Topological Degree Theory in Fractional Order Boundary Value Problem

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Abstract: This paper investigates the existence and uniqueness of a solution to boundary value problems involving the Caputo fractional derivative in the Banach space. It is based on the application of topological degree approach and fixed-point theory with topological structures in some appropriate situations.

Keywords: Fractional derivatives and integrals; Topological properties of mappings and Fixed-point theorems.

1. Introduction

Fractional differential equations were created to be effective modeling of many phenomena in many fields of science. For more details, see [12, 13, 15, 16]. Certainly, the use of topological strategies is very close to evaluating the existence of solutions to fractional differential equations during the past decades, see [10, 14, 19, 20, 21, 22]. Fractional differential equations in Banach space are receiving more attention by many researchers such as Agarwal et al. [2, 3], Balachandran and Park [4], Benchohra et al. [5] and Zhang [25].

Boundary value problems with integral boundary conditions create a very important class of problems. It includes two, three, multipoint and nonlocal boundary value problems as especial cases [8, 9]. Integral boundary conditions appear in cellular systems [1] and population dynamic [6].

In 2006, Zhang [26], considered the existence of positive solutions to nonlinear fractional boundary value problems by applying the properties of the green function and fixed point theorem to cones. In 2009, Benchohra et al. [5], examined the existence and uniqueness of solutions to fractional boundary value problems with nonlocal conditions by fixed point theory. In 2012, Wang et. al [23, 24], got the required and sufficient conditions of fractional boundary value problems by coincidence degree of condensing maps in Banach spaces. In 2015, the result was extended to the case of solutions to the fractional order multipoint boundary value problem by Khan and Shah [11], who intentioned sufficient conditions for existence outcomes for the boundary value problem. In 2017, Samina et al. [18], studied the existence of solutions of nonlinear fractional Hybrid differential equations through some results for the existence of solutions and thus the Kuratowski's measure of non compactness. Li and Zhai [17], investigated the existence and uniqueness of solutions to Langevin equations using two fractional orders using e -positive operators and Altman fixed point theorem. In [7], the authors considered the initial problem of systems of differential equations for fractional order. They Produced regularization problem and were given an algorithm for normal and unique solubility general iterative systems of differential equations with partial derivatives.

Motivated by the above cited results, our aim throughout this paper is to verify some new results about the following boundary value problem (BVP) for fractional differential equations involving Caputo fractional derivative by topological degree theory and fixed-point theorem in Banach space X.

$$\begin{cases} ({}^cD^q \mathrm{x}(\mathrm{t})) = \mathrm{F}(\mathrm{t}, \mathrm{x}(\mathrm{t})) \\ \mathrm{x}(0) = \eta(\mathrm{x}), \ \mathrm{x}(\mathrm{T}) = \mathrm{x}_0, \end{cases} \tag{1}$$

where $t \in J := [0, T]$, $q \in (0,1)$, $({}^cD^q)$ is the Caputo derivative, $F: J \times X \to X$ and $\eta: X \to X$ are given continuous maps.

2. Preliminaries

In this section, we introduce some necessary definitions, lemmas, propositions and theorem which are needed throughout this paper.

We define Banach space $C(J,X)$ as the Banach space of all continuous functions from J to X with the topological norm $\| x \| := sup\{\| x(t) \| : x \in C(J,X),\ t \in J\}$ and $J = [0,T], T > 0$.

Definition 2.1 ([15, 19]) For a given function F in the closed interval $[a,b]$, the qth fractional order integral of F is defined by;

$$I_{a+}^{q}F(t) \quad = \frac{1}{\Gamma(q)}\int_{a}^{t}(t-s)^{q-1}F(s)ds, \tag{2}$$

where Γ is the gamma function.

Definition 2.2 ([15, 19]) For a given function F in the closed interval $[a,b]$, the qth Riemann-Liouville fractional-order derivative of F, is defined by;

$$(D_{a+}^{q}F)(t) \quad = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{n-q-1}F(s)ds. \tag{3}$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 2.3 ([15, 19]) For a given function F in the closed interval $[a,b]$, the Caputo fractional order derivative of F, is defined by;

$$({}^{c}D_{a+}^{q}F)(t) \quad = \frac{1}{\Gamma(n-q)}\int_{a}^{t}(t-s)^{n-q-1}F^{(n)}(s)ds, \tag{4}$$

where $n = [q] + 1$.

Lemma 2.1 ([24]) Fractional order differential equation of order $q > 0$ of the form

$({}^{c}D^{q}h(t)) = 0, n-1 < q \leq n$, has a unique solution of the form

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1}t^{n-1},$$

where $c_i \in X,\quad i = 0,1,2,\ldots,n-1$.

Lemma 2.2 ([24]) Let $n-1 < q \leq n$, then

$$I^{q}({}^{c}D^{q}h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in X,\quad i = 0,1,2,\ldots,n-1, n = [q] + 1$.

Definition 2.4 ([23]) The function $\alpha : B \to X_{+}$ defined by

$$\alpha(M) = inf\{d > 0 : M\ admits\ a\ finite\ cover\ by\ sets\ of\ diameter \leq d\},$$

$M \in B$, is called the Kuratowski measure of non compactness.

Proposition 2.1 ([23]) The following assertions hold:

(i) $\alpha(M) = 0$ iff M is relatively compact.

(ii) $\alpha\ is\ a\ seminorm\ i.e.,\ \alpha(\lambda M) = |\lambda|\alpha(M), \lambda \in R\ and\ \alpha(M_1 + M_2) \leq \alpha(M_1) + \alpha(M_2)$.

(iii) $M_1 \subset M_2\ implies\ \alpha(M_1) \leq \alpha(M_2); \alpha(M_1 + M_2) = max\{\alpha(M_1), \alpha(M_2)\}$.

(iv) $\alpha(convM) = \alpha(M)$.

(v) $\alpha(\overline{M}) = \alpha(M)$.

Definition 2.5 ([20, 27]) Let $\Omega \subset X$ and $F : \Omega \to X$ be a continuous bounded map. One can say that F is α-Lipschitz if there exists $k \geq 0$ such that

$$\alpha(F(B)) \leq k\alpha(B) \quad (\forall)\ B \subset \Omega\ \ bounded\,.$$

In case, $k < 1$, then we call F is a strict α-contraction. One can say that F is α-condensing if

$$\alpha(F(B)) < \alpha(B) \quad (\forall)\ B \subset \Omega\ \ bounded\ with\ \ \alpha(B) > 0.$$

We recall that $F : \Omega \to X$ is Lipschitz if there exists $k > 0$ such that

$$\| F_x - F_y \| \leq k \| x - y \| \quad (\forall)\ x, y \subset \Omega,$$

and if $k < 1$ then F is a strict contraction.

Proposition 2.2 ([21, 27]) If $F, G : \Omega \to X$ are α-Lipschitz maps with constants k, k' respectively, then $F + G : \Omega \to X$ is α-Lipschitz with constant $k + k'$.

Proposition 2.3 ([21, 27]) If $F:\Omega \to X$ is compact, then F is α-Lipschitz with zero constant.

Proposition 2.4 ([21, 27])) If $F:\Omega \to X$ is Lipschitz with constant k, then F is α-Lipschitz with the same constant k.

Theorem 2.1 ([27]) Suppose X is the Banach space, and $F,G:X \to X$ are two operators such that F is a contraction operator, and G is a completely continuous operator then the operator equation $Hx = Fx + Gx = x$ has a solution $x \in X$.

3. Existence and uniqueness result of the system

First, we define the meaning of the solution to the fractional BVP (1).

Definition 3.1 The function $x \in C(J,X)$ is called the solution to the fractional BVP (1), if x satisfies the equation ${}^{c}D^{q}x(t) = F(t,x(t))$ almost everywhere on J and the conditions $x(0) = \eta(x)$, $x(T) = x_0$.

In order to discuss existence and uniqueness solutions in fractional BVP (1), we require the following assumptions:

$[H1]$ $F:J \times X \to X$ is continuous.

$[H2]$ For each $t \in J$ and all $x,y \in X$, there exists constant $\delta_F > 0$ such that

$$\| F(t,x) - F(t,y) \| \leq \delta_F \| x - y \|.$$

$[H3]$ For all $x,y \in C(J,X)$, there exists constant $\delta_\eta > 0$ such that

$$\| \eta(x) - \eta(y) \| \leq \delta_\eta \| x - y \|.$$

$[H4]$ For arbitrary $(t,x) \in J \times X$, there exist $\delta_1, \delta_2 > 0$, $q_1 \in [0,1)$ such that

$$\| F(t,x) \| \leq \delta_1 \| x \|^{q_1} + \delta_2.$$

$[H5]$ For arbitrary $x \in C(J,X)$, there exist $\delta_3, \delta_4 > 0$, $q_2 \in [0,1)$ such that

$$\| \eta(x) \| \leq \delta_3 \| x \|^{q_2} + \delta_4.$$

For the existence of solutions for the BVP (1), we need the following auxiliary lemma, see Faree and Panchal [22].

Lemma 3.1 Let $0 < q < 1$, the fractional integral equation

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s,x(s))ds - \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}F(s,x(s))ds \\ &-(\frac{t}{T}-1)\eta(x) + \frac{t}{T}x_0 \end{aligned} \tag{5}$$

has a solution $x \in C(J,X)$ if and only if x is a solution of the fractional BVP (1).

From expression (5), we can determine the operators:

$$G_1: C(J,X) \to C(J,X), \quad (G_1x)(t) = (1-\frac{t}{T})\eta(x) + \frac{t}{T}x_0, \qquad t \in J,$$

$$G_2: C(J,X) \to C(J,X), \quad (G_2x)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s,x(s))ds, \qquad t \in J,$$

$$G_3: C(J,X) \to C(J,X), \quad (G_3x)(t) = -\frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}F(s,x(s))ds, \qquad t \in J,$$

$$H: C(J,X) \to C(J,X), \quad Hx = G_1x + G_2x + G_3x.$$

It is clear that H is well defined. Then, the fractional integral equation (5) can be written as the following operator

$$x = Hx = G_1x + G_2x + G_3x \tag{6}$$

Thus, the existence of a solution for the fractional BVP (1) is equivalent to the existence of a fixed point for operator H in (6).

In order to get the continuity of operators G_1, G_2, G_3, we need the following assumption:

$[H6]$ For any $\kappa > 0$, there exists a constant $\beta_\kappa > 0$ such that

$$\alpha(F(s, M)) \leq \beta_\kappa \alpha(M)$$

for all $t \in J, M \subset B_\kappa := \{\| x \| \leq \kappa : x \in C(J, X)\}$ and

$$\frac{T^q \beta_\kappa}{\Gamma(q+1)} < 1.$$

Lemma 3.2 The operator $G_1: C(J, X) \to C(J, X)$ is continuous and completely continuous.

Proof. Let $\{x_n\}$ be a sequence of a bounded set $B_\kappa \subseteq C(J, X)$ such that $\| x_n - x \| \to 0$ in $B_\kappa (\kappa > 0)$. Due to the continuity of η, it is easy to see that

$$\| G_1 x_n)(t) - (G_1 x)(t) \| = |1 - \frac{t}{T}| \, \| \eta(x_n) - \eta(x) \| \to 0 \ \ as \ \ n \to \infty.$$

Therefore, $G_1 x_n \to G_1 x$ as $n \to \infty$ which implies that G_1 is continuous.

Let $\{x_n\}$ be a sequence on $M \subseteq B_\kappa$. Using (H5) and (H6), for every $x_n \in M$ and $0 \leq t \leq T$, we have

$$\| (G_1 x_n)(t) \| = \| (1 - \tfrac{t}{T}) \eta(x) + \tfrac{t}{T} x_0 \| \leq |1 - \tfrac{t}{T}| [\delta_3 |\kappa|^{q_2} + \delta_4] + |\tfrac{t}{T}| x_0 := K_1$$

Therefore, for every $x_n \in M$, so $G_1(M)$ is bounded in B_κ.

For $0 \leq t_1 < t_2 \leq T$ and as $t_2 \to t_1$, we obtain

$$\begin{aligned} &\| (G_1 x_n)(t_2) - (G_1 x_n)(t_1) \| = \| -\tfrac{t_2}{T} \eta(x_n) + \tfrac{t_2}{T} x_0 + \tfrac{t_1}{T} \eta(x_n) - \tfrac{t_1}{T} x_0 \| \\ &\leq |\tfrac{t_2 - t_1}{T}| \, \| \eta(x_n) + x_0 \| \to 0. \end{aligned}$$

Therefore, $G_1 x_n$ is equicontinuous.

Since G_1 is a continuous, bounded map and equicontinuous as a result of the Arzela Ascoli theorem, we get $G_1: C(J, X) \to C(J, X)$ is completely continuous.

Lemma 3.3 The operators $G_2, G_3: C(J, X) \to C(J, X)$ are continuous. Consequently, $G_2 + G_3$ is continuous. Moreover, $G_2 + G_3$ satisfies the following condition:

$$\| G_2 x \| + \| G_3 x \| \leq (1 + |\tfrac{t}{T}|) \frac{T^q (\delta_1 \|x\|^{q_1} + \delta_2)}{\Gamma(q+1)} \tag{7}$$

for every $x \in C(J, X)$.

Proof. Let $\{x_n\}$ be a sequence of a bounded set $B_\kappa \subseteq C(J, X)$ such that $\| x_n - x \| \to 0$ as $n \to \infty$. we have to show that $\| G_2 x_n - G_2 x \| \to 0$ as $n \to \infty$. It is obvious that $\| F(s, x_n(s)) - F(s, x(s)) \| \to 0$ as $n \to \infty$ due to the continuity of F. Using (H4) and (H6), we get for each $t \in J$,

$$\| F(s, x_n(s)) - F(s, x(s)) \| \leq \| F(s, x_n(s)) \| + \| F(s, x(s)) \| \leq 2(\delta_1 |\kappa|^{q_1} + \delta_2).$$

As the function $s \to 2(\delta_1 |\kappa|^{q_1} + \delta_2)$ is integrable for $s \in [0, t], t \in J$, by means of the Lebesgue Dominated Convergence theorem

$$\int_0^t (t - s)^{q-1} \| F(s, x_n(s)) - F(s, x(s)) \| \, ds \to 0 \ \ as \ \ n \to \infty$$

Then, for all $t \in J$,

$$\| G_2 x_n - G_2 x \| \leq \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| F(s, x_n(s)) - F(s, x(s)) \| \, ds \to 0 \ \ as \ \ n \to \infty$$

Which means that G_2 is continuous. By repeating the same process of the G_2 operator for the G_3 operator, one can get that G_3 is also continuous.

Consequently, by the proposition of continuity maps, since $G_2, G_3: C(J, X) \to C(J, X)$ are continuous then $G_2 + G_3$ is continuous.

The relation (7) is obtained as a simple result of the definitions G_2, G_3, and (H4)

$$\| G_2x \| + \| G_3x \| = \| \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s,x(s))ds \|$$
$$+\| \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}F(s,x(s))ds \|$$
$$\leq (1+|\frac{t}{T}|)\frac{T^q(\delta_1\|x\|^{q_1}+\delta_2)}{\Gamma(q+1)}$$

for every $x \in C(J,X)$.

Lemma 3.4 The operators $G_2, G_3: C(J,X) \to C(J,X)$ are compact. Consequently G_2, G_3 are $\alpha -$ Lipschitz with zero constant. Further, $G_2 + G_3$ is $\alpha -$ Lipschitz with zero constant.

Proof. In order to demonstrate the compactness of G_2, G_3 , we consider a bounded set $M \subseteq C(J,X)$ and we have to show that $G_2(M), G_3(M)$ are relatively compact in $C(J,X)$.

Suppose $\{x_n\}$ be a sequence on $M \subset B_\kappa$, for every $x_n \in M$. By assumption [H6] with relation (7), we have

$$\| G_2x_n \| + \| G_3x_n \| \leq (1+|\frac{t}{T}|)[\frac{T^q\delta_1|\kappa|^{q_1}+\delta_2}{\Gamma(q+1)}] := K_2$$

for every $x_n \in M$, so $G_2(M)$ and $G_3(M)$ are bounded in B_κ.

Now, we prove that $\{G_2x_n\}$ is equicontinuous. For $0 \leq t_1 < t_2 \leq T$, we get

$$\| (G_2x_n)(t_2) - (G_2x_n)(t_1) \| \leq \| \frac{1}{\Gamma(q)}\int_0^{t_2} (t_2-s)^{q-1}F(s,x_n(s))ds - \frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1}F(s,x_n(s))ds \|$$
$$\leq \frac{1}{\Gamma(q)}\int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] \| F(s,x_n(s)) \| ds + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2} (t_2-s)^{q-1} \| F(s,x_n(s)) \| ds$$
$$\leq \frac{1}{\Gamma(q)}\int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}](\delta_1 \| x_n \|^{q_1} + \delta_2)ds + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2} (t_2-s)^{q-1}(\delta_1 \| x_n \|^{q_1} + \delta_2)ds$$
$$\leq \frac{(\delta_1|\kappa|^{q_1}+\delta_2)}{\Gamma(q)}[\frac{t_1^q}{q} + \frac{(t_2-t_1)^q}{q} - \frac{t_2^q}{q}] + \frac{(\delta_1|\kappa|^{q_1}+\delta_2)(t_2-t_1)^q}{\Gamma(q+1)}$$
$$\leq \frac{2(\delta_1|\kappa|^{q_1}+\delta_2)(t_2-t_1)^q}{\Gamma(q+1)}$$

As $t_2 \to t_1$ the above inequality with $q_1 \in [0,1)$ tends to zero. Therefore, G_2x_n is equicontinuous.

Therefore, $G_2(M) \subset C(J,X)$ satisfies the Arzela Ascoli hypothesis, so $G_2(M)$ is relatively compact in $C(J,X)$. By repeating the same process of the G_2 operator for the G_3 operator, one can get that G_3 is also relatively compact in $C(J,X)$.

By proposition (2.3), G_2, G_3 are $\alpha -$ Lipschitz with zero constant. Moreover, by proposition (2.2), $G_2 + G_3$ is $\alpha -$ Lipschitz with zero constant.

Theorem 3.1 Assume that $(H1) - (H6)$ hold then the fractional BVP (1) has at least one solution $x \in C(J,X)$ and the set of solutions for the fractional BVP (1) is bounded in $C(J,X)$.

Proof. Let $G_1, G_2, G_3, H: C(J,X) \to C(J,X)$ be the operators defined in the beginning of this section. They are continuous and bounded. Morever G_1 is completely continuous operator and $G_2 + G_3$ is $\alpha -$ Lipschitz with zero constant which means that $G_2 + G_3$ is a strict $\alpha -$ contraction with zero constant. Therefore, Krasnoselskii's fixed point theorem (2.1) shows that H has a fixed point on $C(J,X)$, so the fractional BVP (1) has a solution $x = Hx = G_1x + G_2x + G_3x$ for all $t \in J$.

Consider the following set of solutions for the system (1)

$$S = \{x \in C(J,X): there\ \ exists\ \ \lambda \in [0,1]\ \ such\ \ that\ \ x = \lambda Hx\}.$$

We shall prove that S is bounded in $C(J,X)$. For $x \in S$ and $\lambda \in [0,1]$, we have

$$\| x(t) \| = \| \lambda Hx(t) \| \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \| F(s,x(s)) \| ds + \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1} \| F(s,x(s)) \| ds$$
$$+(1-\frac{t}{T}) \| \eta(x) \| + \frac{t}{T}x_0$$
$$\leq \frac{t(t^{q-1}+T^{q-1})}{\Gamma(q+1)}[\delta_1|\kappa|^{q_1} + \delta_2] + (1-\frac{t}{T})[\delta_3|\kappa|^{q_2} + \delta_4] + \frac{t}{T}x_0 := \iota$$

The above inequality with $q_1, q_2 \in [0,1)$ and $t \in [0,T]$ shows that S is bounded in $C(J,X)$. As a result of Schaefer's fixed-point theorem, we conclude that H has at least one fixed point and the set of fixed points of H is bounded which is a solution of the fractional BVP (1).

Theorem 3.2 Assume that (H1) -(H6) hold, if

$$\frac{2\delta_F T^q}{\Gamma(q+1)} + \delta_\eta < 1.$$

then the fractional BVP (1) has a unique solution $x \in C(J, X)$.

Proof. By Theorem (3.1), the fractional BVP (1) has a solution $x(.)$ in $C(J, X)$. Let $y(.)$ be another solution of the fractional BVP (1) with boundary conditions $x(0) = \eta(x),\ x(T) = x_0$. By (H2), we obtain

$$\begin{aligned}
&\| x(t) - y(t) \| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| F(s, x(s)) - F(s, y(s)) \| ds \\
&+ \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \| F(s, x(s)) - F(s, y(s)) \| ds + (\frac{t}{T} - 1) \| \eta(x) - \eta(y) \| \\
&\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \delta_F \| x - y \| ds + \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \delta_F \| x - y \| ds \\
&+ (\frac{t}{T} - 1) \delta_\eta \| x - y \| \\
&\leq [\frac{\delta_F t (t^{q-1} + T^{q-1})}{\Gamma(q+1)} + \delta_\eta (\frac{t}{T} - 1)] \| x - y \|
\end{aligned} \tag{8}$$

as $0 \leq t \leq T$ then,

$$\| x(t) - y(t) \| \leq [\frac{2\delta_F T^q}{\Gamma(q+1)} + \delta_\eta] \| x - y \|$$

By the standard singular Gronwall inequality, we obtain $\| x(.) - y(.) \| \leq 0$, which yields the uniqueness of $x(.)$.

4.Conclusion

We confirmed some sufficient conditions for the existence and uniqueness of a solution to the fractional BVP (1). We have relied on the fixed-point theory along with the topological technique for solutions. In addition, we studied some topological properties of the solution set.

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