Operators Preserving *K-g-* Frames

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Abstract

K-frames were recently introduced by Găvruţa in Hilbert spaces to study atomic systems with respect to a bounded linear operator. K-g-frames are more general than of g-frames in Hilbert spaces.Results on k-g- frames have been proved through operator- theoretic results of bounded operators.

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1. Introduction

Frames for Hilbert spaces were introduced by R.J. Duffin and A.C. Schaeffer in 1952, while discussing some problems in the theory of non-harmonic Fourier series. Frame theory was developed by Peter G. Casazzaand O.Christensen [6,7].A.Najati and A. Rahimi [1] have developed the generalized frame theory and introduced methods for generating g-frames of a Hilbert space.

The notion of K-frames has been introduced by L.Gavruta [5] to study yhe atomic systems with respect to a bounded linear operator K in Hilbert space H. K-frames are more more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K.Dingli Hua and Yongdong Huang [2] are proposed for construction methods for K-g-frames.Results on K-frames have been proved through operator-theoritic results on quotient of bounded operators byG. Ramu and P.Johnson[4].

In this paper some results on k-g- frames have been proved through operator- theoretic results of bounded operators. Some results on K-g frames are studied by GU Reddy [8] and $(K_1 \otimes K_2)$-g frame for the tensor product of Hilbert space $H_1 \otimes H_2$ is introduced and some results on it are established.

2. Notations and Preliminaries

The basics of frame theory and related topics, we refer to the book by Christian [6]. Here we recall a few basic definition and results needed in the sequel[1,5 and 7].

Definition2.1: a family $\{f_i\}_{i=1}^{\infty}$ of vectors in H is called a Bessel sequence if there exists a constant A>0 such that $\sum_{j\in J} \left|\langle f, f_j \rangle\right|^2 \le A\|f\|^2 \ \forall f \in H$

Definition 2.2 A sequence $\{f_j\}_{j\in J}$ of vectors in a Hilbert space H is called a frame if there exist two constants $0 < \mathrm{A} \le \mathrm{B} < \infty$, such that

$$A\|f\|^2 \le \sum_{j\in J}\left|\langle f, f_j\rangle\right|^2 \le B\|f\|^2 \;\; \forall\, f \in H$$

The above inequality is called a frame inequality. The numbers A and B are called the lower and upper frame bounds respectively. If A=B then $\{f_j\}_{j\in J}$ is called tight frame, if A=B=1 then $\{f_j\}_{j\in J}$ is called normalized tight frame. A synthesis operator T:$l_2 \to$ H is defined as $Te_j = f_j$ where $\{e_j\}$ is an orthonormal basis for l_2. The analysis operator T^* : H $\to l_2$ is an adjoint of synthesis operator T and is defined as $T^* f = \sum_{j\in J} < f, f_j > e_j \;\; \forall\, f \in H$. A frame operator $S = TT^* : H \to H$ is defined as $Sf = \sum_j < f, f_j > f_j \;\; \forall f \in H$

Throughout this paper $\{H_j, j \in J\}$ will denote a sequence of Hilbert spaces. Let $L(H, H_j)$ be a collection all bounded linear operators from H to H_j and $\{\Lambda_j \in L(H, H_j) : j \in J\}$.

Definition 2.3. A sequence of operators $\{\Lambda_j\}_{j\in J}$ is said to be g-frame for Hilbert space H with respect to sequence of Hilbert spaces $\{H_j, j \in J\}$, if there exist two constants $0 < A \le B < \infty$, such that $A\|f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le B\|f\|^2 \;\; \forall\, f \in H$.

The above inequality is called a g-frame inequality. The numbers A and B are called the lower frame bound and upper frame bound respectively. A g-frame $\{\Lambda_j\}_{j\in J}$ for H is said to be g-tight frame if A = B and g-normalized tight frame for H if A = B = 1.

Definition 2.4. Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for Hilbert space H. A g-frame operator S: H $\to$H is defined as $Sf = \sum_{j\in J}\Lambda_j^*\Lambda_j f \;\; \forall f \in H$.

By using above definitions, the following theorem on g-frame operator can be derived easily, so left to reader.

***Theorem* 2.5.** If S is a g- frame operator, then we have

(i)< Sf, f> = $\sum_{j\in J}\|\Lambda_j f\|^2$, for all f $\in$ H.

(ii) S is a positive operator.

(iii) S is a self-adjoint operator.

***Theorem* 2.6.**(Douglas' factorization theorem)[3]. Let H be a Hilbert space and $A, B \in \mathcal{B}(H)$. Then the following are equivalent:

1. $R(A) \subseteq R(B)$.
2. $AA^* \le \alpha^2 BB^*$ for some $\alpha > 0$.
3. $A = BX$ for some $X \in \mathcal{B}(H)$.

***Theorem* 2.7[3].**Let $A, B, C \in \mathcal{B}(H)$. Then the following are equivalent:

(i)$R(A) \subseteq R(B) + R(C)$.

(ii)$AA^* \le \alpha^2(BB^* + CC^*)$ for some $\alpha > 0$.

(iii)$A = BX + CY$ for some $X, Y \in \mathcal{B}(H)$.

3. Operator frames

Definition 3.1. Let $K \in B(H)$. A sequence $\{f_j\}_{j\in J}$ in Hilbert space H is said to be a K-frame for H if there existtwo constants $0 < A \leq B < \infty$, such that $A\|K^* f\|^2 \leq \sum_{j\in J} |< f, f_j >|^2 \leq B\|f\|^2, \ \forall f \in H$.

Where A and B are called lower and upper frame bounds for k-frame respectively. If K=I, then K-frames are just ordinary frames.

Definition 3.2: Let $\{f_j\}_{j\in J}$is a K- frame for H. Obviously it is a Bessel sequence, so we can define the following operator $T: l^2 \to H$ by

$T(c_j) = \sum_{j} c_j f_j \qquad \forall\{c_j\} \in l^2$

is called Synthesis operator for K- frame$\{f_j\}_{j\in J}$. Also, we have

$T^*: H \to l^2$by$T^*(f) = \{< f, f_j >\}_{j\in J} \in l^2$ is called Analysis operator for K- frame$\{f_j\}_{j\in J}$.

The frame operator is given by S^k : H →H is defined as S^k f = $\sum_{j\in J} \langle f, f_j \rangle f_j$, for all f∈ H.

Definition3.3. Let $K \in L(H)$ and $\Lambda_j \in L(H, H_j)$ $_{j\in J}$. A sequence of operators $\{\Lambda_j\}_{j\in J}$ is said to be K-g-frame for Hilbert space H with respect to sequence of Hilbert spaces $\{H_j\}_{j\in J}$ if there exist two constants $0 < A \leq B < \infty$, such that

$A\|K^* f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \ \forall \ f \in H.$.

The above inequality is called a K-g-frame inequality. The numbers A and B are called the lower and upper frame bounds of K-g-frame respectively. When K=I, K-g-frame is a g-frame.

A k-g- frame is said to be tight if there exist a positive constant A such that

$\sum_{j\in J} \|\Lambda_j f\|^2 = A\|K^* f\|^2, \ \forall \ f \in H.$

If A=1 then $\{\Lambda_j\}_{j\in J}$ is said to be parseval tight k-g-frame.

Definition3.4. Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for H. A synthesis operator $T : l^2\left(\{H_j\}_{j\in J}\right) \to H$ is defined as $T\left(\{g_j\}_{j\in J}\right) = \sum_{j\in J} \Lambda_j^* g_j \qquad \forall \{g_j\}_{j\in J} \in l^2\left(\{H_j\}_{j\in J}\right)$.

Definition3.5. Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for H.The analysis operator $T^* \ H \to: l^2\left(\{H_j\}_{j\in J}\right)$ is the adjoint of synthesis operator T and is defined as $T^* f = \{\Lambda_j f\}_{j\in J} \ \forall f \in H$

Definition 3.6. Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for Hilbert space H. A K- g-frame operator $S: H \to H$ is defined as $S f = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \qquad \forall f \in H$.

Note that $\langle Sf, f \rangle = \sum_{j\in J} \|\Lambda_j f\|^2$.

***Proposition* 3.7[5]**. Let $\{f_j\}_{j=1}^{\infty}$ be a Bessel sequence in H. Then $\{f_j\}_{j=1}^{\infty}$ is a K -frame for H if and only if there exists constant $A > 0$ such that $S \geq AKK^*$, where S is the frame operator for $\{f_j\}_{j=1}^{\infty}$

Theorem 3.8[2].If $K \in L(H)$ and $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for Hilbert space H with respect to $\{H_j\}_{j\in J}$ then $S \geq AKK^*$.

Proof. Suppose $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H

$$\Rightarrow \qquad A\|K^* f\|^2 \leq \sum_{j\in J}\|\Lambda_j f\|^2 \leq B\|f\|^2,\ \forall\ f \in H.$$

$$\Rightarrow \qquad A\langle K^* f, K^* f\rangle \leq \langle Sf, f\rangle\ \forall\ f \in H$$

$$\Rightarrow \qquad \langle AKK^* f, f\rangle \leq \langle Sf, f\rangle\ \forall\ f \in H$$

$$\Rightarrow \qquad S \geq AKK^*.$$

□

4. Operators preserving K-g- Frames

Results on K-frames have been proved through operator-theoretic results on quotient of bounded operators byG. Ramu and P.Johnson[4]. In this section theorem 3.4, 3.8 and proposition 3.3 were discussed in [4] are extended to K-g- frames.

***Proposition*4.1:** Let $\{\Lambda_j\}_{j\in J}$be a K-g- frame for H. Let $T \in \mathcal{B}(H)$ with $R(T) \subseteq R(K)$ then $\{\Lambda_j\}_{j\in J}$is a T-g- frame for H.

***Proof*:**Suppose $\{\Lambda_j\}_{j=1}^{\infty}$ is a K -g-frame for H. Then there existtwo positive constants λ and μ such that

$$\lambda\|K^* f\|^2 \leq \sum_{j=1}^{\infty}\|\Lambda_j f\|^2 \leq \mu\|f\|^2\ \forall f \in H \quad \ldots(1)$$

Since $R(T) \subset R(K)$, by Douglas' factorization theorem, there exists $\alpha > 0$ such that $TT^* \leq \alpha^2 KK^*$then $\forall f \in H$we have

$$< TT^* f, f > \leq < \alpha^2 KK^* f, f > \forall f \in H$$

$$\Rightarrow < TT^* f, f > \leq \alpha^2 < KK^* f, f > \forall f \in H$$

$$\Rightarrow < T^* f, T^* f > \leq \alpha^2 < K^* f, K^* f >$$

$$\Rightarrow \|T^* f\|^2 \leq \alpha^2 \|K^* f\|^2$$

$$\Rightarrow \frac{1}{\alpha^2}\|T^* f\|^2 \leq \|K^* f\|^2$$

$$\Rightarrow \frac{\lambda}{\alpha^2}\|T^* f\|^2 \leq \lambda\|K^* f\|^2 \quad \text{since } \lambda > 0$$

$$\Rightarrow \leq \sum_{j=1}^{\infty} \left\| \Lambda_j f \right\|^2$$

$$\Rightarrow \leq \mu \| f \|^2 \textit{ by (1)}$$

$$\Rightarrow \frac{\lambda}{\alpha^2} \left\| T^* f \right\|^2 \leq \sum_{j=1}^{\infty} \left\| \Lambda_j f \right\|^2 \leq \mu \| f \|^2 \text{ for all } f \in H$$

Hence $\{\Lambda_j\}_{j \in J}$ is a T-g- frame for H

Theorem4.2: Let $K \in \mathcal{B}(H)$be with a dense range. Let $\{\Lambda_j\}_{j=1}^{\infty}$ be a K -g-frame and $T \in \mathcal{B}(H)$ have closed range. If $\{T\Lambda_j\}_{j=1}^{\infty}$ and $\{T^*\Lambda_j\}_{j=1}^{\infty}$are a K -g-frames for H, then TT^* is invertible.

Proof*:* Suppose $\{T\Lambda_j\}_{j=1}^{\infty}$ is a K -g-frame for H with frame bounds λ and μ. Then for any $f \in H$ we have

$$\lambda \left\| K^* f \right\|^2 \leq \sum_{j=1}^{\infty} \left\| T\Lambda_j f \right\|^2 \leq \mu \| f \|^2$$

$$\Rightarrow \lambda \left\| K^* f \right\|^2 \leq \sum_{j=1}^{\infty} \left\langle T\Lambda_j f, T\Lambda_j f \right\rangle \leq \mu \| f \|^2 \ \forall f \in H$$

$$\Rightarrow \lambda \left\| K^* f \right\|^2 \leq \sum_{j=1}^{\infty} \left\langle T^* T\Lambda_j f, \Lambda_j f \right\rangle \leq \mu \| f \|^2 \ \forall f \in H \quad \ldots(1)$$

As K is with a dense range, K^* is injective. Then from (1), T^*T is injective since $N(T^*T) \subset N(K^*)$. Moreover, $R(TT^*) = N(T^*T)^{\perp} = H$. Thus T^*T is surjective.

Suppose $\{T^*\Lambda_j\}_{j=1}^{\infty}$ is a K-g- frame for H with bounds α and β. Then for any $f \in H$

$$\alpha \left\| K^* f \right\|^2 \leq \sum_{j=1}^{\infty} \left\| T^* \Lambda_j f \right\|^2 \leq \beta \| f \|^2 \ \forall f \in H$$

$$\Rightarrow \alpha \left\| K^* f \right\|^2 \leq \sum_{j=1}^{\infty} \left\langle TT^* \Lambda_j f, \Lambda_j f \right\rangle \leq \beta \| f \|^2 \ \forall f \in H \quad \ldots(2)$$

As K has a dense range, K^* is injective. Then from (2) TT^* is injective since

$$N(TT^*) \subset N(K^*).$$

$\Longrightarrow T^*T$ is bijective. By bounded inverse theorem T^*T invertible.

Theorem 4.3: Let $K \in \mathcal{B}(H)$ and let $\{\Lambda_j\}_{j=1}^{\infty}$ be a K -g-frame for H. And let $T \in \mathcal{B}(H)$be isometry then $\{T\Lambda_j\}_{j=1}^{\infty}$is a K-g-frame for H.

Proof: **S**uppose $\{\Lambda_j\}_{j=1}^{\infty}$is a K-g- frame for H. then for each $f \in H$ we have

$$\lambda \left\| K^* f \right\|^2 \leq \sum_{j=1}^{\infty} \left\| \Lambda_j f \right\|^2 \leq \mu \| f \|^2 \ldots (1)$$

Consider $\sum_{j=1}^{\infty}\|T\Lambda_j f\|^2 = \sum_{j=1}^{\infty}\|\Lambda_j f\|^2$ since T is isometry

$\geq \lambda\|K^* f\|^2$ ….(2)

Consider $\sum_{j=1}^{\infty}\|T\Lambda_j f\|^2 = \sum_{j=1}^{\infty}\|\Lambda_j f\|^2$

$\leq \mu\|f\|^2$ ….(3)

From (2) and (3) we have

$\lambda\|K^* f\|^2 \leq \sum_{j=1}^{\infty}\|T\Lambda_j f\|^2 \leq \mu\|f\|^2 \quad f \in H$

Which shows that $\{T\Lambda_{j_i}\}_{i=1}^{\infty}$ is a K-g-frame for H.

Let $\{f_j\}_{j=1}^{\infty}$ be a K-frame for H with the frame operator S and let A be a positive operator then $\{f_j + Af_j\}_{j=1}^{\infty}$ is K-frame for H have been discussed in [4]. We extend this result to K-g frames in the fallowing Proposition.

***Proposition 4.4*:** Let $\{\Lambda_j\}_{j=1}^{\infty}$ be a K-g- frame for H with frame operator S and let A be a positive operator which commutes with Λ_j for every j then
$\{\Lambda_j + A\Lambda_j\}_{j=1}^{\infty}$ is a K-g- frame for H.

Proof: Suppose $\{\Lambda_j\}_{j=1}^{\infty}$ is K-g-frame for H. then by the theorem 3.8, there exists $\alpha > 0$ such that $S \geq \alpha KK^*$

For each $f \in H$ consider

$$\sum_{j=1}^{\infty}(\Lambda_j + A\Lambda_j)^*(\Lambda_j + A\Lambda_j)f = \sum_{j=1}^{\infty}((I+A)\Lambda_j)^*(I+A)\Lambda_j f$$

$$= \sum_{j=1}^{\infty}\Lambda^*_j(I+A)^*(I+A)\Lambda_j f$$

$$= (I+A)^*(I+A)\sum_{j=1}^{\infty}\Lambda^*_j\Lambda_j f$$

$$= (I+A)^*(I+A)Sf$$ by the definition of frame operator

$$\geq S$$

$$\geq \alpha KK^*$$

Hence by the theorem 3.7 we can conclude that $\{\Lambda_j + A\Lambda_j\}_{j=1}^{\infty}$ is a K-g- frame for H.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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We declare that this manuscript is original, has not been published before and is not currently being considered for publication elsewhere.

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