)$, then $G\xi \in S_r^T$. Moreover, $\alpha \le G\xi \le \beta$ on $[-r,T]$ and, hence, $G\xi \in [\alpha,\beta]$.

Claim 1. G is nondecreasing. Let $\xi_1,\xi_2 \in [\alpha,\beta]$ be such that $\xi_1 \le \xi_2$. Since B is nondecreasing, for all $\theta \in [-r,0]$ we have that

$$G\xi_1(\theta,\omega) = B\xi_1(\theta,\omega) \le B\xi_2(\theta,\omega) = G\xi_2(\theta,\omega)$$

In particular, we have that $B\xi_1(0,\omega) \le B\xi_2(0,\omega)$.

On the other hand, by the definition of G and the condition (f3), for a.e. $t \in I$ we have that

$$(G\xi_2)'(t,\omega) = (u+v)(t,G\xi_2(t,\omega),\xi_2,\omega) \ge (u+v)(t,G\xi_2(t,\omega),\xi_1,\omega)$$

which implies that $G\xi_{2|I}$ is an upper solution (in the sense of Definition 2.3) of (3.3) with $\xi = \xi_1$. Moreover, $G\xi_2 \ge \alpha$ on I and α_1 is a lower solution of (3.3) with $\xi = \xi_1$. Thus, by Theorem 2.4, the problem (3.3) with $\xi = \xi_1$ has solutions between α_1 and $G\xi_{2|I}$. Finally, since $G\xi_{1|I}$ is the minimal random solution of that problem between α_1 and β_1, we conclude that $G\xi_1 \le G\xi_2$ on I.

Claim 2. $\{G\xi_n\}$ converges in S_r^T whenever $\{\xi_n\} \subset [\alpha, \beta]$ is a monotone sequence.

Let $\{\xi_n\} \subset [\alpha, \beta]$ be a monotone sequence. Since G is nondecreasing and maps $[\alpha, \beta]$ into itself, then $\{G\xi_n\}$ is monotone and bounded. Hence we can define

$$q(t,\omega) = \lim_{n\to\infty} G\xi_n(t,\omega) \text{ for all } t \in [-r, T].$$

Obviously, $\alpha \le q \le \beta$ on $[-r, T]$. Let us prove that $q \in S_r^T$.

First, $q_{|[-r,0]}$ is measurable on $[-r, 0]$ because it is a point wise limit of measurable functions on $[-r, 0]$ Thus $q_{|[-r,0]} \in L^\infty([-r, 0])$.

On the other hand, by (3.1) and the definition of $G\xi_n$, for all $n \in \aleph$ and all $t, s \in [0, T]$ we have

$$| G\xi_n(t,\omega) - G\xi_n(s,\omega) | \le \left| \int_s^t \psi(r,\omega)dr \right|$$

which implies that for all $t, s \in [0, T]$ we have

$$| q(t,\omega) - q(s,\omega) | \le \left| \int_s^t \psi(r,\omega)dr \right|$$

Therefore $q_{|I} \in AC(I)$ and, thus, $q \in S_r^T$.

By Lemma 3.1 G has a minimal fixed point $p_* \in [\alpha, \beta] \cap Y$ which satisfies

$$p_* = \min \{\xi \in [\alpha, \beta] \cap Y : G\xi \le \xi\}. \tag{3.4}$$

It is easy to verify that the fixed points of G, and in particular p_* are solutions of (P)

Claim 3. p_* is the minimal solution of (P) in $[\alpha, \beta]$. Assume that $\mu \in [\alpha, \beta]$ is a solution of (P). Since μ is a solution of (P), we have that $\mu(\theta,\omega) = B\mu(\theta,\omega) = G\mu(\theta,\omega)$ for all $\theta \in [-r, 0]$. Moreover, μ is a solution between $\alpha_{|I}$ and $\beta_{|I}$ of the initial value problem (3.3) with $\xi = \mu$, hence $\mu(t,\omega) \ge G\mu(t,\omega)$ for all $t \in I$, because $G\mu_{|I}$ is the minimal solution of that problem between α_1 and β_1. Therefore we conclude that $G\mu \le \mu$ on $[-r, T]$ which, together with (3.4), implies that $p_* \le \mu$ on $[-r, T]$.

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