Periodic Solution for Nonlinear Second Order Differential Equation System

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Abstract

In this work, we investigate the periodic solutions for non-linear system of differential equations by using the method of periodic solutions of ordinary differential equations which are given by A.M.Samoilenko. Additionally, the existence and uniqueness theorem have been proved for second differential equations system by using Banach fixed point theorem.

Keywords: Periodic solution, Nonlinear system of differential equation, Banach fixed point.

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I.Introduction

According to the differential equation system which is:

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right). \tag{1}$$

Where $x \in D$, D is a closed and bounded subset of R^n, D_1, D_2 are closed and bounded subset of R^m[1]. Let the vector function $f(t, x, \dot{x}, \ddot{x})$ be defined and continuous on the domain:

$$(t, x, \dot{x}, \ddot{x}) \in R^1 \times D \times D_1 \times D_2 = (-\infty, \infty) \times D \times D_1 \times D_2, \tag{2}$$

where $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$ are periodic in t of periodic T.

Assume that the vector functions $f(t, x, \dot{x}\ddot{x},)$ satisfying the following inequalities [2]:

$$\|f(t, x, \dot{x}, \ddot{x})\| \le M, \tag{3}$$

$$\|f(t, x_1, \dot{x}_1, \ddot{x}_1) - f(t, x_2, \dot{x}_2, \ddot{x}_2)\| \le K_1\|x_1 - x_2\| + K_2\|\dot{x}_1 - \dot{x}_2\| + K_3\|\ddot{x}_1 - \ddot{x}_2\|, \tag{4}$$

for all $t \in R^1, x, x_1, x_2 \in D,\ \dot{x}, \dot{x}_1, \dot{x}_2 \in D_1,\ \ddot{x}, \ddot{x}_1, \ddot{x}_2 \in D_2$, where M is a positive vector constant and $K_1 = (K_{1ij}), K_2 = (K_{2ij}), K_3 = (K_{3ij})$ are $n \times n$ non-negative matrices, where $ij = 1,2, \dots n$.

From [3], we have

$$L^2 f(t) = \int_0^t \left\{\int_0^t f(t) - \overline{f(t)}dt - \overline{\int_0^t f(t) - \overline{f(t)}dt}\right\} dt, \tag{5}$$

where L is a mapping from D to D_1 , and by $\overline{f(t)}$ the time average over the interval $[0, T]$,

$\overline{f(t)} = \frac{1}{T}\int_0^t f(s)ds$. (6)

Define a non-empty set as:

$$\left.\begin{aligned} D_f &= D - \frac{T}{2}M \\ D_{1f} &= D_1 - \frac{5T}{6}M \\ D_{2f} &= D_2 - 2M \end{aligned}\right\} \quad (7)$$

Functional sequence $\{x_m(t,x_0)\}_{m=0}^{\infty}$, $\{\dot{x}_m(t,x_0)\}_{m=0}^{\infty}$ and $\{\ddot{x}_m(t,x_0)\}_{m=0}^{\infty}$ defined as [4]:

$$x_{m+1}(t,x_0) = x_0 + L^2 f(t, x_m(t,x_0), \dot{x}_m(t,x_0), \ddot{x}_m(t,x_0)) , \quad (8)$$

where $m = 0,1,2,\ldots$

And

$$\dot{x}_{m+1}(t,x_0) = \dot{x}_0 + Lf(t, x_m(t,x_0), \dot{x}_m(t,x_0), \ddot{x}_m(t,x_0)) - \overline{Lf(t, x_m(t,x_0), \dot{x}_m(t,x_0), \ddot{x}_m(t,x_0))} \quad (9)$$

where $m = 0,1,2,\ldots$

And

$$\ddot{x}_{m+1}(t,x_0) = \ddot{x}_0 + f(t, x_m(t,x_0), \dot{x}_m(t,x_0), \ddot{x}_m(t,x_0)) - \overline{Lf(t, x_m(t,x_0), \dot{x}_m(t,x_0), \ddot{x}_m(t,x_0))} \quad (10)$$

where $m = 0,1,2,\ldots$

2. Approximation of a Periodic Solution of (1).

The study of the approximate periodic solution of the problem (1) is introduced in the following theorem: -

Lemma 1. [5] Let the vector function $f(t,x)$ is defined and continuous on the interval $[0,T]$, then the following inequality

$$\left\| \int_0^t \left(f\big(s,x(s)\big) - \frac{1}{T}\int_0^T f\big(s,x(s)\big)ds \right) ds \right\| \le \alpha(t)M.$$

Is hold, where $\alpha(t) = 2t\left(1 - \frac{t}{T}\right)$ and $M = \max_{t\in[0,T]} |f(t,x)|$.

Theorem 2. Let the vector function $f(t,x,\dot{x},\ddot{x})$ be defined and continuous on the domain (2) satisfy the inequalities (3),(4). Then there exists a sequence of functions (8),(9), and (10) are periodic in t of period T, converge uniformly as $m \to \infty$ in the domain

$$x(t,x_0) \in [0,T] \times D_f \quad (11)$$

To the limit functions $x(t,x_0), \dot{x}(t,x_0)$ and satisfy the following integral equations:

$$x(t,x_0) = x_0 + L^2 f(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)) \quad (12)$$

And

$$\dot{x}(t,x_0) = \dot{x}_0 + Lf(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)) - \underline{Lf\big(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)\big)} \quad (13)$$

And

$$\ddot{x}(t,x_0)=\ddot{x}_0+f(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0))-\underline{f\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)} \quad (14)$$

Which is a periodic solution of the problem (1).

Provided that: -

$$\left.\begin{array}{c}\|x^0(t,x_0)-x_0\|\le \frac{T^2}{4}M\\ \|x^0(t,x_0)-x_m(t,x_0)\|\le Q^m(E-Q)^{-1}\,\frac{T^2}{4}M\end{array}\right\} \quad (15)$$

Proof. By the lemma (1) and using the sequence of functions (8),(9) and (10) when $m=0$, we get

$$\begin{pmatrix}\|x_1(t,x_0)-x_0\|\\ \|\dot{x}_1(t,x_0)-\dot{x}_0\|\\ \|\ddot{x}_1(t,x_0)-\ddot{x}_0\|\end{pmatrix} = \begin{pmatrix}\|x_0+L^2f(t,x_0,\dot{x}_0,\ddot{x}_0)-x_0\|\\ \left\|\dot{x}_0+Lf(t,x_0,\dot{x}_0,\ddot{x}_0)-\underline{Lf\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)}-\dot{x}_0\right\|\\ \ddot{x}_0+f(t,x_0,\dot{x}_0,\ddot{x}_0)-\underline{f(t,x_0,\dot{x}_0,\ddot{x}_0)}-\ddot{x}_0\end{pmatrix}$$

And

$$\begin{pmatrix}\|x_1(t,x_0)-x_0\|\\ \|\dot{x}_1(t,x_0)-\dot{x}_0\|\\ \|\ddot{x}_1(t,x_0)-\ddot{x}_0\|\end{pmatrix} \le \begin{pmatrix}\|L\big(Lf(t,x_0,\dot{x}_0,\ddot{x}_0)\big)\|\\ \|Lf(t,x_0,\dot{x}_0,\ddot{x}_0)-\underline{Lf(t,x_0,\dot{x}_0,\ddot{x}_0)}\|\\ \|2f(t,x_0,\dot{x}_0,\ddot{x}_0)\|\end{pmatrix}$$

And

$$\begin{pmatrix}\|x_1(t,x_0)-x_0\|\\ \|\dot{x}_1(t,x_0)-\dot{x}_0\|\\ \|\ddot{x}_1(t,x_0)-\ddot{x}_0\|\end{pmatrix} \le \begin{pmatrix}\dfrac{T^2}{4}M\\ \dfrac{T}{2}M+\dfrac{T}{3}M\\ 2M\end{pmatrix}$$

Therefore, $x_1(t,x_0)\in D$, $\dot{x}_1(t,x_0)\in D_1$ and $\ddot{x}_1(t,x_0)\in D_2$, for all $t\in[0,T]$.

Then, by mathematical induction we can prove that: -

$$(\|x_m(t,x_0)-x_0\|\ \|\dot{x}_m(t,x_0)-\dot{x}_0\|\ \|\ddot{x}_m(t,x_0)-\ddot{x}_0\|\ )\le\left(\tfrac{T^2}{4}M\ \ \tfrac{5T}{6}M\ 2M\ \right) \quad (16)$$

i.e.

$x_m(t,x_0)\in D$, $\dot{x}_m(t,x_0)\in D_1$ and $\ddot{x}_m(t,x_0)\in D_2$ when $x_0\in D_f$, $\dot{x}_0\in D_{1f}$ and $\ddot{x}_0\in D_{2f}$.

Then we can prove that: -

$$\begin{pmatrix}\|x_1(t,x_0)-x_0\|\\ \|\dot{x}_1(t,x_0)-\dot{x}_0\|\\ \|\ddot{x}_1(t,x_0)-\ddot{x}_0\|\end{pmatrix} \le \begin{pmatrix}\dfrac{T^2}{4}M\\ \dfrac{5T}{6}M\\ 2M\end{pmatrix}$$

i.e.

$x_m(t,x_0) \in D$, $\dot{x}_m(t,x_0) \in D_1$ and $\ddot{x}_m(t,x_0) \in D_2$ when $x_0 \in D_f$, $\dot{x}_0 \in D_{1f}$ and $\ddot{x}_0 \in D_{2f}$.

Next, we prove that the sequence of functions (8), (9), and (10) convergent uniformly on the domain (2).

When $m = 1$, we get

$$\begin{pmatrix} \|x_1(t,x_0) - x_0(t,x_0)\| \\ \|\dot{x}_1(t,x_0) - \dot{x}_0(t,x_0)\| \\ \|\ddot{x}_1(t,x_0) - \ddot{x}_0(t,x_0)\| \end{pmatrix} =$$

$$\begin{pmatrix} \|x_0 + \int_0^t L\left(f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big) - \underline{f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big)}\right)ds - x_0 - \int_0^t L\left(f(t,x_0,\dot{x}_0,\ddot{x}_0) - \underline{f(t,x_0,\dot{x}_0,\ddot{x}_0)}\right)ds\| \\ \|\dot{x}_0 + L\left(f(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)) - \underline{f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big)}\right) - \dot{x}_0 - L\left(f(t,x_0,\dot{x}_0,\ddot{x}_0) - \underline{f(t,x_0,\dot{x}_0,\ddot{x}_0)}\right)\| \\ \|\ddot{x}_0 + f(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)) - \underline{f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big)} - \ddot{x}_0 - f(t,x_0,\dot{x}_0,\ddot{x}_0) - \underline{f(t,x_0,\dot{x}_0,\ddot{x}_0)}\| \end{pmatrix}$$

$$\begin{pmatrix} \|x_1(t,x_0) - x_0(t,x_0)\| \\ \|\dot{x}_1(t,x_0) - \dot{x}_0(t,x_0)\| \\ \|\ddot{x}_1(t,x_0) - \ddot{x}_0(t,x_0)\| \end{pmatrix}$$

$$\leq \begin{pmatrix} \alpha(t)\frac{T}{2}\|f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big) - f(t,x_0,\dot{x}_0,\ddot{x}_0)\| \\ \alpha(t) + \frac{T}{3}\|f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big) - f(t,x_0,\dot{x}_0,\ddot{x}_0)\| \\ 2\|f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big) - f(t,x_0,\dot{x}_0,\ddot{x}_0)\| \end{pmatrix}$$

And

$$\begin{pmatrix} \|x_1(t,x_0) - x_0(t,x_0)\| \\ \|\dot{x}_1(t,x_0) - \dot{x}_0(t,x_0)\| \\ \|\ddot{x}_1(t,x_0) - \ddot{x}_0(t,x_0)\| \end{pmatrix} \leq$$

$$\begin{pmatrix} \frac{T^2}{4}(K_1\|x_1(t,x_0) - x_0\| + K_2\|\dot{x}_1(t,x_0) - \dot{x}_0\| + K_3\|\ddot{x}_1(t,x_0) - \ddot{x}_0\|) \\ \alpha(t) + \frac{T}{3}(K_1\|x_m(t,x_0) - x_{m-1}(t,x_0)\| + K_2\|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| + K_3\|\ddot{x}_m(t,x_0) - \ddot{x}_{m-1}(t,x_0)\|) \\ 2(K_1\|x_m(t,x_0) - x_{m-1}(t,x_0)\| + K_2\|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| + K_3\|\ddot{x}_m(t,x_0) - \ddot{x}_{m-1}(t,x_0)\|) \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \\ \|\ddot{x}_{m+1}(t,x_0) - \ddot{x}_m(t,x_0)\| \end{pmatrix} =$$

$$\begin{pmatrix} \|x_0 + \int_0^t L\left(f(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)) - \underline{f\left(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\right)}\right) ds - x_0 - \int_0^t L\left(f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0)) - \underline{f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))}\right) ds\| \\ \|\dot{x}_0 + \left(Lf(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)) - \underline{Lf(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0))}\right) - \dot{x}_0 - \left(Lf(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0)) - \underline{Lf(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))}\right)\| \\ \|\ddot{x}_0 + f(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)) - \underline{f(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0))} - \ddot{x}_0 - f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0)) - \underline{f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))}\| \end{pmatrix}$$

And

$$\begin{pmatrix} \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \\ \|\ddot{x}_{m+1}(t,x_0) - \ddot{x}_m(t,x_0)\| \end{pmatrix} \leq$$

$$\begin{pmatrix} \alpha(t)\frac{T}{2}\|f\left(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\right) - f\left(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0)\right)\| \\ \alpha(t) + \frac{T}{3}\|f\left(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\right) - f\left(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0)\right)\| \\ 2\|f\left(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\right) - f\left(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0)\right)\| \end{pmatrix}$$

$$\begin{pmatrix} \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \\ \|\ddot{x}_{m+1}(t,x_0) - \ddot{x}_m(t,x_0)\| \end{pmatrix} \leq$$

$$\begin{pmatrix} \alpha(t)\frac{T}{2}(K_1\|x_m(t,x_0) - x_{m-1}(t,x_0)\| + K_2\|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| + K_3\|\ddot{x}_m(t,x_0) - \ddot{x}_{m-1}(t,x_0)\|) \\ \alpha(t) + \frac{T}{3}(K_1\|x_m(t,x_0) - x_{m-1}(t,x_0)\| + K_2\|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| + K_3\|\ddot{x}_m(t,x_0) - \ddot{x}_{m-1}(t,x_0)\|) \\ 2(K_1\|x_m(t,x_0) - x_{m-1}(t,x_0)\| + K_2\|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| + K_3\|\ddot{x}_m(t,x_0) - \ddot{x}_{m-1}(t,x_0)\|) \end{pmatrix}$$

Hence

$$\begin{pmatrix} \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \\ \|\ddot{x}_{m+1}(t,x_0) - \ddot{x}_m(t,x_0)\| \end{pmatrix} \leq$$

$$\begin{pmatrix} \alpha(t)\frac{T}{2}K_1 & \alpha(t)\frac{T}{2}K_2 & \alpha(t)\frac{T}{2}K_3 \\ \left(\alpha(t) + \frac{T}{3}\right)K_1 & \left(\alpha(t) + \frac{T}{3}\right)K_2 & \left(\alpha(t) + \frac{T}{3}\right)K_3 \\ 2K_1 & 2K_2 & 2K_3 \end{pmatrix} \begin{pmatrix} \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \\ \|\ddot{x}_{m+1}(t,x_0) - \ddot{x}_m(t,x_0)\| \end{pmatrix} \quad (17)$$

Rewrite inequality (17) in vector form

$$Z_{m+1}(t) \leq Q(t)Z_m^0 \qquad (18)$$

Where

$$Z_{m+1}(t) = \begin{pmatrix} \|x_{m+1}(t,x_0) - x_m\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m\| \\ \|\ddot{x}_{m+1}(t,x_0) - \ddot{x}_m\| \end{pmatrix}$$

$$Q(t) = \begin{pmatrix} \alpha(t)\frac{T}{2}K_1 & \alpha(t)\frac{T}{2}K_2 & \alpha(t)\frac{T}{2}K_3 \\ \left(\alpha(t)+\frac{T}{3}\right)K_1 & \left(\alpha(t)+\frac{T}{3}\right)K_2 & \left(\alpha(t)+\frac{T}{3}\right)K_3 \\ 2K_1 & 2K_2 & 2K_3 \end{pmatrix}$$

$$Z_m^0 = \begin{pmatrix} \|x_{m+1}(t,x_0) - x_m\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m\| \\ \|\ddot{x}_{m+1}(t,x_0) - \ddot{x}_m\| \end{pmatrix}$$

$$Z_1^0 \leq \begin{pmatrix} \frac{T^2}{4}M \\ \frac{5T}{6}M \\ 2M \end{pmatrix}$$

It follows from inequality (18) that

$$Z_{m+1}^0 \leq Q_0\, Z_m^0 \tag{19}$$

$$Q_0 = \begin{pmatrix} \frac{T^2}{4}K_1 & \frac{T^2}{4}K_2 & \frac{T^2}{4}K_3 \\ \frac{5T}{6}K_1 & \frac{5T}{6}K_2 & \frac{5T}{6}K_3 \\ 2K_1 & 2K_2 & 2K_3 \end{pmatrix} \tag{20}$$

By iterating inequality (19) we have

$$Z_{m+1}^0 \leq Q_0^m\, Z_1^0 \tag{21}$$

This leads to the estimate

$$\sum_{i=1}^m Z_i^0 \leq \sum_{i=1}^m Q_0^{i-1}\, Z_1^0 \tag{22}$$

Since the matrix Q_0 has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \frac{T^2}{4}K_1 + \frac{5T}{6}K_2 + 2K_3 < 1$, the series (22) is uniformly convergent:

$$\lim_{m\to\infty} \sum_{i=1}^m Q_0^{i-1} = \sum_{i=1}^\infty Q_0^{i-1} Z_1^0 = (E - Q_0)^{-1} Z_1^0 \tag{23}$$

The limiting relation (23) signifies a uniform convergence of the sequence $[x_m(t,x_0)$ $,\dot{x}_m(t,x_0), \ddot{x}_m(t,x_0)]$

$$\left.\begin{aligned} \lim_{m\to\infty} x_m(t,x_0) &= x^0(t,x_0) \\ \lim_{m\to\infty} \dot{x}_m(t,x_0) &= \dot{x}^0(t,x_0) \\ \lim_{m\to\infty} \ddot{x}_m(t,x_0) &= \ddot{x}^0(t,x_0) \end{aligned}\right\} \tag{24}$$

By inequality (17), the estimate

$$\begin{pmatrix} \|x^0(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}^0(t,x_0) - \dot{x}_m(t,x_0)\| \\ \|\ddot{x}^0(t,x_0) - \ddot{x}_m(t,x_0)\| \end{pmatrix} \leq Q_0^m (E - Q_0)^{-1} Z_1^0 \tag{25}$$

3. Existence of periodic solution of problem (1).

The problem of the existence of a periodic solution of a period T of (1) is a uniquely connected with the existence of zero of the function $\Delta(t,x_0)$ which has the form:-

$\Delta: D_f \to R^1$

$$\Delta(t,x_0) = \frac{1}{T}\int_0^T f(s,x^0(s,x_0),\dot{x}^0(s,x_0),\ddot{x}^0(s,x_0))ds \qquad (26)$$

Where $x^0(t,x_0)$ is the limiting function of (8) and the equation (26) is an approximation determined from the sequence of functions:

$\Delta_m: D_f \to R^1$

$$\Delta_m(t,x_0) = \frac{1}{T}\int_0^T f\big(s,x_m(s,x_0),\dot{x}_m(s,x_0),\ddot{x}_m(s,x_0)\big)ds \qquad (27)$$

Where $m = 0,1,2,\ldots$

Theorem 3. Under the hypothesis of theorem (1), the following inequality:

$$\|\Delta(t,x_0) - \Delta_m(t,x_0)\| \le d_m$$

Is hold for all ≥ 0. Where $d_m = \langle \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}, Q_0^m (E - Q_0)^{-1} Z_1^0 \rangle$

Proof.

$$\|\Delta(t,x_0) - \Delta_m(t,x_0)\| = \|\frac{1}{T}\int_0^T f(s,x^0(s,x_0),\dot{x}^0(s,x_0),\ddot{x}^0(s,x_0))\,ds - \frac{1}{T}\int_0^T f\big(s,x_m(s,x_0),\dot{x}_m(s,x_0),\ddot{x}_m(s,x_0)\big)ds\|$$

And

$$\|\Delta(t,x_0) - \Delta_m(t,x_0)\| \le \frac{1}{T}\int_0^T (K_1\|x^0(s,x_0) - x_m(s,x_0)\| + K_2\|\dot{x}^0(s,x_0) - \dot{x}_m(s,x_0)\| + K_3\|\ddot{x}^0(s,x_0) - \ddot{x}_m(s,x_0)\|)\,ds$$

$$\|\Delta(t,x_0) - \Delta_m(t,x_0)\| \le K_1\|x^0(s,x_0) - x_m(s,x_0)\| + K_2\|\dot{x}^0(s,x_0) - \dot{x}_m(s,x_0)\| + K_3\|\ddot{x}^0(s,x_0) - \ddot{x}_m(s,x_0)\|$$

Hence

$$\|\Delta(t,x_0) - \Delta_m(t,x_0)\| \le \langle \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}, Q_0^m (E - Q_0)^{-1} Z_1^0 \rangle = d_m$$

Theorem 4. Let a T-system $\frac{dx(t)}{dt} = f\left(t,x,\frac{dx}{dt},\frac{d^2x}{dt^2}\right)$ be defined on an interval $a \le x \le b$ of a straight line R^1

Assume that for a real t and an integral $m \ge 1$ function

$$\Delta_m\,(t,x_0) = \frac{1}{T}\int_0^T f\big(s,x_m(s,x_0),\dot{x}_m(s,x_0),\ddot{x}_m(s,x_0)\big)\,ds$$

Satisfies the inequalities

$\Delta_m(x) \le -d_m$

$\Delta_m(x) \ge d_m$

Where

$$d_m = \langle \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}, Q_0^m (E - Q_0)^{-1} Z_1^0 \rangle \qquad (28)$$

Proof. Let x_1 and x_2 be points of the interval $[a + \frac{T^2}{2} M \le x \le b - \frac{T^2}{2} M]$ such that

$$\left.\begin{array}{l} \Delta_m(x_1) = \Delta_m(x) \\ \Delta_m(x_2) = \Delta_m(x) \end{array}\right\} \qquad (29)$$

From the inequalities (22) and (23), we have

$$\left.\begin{array}{l} \Delta(x_1) = \Delta_m(x_1) + \big(\Delta(x_1) - \Delta_m(x_1)\big) \le 0 \\ \Delta(x_2) = \Delta_m(x_2) + \big(\Delta(x_2) - \Delta_m(x_2)\big) \ge 0 \end{array}\right\} \qquad (30)$$

It follows from (26) in virtue of the continuity of the Δ-constant that there exists a point $x^0, x^0 \in [x_1, x_2]$ such that $\Delta(x^0) = 0$. This means that the system

$\frac{dx(t)}{dt} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)$ has a periodic solution $x = x(t, x_0)$.

Theorem 5. Let the right hand side of the system $\frac{dx(t)}{dt} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)$ be defined in the domain:

$(t, x, \dot{x}, \ddot{x}) \in R^1 \times D \times D_1 \times D_2$ and satisfying the following conditions:

1- the functions $f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)$ is periodic in t of periodic T, bounded, and satisfies a Lipschitz condition with a matrix K

$\|f(t, x, y, z)\| \le M$

$\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \le K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| + K_3\|z_1 - z_2\|$

For all $t \in R^1, x, x_1, x_2 \in D. \ y, y_1, y_2 \in D_1 \ . z, z_1, z_2 \in D_2$, and

2- for any (t, x, y, z) from the domain (2)

$f(-t, x, y) = -f(t, x, y)$

Then all the solutions $x = x(t)$ of the system (1) for which $x(0) \in D_f$ and periodic in t of periodic T.

Proof. Given $x_0 \in D_f$, consider the successive approximation

$$x_{m+1}(t) = x_0 + \int_0^t Lf(s, x_m(s, x_0), y_m(s, x_0), z_m(s, x_0))\, ds$$

$m = 0,1,2, \dots \dots$

Since $f(t,x_0)$ is an odd function $\underline{f(t,x_0)} = 0$

Hence

$$x_1(t) = x_0 + \int_0^t \left(Lf\left(s,x_0\right) - \underline{Lf(s,x_0)}\right) ds = x_1(t+T)$$

i.e.

The function $x_1(t)$ is periodic of T in t. Moreover,

$$\|x_1\,(t) - x_o\| \leq \frac{T^2}{2} M$$

i.e.

The function $x_1(t) \in D$. Finally,

$x_1(t) = x_1(-t)$ as the integral of an odd function

$$\|\Delta_m(t,x_1)\| \leq \|\Delta(t,x_1)\| + \langle \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}, Q_0^m (E - Q_0)^{-1} Z_1^0 \rangle$$

4. Existence and uniqueness of periodic solution of (1).

In this part, we prove the existence and uniqueness theorem for (1) by using Banach fixed point theorem.

Theorem 6. Let the vector functions $f(t,x,y,z)$ be defined and continuous on the domain (2) and satisfy the assumptions and conditions of theorem (2), then the problem (1) will have a unique periodic continuous solution on the domain (2).

Proof: Let $(C[0,T], \|.\|)$ Be a Banach space and T^* be a mapping on $C[0,T]$ as follows:

$$T^* x(t,x_0) = x_0 + L^2 f(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0))$$

And

$$T^* \dot{x}(t,x_0) = \dot{x}_0 + Lf(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)) - \underline{Lf\left(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)\right)}$$

And

$$T^* \ddot{x}(t,x_0) = \ddot{x}_0 + f(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)) - \underline{f\left(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)\right)}$$

Since

$$\int_0^t \left(Lf(s, x(s,x_0), \dot{x}(s,x_0), \ddot{x}(t,x_0)) - \underline{Lf(s, x(s,x_0), \dot{x}(s,x_0), \ddot{x}(t,x_0))}\right) ds$$

is continuous on the domain (2).

And also

$$Lf(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)) - Lf\left(t, x(t,x_0), \dot{x}(t,x_0), \ddot{x}(t,x_0)\right) \;,$$

$$f(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)) - \underline{f(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0))}$$

are continuous on the same domain (2).

Therefore: $T^*: C[0,T] \longrightarrow C[0,T]$

Now, we shall to prove that T^* is a contraction mapping on $[0,T]$.

Let $x(t,x_0), z(t,x_0)$ be a vector function on $[0,T]$, then

$$\|T^*x(t,x_0) - T^*z(t,x_0)\| = \max_{t\in[0,T]}\{|T^*x(t,x_0) - T^*z(t,x_0)|\}$$

And

$$\|T^*\dot{x}(t,x_0) - T^*\dot{z}(t,x_0)\| = \max_{t\in[0,T]}\{|T^*\dot{x}(t,x_0) - T^*\dot{z}(t,x_0)|\}$$

And

$$\|T^*\ddot{x}(t,x_0) - T^*\ddot{z}(t,x_0)\| = \max_{t\in[0,T]}\{|T^*\ddot{x}(t,x_0) - T^*\ddot{z}(t,x_0)|\}$$

$$\begin{pmatrix} \|T^*x(t,x_0) - T^*z(t,x_0)\| \\ \|T^*\dot{x}(t,x_0) - T^*\dot{z}(t,x_0)\| \\ \|T^*\ddot{x}(t,x_0) - T^*\ddot{z}(t,x_0)\| \end{pmatrix} =$$

$$\begin{pmatrix} \max\limits_{t\in[0,T]}\left\{\left|x_0 + \int_0^t L\left(f\left(s,x(s,x_0),\dot{x}(s,x_0),\ddot{x}(t,x_0)\right) - \underline{f\left(s,x(s,x_0),\dot{x}(s,x_0),\ddot{x}(t,x_0)\right)}\right)ds - x_0 - \int_0^t L\left(f\left(s,z(s,x_0),\dot{z}(s,x_0),\ddot{z}(t,x_0)\right) - \underline{f\left(s,z(s,x_0),\dot{z}(s,x_0),\ddot{z}(t,x_0)\right)}\right)ds\right|\right\} \\ \max\limits_{t\in[0,T]}\left\{\left|\dot{x}_0 + Lf\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right) - \underline{Lf\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right)} - \dot{x}_0 - Lf\left(t,z(t,x_0),\dot{z}(t,x_0),\ddot{z}(t,x_0)\right) - \underline{Lf\left(t,z(t,x_0),\dot{z}(t,x_0),\ddot{z}(t,x_0)\right)}\right|\right\} \\ \max\limits_{t\in[0,T]}\left\{\left|\ddot{x}_0 + f\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right) - \underline{f\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right)} - \ddot{x}_0 - f(t,z(t,x_0),\dot{z}(t,x_0),\ddot{z}(t,x_0)) - \underline{f(t,z(t,x_0),\dot{z}(t,x_0),\ddot{z}(t,x_0))}\right|\right\} \end{pmatrix}$$

And

$$\begin{pmatrix} \|T^*x(t,x_0) - T^*z(t,x_0)\| \\ \|T^*\dot{x}(t,x_0) - T^*\dot{z}(t,x_0)\| \\ \|T^*\ddot{x}(t,x_0) - T^*\ddot{z}(t,x_0)\| \end{pmatrix} \leq$$

$$\begin{pmatrix} \alpha(t)\frac{T}{2}\left\|f\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right) - f\left(t,z(t,x_0),\dot{z}(t,x_0),\ddot{z}(t,x_0)\right)\right\| \\ \alpha(t) + \frac{T}{3}\left\|f\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right) - f\left(t,z(t,x_0),\dot{z}(t,x_0),\ddot{z}(t,x_0)\right)\right\| \\ 2\|f\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right) - f\left(t,z(t,x_0),\dot{z}(t,x_0),\ddot{z}(t,x_0)\right)\| \end{pmatrix}$$

And

$$\begin{pmatrix} \|T^*x(t,x_0) - T^*z(t,x_0)\| \\ \|T^*\dot{x}(t,x_0) - T^*\dot{z}(t,x_0)\| \\ \|T^*\ddot{x}(t,x_0) - T^*\ddot{z}(t,x_0)\| \end{pmatrix} \leq$$

$$\begin{pmatrix} \alpha(t)\frac{T}{2}(K_1\|x(t,x_0) - z(t,x_0)\| + K_2\|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| + K_3\|\ddot{x}(t,x_0) - \ddot{z}(t,x_0)\|) \\ \alpha(t) + \frac{T}{3}(K_1\|x(t,x_0) - z(t,x_0)\| + K_2\|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| + K_3\|\ddot{x}(t,x_0) - \ddot{z}(t,x_0)\|) \\ 2(K_1\|x(t,x_0) - z(t,x_0)\| + K_2\|\dot{x}(t,x_0) - \dot{z}(t,x_0)\| + K_3\|\ddot{x}(t,x_0) - \ddot{z}(t,x_0)\|) \end{pmatrix}$$

And

$$\begin{pmatrix} \|T^*x(t,x_0)-T^*z(t,x_0)\| \\ \|T^*\dot{x}(t,x_0)-T^*\dot{z}(t,x_0)\| \\ \|T^*\ddot{x}(t,x_0)-T^*\ddot{z}(t,x_0)\| \end{pmatrix} \leq$$

$$\begin{pmatrix} \alpha(t)\frac{T}{2}K_1 & \alpha(t)\frac{T}{2}\ K_2 & \alpha(t)\frac{T}{2}K_3 \\ \left(\alpha(t)+\frac{T}{3}\right)K_1 & \left(\alpha(t)+\frac{T}{3}\right)K_2 & \left(\alpha(t)+\frac{T}{3}\right)K_3 \\ 2K_1 & 2K_2 & 2K_3 \end{pmatrix} \begin{pmatrix} \|x(t,x_0)-z(t,x_0)\| \\ \|\dot{x}(t,x_0)-\dot{z}(t,x_0)\| \\ \|\ddot{x}(t,x_0)-\ddot{z}(t,x_0)\| \end{pmatrix}$$

By the condition $\lambda_{max}\Lambda < 1$, then T^* is a contraction mapping.

Thus by Banach fixed point theorem then there exists a fixed point $(t,x_0),\dot{x}(t,x_0)$, in $C[0,T]$ such that

$$T^*x(t,x_0) = x(t,x_0)$$

$$T^*\dot{x}(t,x_0) = \dot{x}(t,x_0)$$

$$T^*\ddot{x}(t,x_0) = \ddot{x}(t,x_0)$$

Therefore,

$$x(t,x_0) = x_0 + L^2 f(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0))$$

And

$$\dot{x}(t,x_0) = \dot{x}_0 + Lf(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)) - \underline{Lf\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)}$$

And

$$\ddot{x}(t,x_0) = \ddot{x}_0 + f(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)) - \underline{f\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)} \quad .$$

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