A Markov Decision Process Using In Stochastic Service System

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Abstract: A service facility modeled as a queueing system with finite or infinite capacity. Arriving customers enter if there is room in the facility and if they are willing to pay the price posted by the service provider. Customers belong to one of a finite number of classes that have different willingnesses-to-pay. Moreover, there is a penalty for congestion in the facility in the form of state-dependent holding costs. The service provider may advertise class-specific prices that may fluctuate over time. The existence of a unique optimal stationary pricing policy in a continuous and unbounded action space that maximizes the long-run average profit per unit time. To determine an expression for this policy under certain conditions and also analyze the structure and the properties of this policy.

Keywords: Dynamic pricing; queueing; Markov decision process; revenue management
MSC Code : 90C40

1. Introduction

A decision maker wishes to quote prices at the most profitable level. When a customer arrives, the customer decides whether to pay the quoted price and enter the service system or to depart without obtaining service[1]. Rather than being restricted to a single price that is offered to all customers, the decision maker has a great deal of flexibility in setting prices[3]. The decision maker is allowed to use two pieces of information when making a quotation. The decision maker knows the number of customers currently in the service system, which is a measure of the congestion in the system[5]. Allowing the price to depend upon the level of congestion will be called *congestion-dependent pricing*. In addition, the decision maker is able to classify the customers into different types, and the decision maker knows the probability of a customer of a particular type accepting a particular price. Allowing the price to depend upon information about a customer will be referred to as *precision pricing*.

Thus, the decision maker can use a congestion dependent, precision pricing strategy. If all customers are classified as the same type then the decision maker uses only congestion-dependent pricing[7]. If the decision maker is not allowed to use information about the current level of congestion when setting prices then the decision maker will be using *static pricing*.

We assume that the probability of a particular type of customer accepting a price does not increase as the price increases. Already the decision maker faces a trade-off. If prices are high, each customer pays a lot, but few customers pay; if prices are low, each customer pays a little, but many pay[9]. If there is no limit on the number of customers that can be in the service system simultaneously and the decision maker has no reason to keep the number of customers in the system at a low level, then the most profitable prices could easily result in a large average number of customers in the system. In most applications this would be unacceptable[11]. To give the decision maker an incentive to reduce congestion, we assume that the decision maker incurs a cost at rate h_s when there are s customers in the service system. These costs will be nonnegative and no decreasing in s.

2. Mathematical Model

We model the service facility as a queueing system of capacity $N \leq \infty$; that is, no more than N customers are allowed in the system at any time. There are I classes of customers and customers from class $i = 1, \ldots, I$ arrive according to a Poisson process with parameter $\lambda_i > 0$. The arrival processes from customer classes are independent of each other. This formulation is equivalent to having arriving customers randomly assigned to a specific class, independent of everything else[13].

The maximum amounts that successive class $i \ 1, \ldots, I$ customers are willing to pay are independent, identically distributed random variables with distribution Fi. The amount a class $i = 1, \ldots, I$ customer is willing to pay is independent of the amount a class $\mathbf{j} = 1, \ldots, I$ customer is willing to pay for $i \neq j$. For all $i = 1, \ldots, I$, we assume that the cumulative distribution function $F_i(.)$ is absolutely continuous with density $f_i(.)$ support (α_i, β_i) and finite mean. Let $r_i(.)$ denote the hazard rate function of $F_i(.)$

$$r_i(z) = \frac{f_i(z)}{1-f_i(z)} \quad \text{for } \alpha_i < z < \beta_i \qquad \ldots\ldots\ (1)$$

We assume that F_i has **Increasing Generalized Hazard Rate (IGHR)** that is, $zr_i(z)$ is strictly increasing for all z in $[\alpha_i, \beta_i]$. we can interpret $zr_i(z)$ as the price elasticity of the demand function for class-i customers[15]. The service provider can advertise different prices to different classes. Without loss of generality, only prices in $[\alpha_i, \beta_i]$ can be advertised to class-i customers.

3. **An Optimal Stationary Policy**

We use a **Markov Decision Process (MDP)** approach to exhibit an optimal stationary policy. The MDP associated with our system behaves as abirth–death process. Since the death rates are strictly positive, the MDP is unichain for any stationary policy. We set up the system of **Average-Cost Optimality Equations (ACOE)** as

$$l(-1) = 0.$$

$$l(s) = \frac{\sup}{z_0, \ldots, z_1} \left\{ \frac{\sum_{i=1}^{I} \lambda_i(z_i)(z_i + 1(s+1) + \mu_s l(s-1) - g - h_s}{\sum_{i=1}^{I} \lambda_i(z_i) + \mu_s} \right\} \quad 0 \leq s \leq N-1.$$

$$l(N) = l(N-1) - \frac{g + h_N}{\mu_s} \qquad \ldots\ldots\ (2)$$

where g is the gain and $l(\cdot)$ is the bias vector. Since the value of μ_0 does not matter as long as it is positive, we will consider $\mu_0 = \mu_1$ without loss of generality[17]. In this system we are solving for g and $l(\cdot)$. We can transform these equations into a simpler equivalent form by letting $G(-1) = 0$ and $G(s) = l(s) - l(s+1)$ for $s = 0, \ldots, N-1$. Then

$$G(-1) = 0 \qquad \ldots\ldots(3)$$

$$g + h_i - \mu_i G(s-1) = \sum_{i=1}^{I} \frac{\sup}{z} \{ z - G(s)) \lambda_i(z) \} \quad \text{if } s = 0, \ldots, N-1, \qquad \ldots\ldots(4)$$

$$G(N-1) = \frac{g + h_N}{\mu_N} \quad \ldots\ldots (5)$$

If a solution (g,.G(.), **z**) to the system of ACOE exists, we call it a canonical triplet, where **z** are prices that achieve the supreme in (4). Precisely, for s = 0,...,N-1 and i =1,...,I, the component $z_{i,s}$ of **z** satisfies $z_{i,s} = \arg\sup \{(z - G(s))\lambda_i(z)\}$.

Example : 3.1

We assume that the probability of a particular type of customer accepting a price does not increase as the price increases. Already the decision maker faces a trade-off[19] .

Let x-axis considered as h_N

y-axis considered as μ_N and g –different values

G (N-1) = $\frac{g+h_N}{\mu_N}$ (using normal distribution)

Therefore, G(s) = $\frac{g+h_N}{\mu_N}$ (where S =0,1,….(N-1)

If price are high, each customers pays a lot, but few customers pay

Table 3.1(a)

X **Price are High (inmillions)**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Y **Customers Pays a Lot**	**3.48**	**3.51**	**3.53**	**3.56**	**3.58**	**3.60**	**3.63**	**3.65**	**3.68**	**3.70**	**3.73**

Graph 3.1(a)

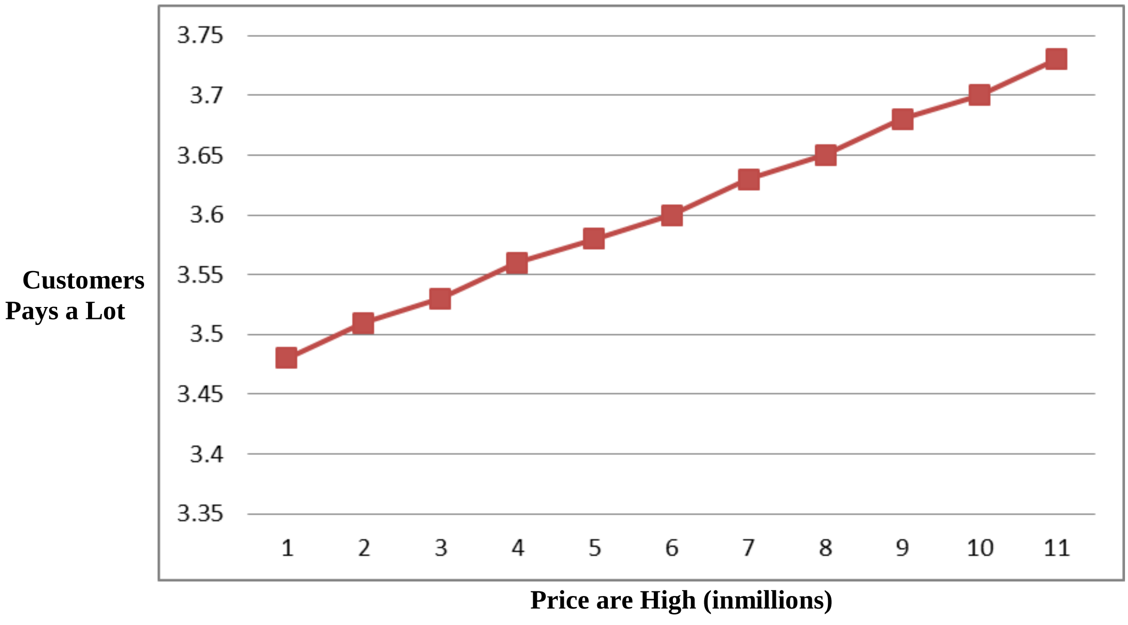
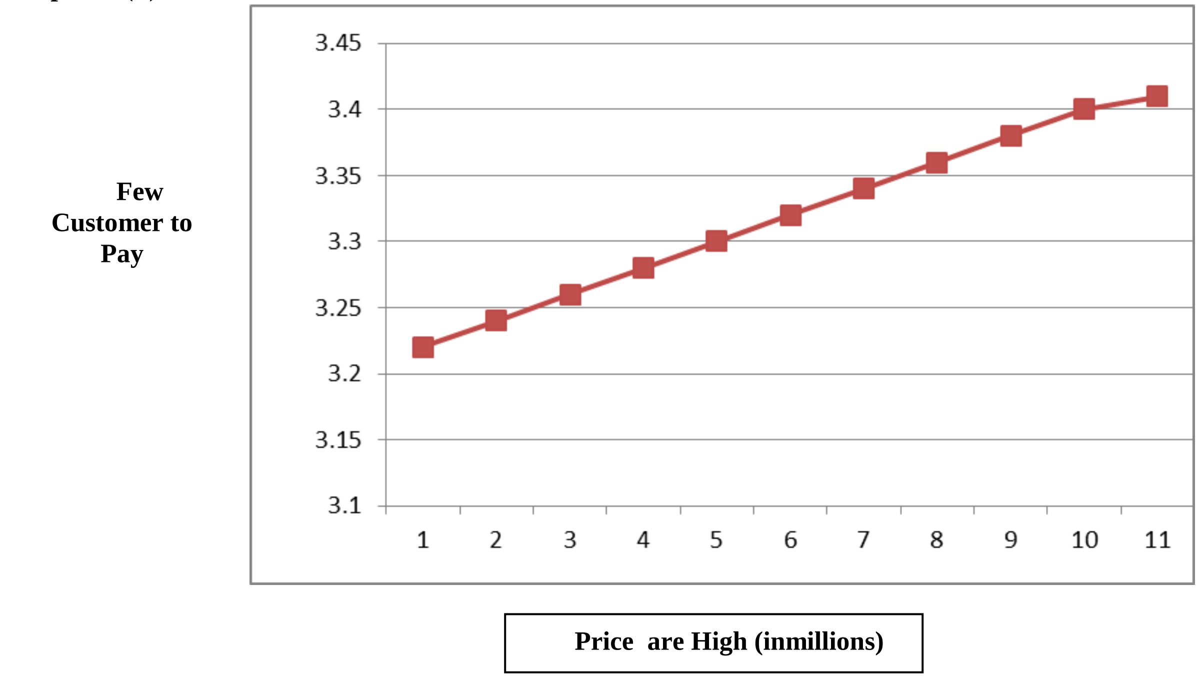


Table 3.1(b)

X Price are High (inmillions)	1	2	3	4	5	6	7	8	9	10	11
Y **Few customers to Pays**	**3.22**	**3.24**	**3.26**	**3.28**	**3.30**	**3.32**	**3.34**	**3.36**	**3.38**	**3.40**	**3.41**

Graph 3.1(b)



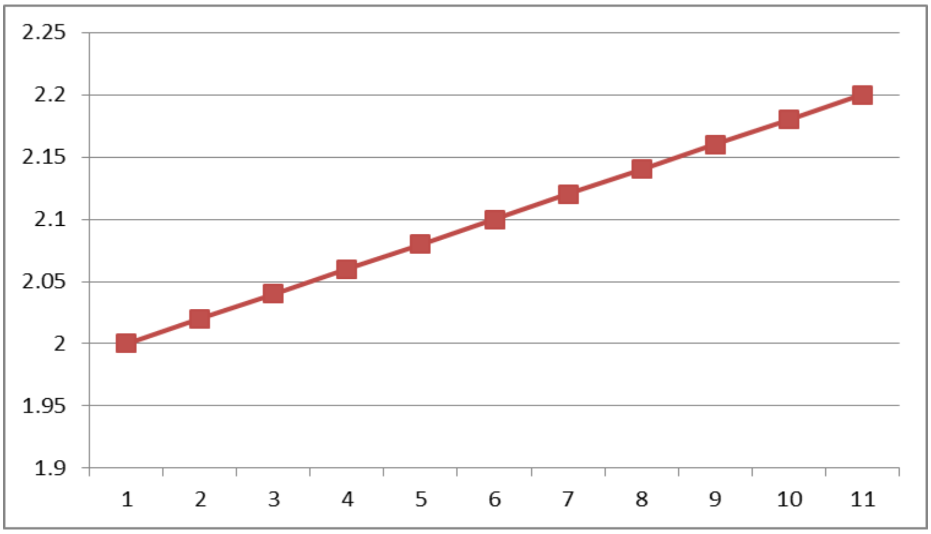
If prices are low each customers pays a little , but many Customers to pay

Table 3.2(c)

X Price are Low (inmillions)	1	2	3	4	5	6	7	8	9	10	11
Y **Customers Pays a Little**	**2.00**	**2.02**	**2.04**	**2.06**	**2.08**	**2.10**	**2.12**	**2.14**	**2.16**	**2.18**	**2.20**

Graph 3.2(c)

Many customers to pay



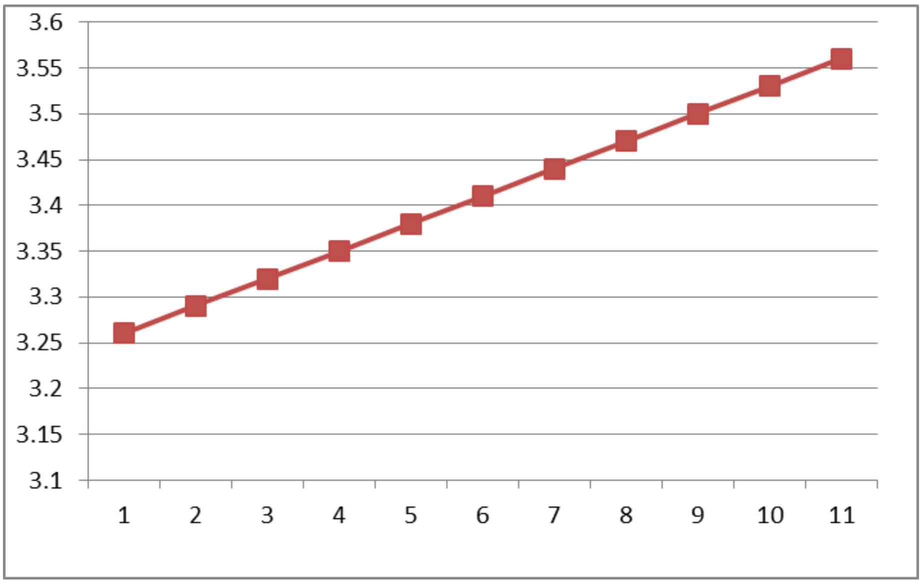
Prices are Low (in millions)

Table 3.2(d)

X Price are Low (inmillions)	1	2	3	4	5	6	7	8	9	10	11
Y Many Customers to Pay	3.26	3.29	3.32	3.35	3.38	3.41	3.44	3.47	3.50	3.53	3.56

Graph 3.2(d)

Many customers to pay



Prices are Low (in millions)

Theorem 3.1. There exists a canonical triplet (g, G(·), **z**) for the system of ACOE (3)–(5). Moreover, the optimal long-run average reward is $g_N^* = g$, and $z^* = z$ is a unique optimal stationary policy, where, for $s = 0,\ldots,N-1$ and $i = 1,\ldots,I$,

$$g_{i,s}^* = \inf\{z : r_i(z)\,(z - G(s)) \geq 1\}$$

Lemma 3.1. For all $s = -1,\ldots,N-1$, $G(s,\cdot)$ is nondecreasing and continuous[2]. Moreover, there exists $g \geq 0$ such that $G(-1, g) = 0$.

Proof. Let $G(N-1, g) = (g + h_N)\,\mu_N$ is continuous and nondecreasing in g. Suppose that G(s, g) is nondecreasing and continuous in g for some state s between 0 and N - 1. As $\sup\{\lambda_i(z)\,(z-G(s,g))\}$ is the supremum of a bounded continuous function of z, we can claim that

$$\mu_s G(s-1,g) = g - \sum_{i=1}^{I} \sup\{\lambda_i(z)\,(z - G(s)\}$$

is continuous and nondecreasing in *g*. By induction, for all $s = -1,\ldots,N-1$ $G(s,\cdot)$ is nondecreasing and continuous[4].

Hence the lemma

Lemma 3.2. Let (g,G(·),**z**) be a canonical triplet. Then, for all $s = -1,\ldots,N-1$, $0 \leq G(s) \leq (g + h_s + 1) / \mu_s + 1$.

Proof. For all $s = 0,\ldots, N-1$, $\sup\{(z-G(s))\,\lambda_i(z)\} \geq 0$. Therefore, we have $G(s-1) \leq (g + h_s)/\mu_s$ from (4). Using (3) and (5) as well, $G(s) \leq (g+h_{s-1})/\mu_{s+1}$ for all $s = -1,\ldots,N-1$.

Now suppose that there exists $s = 0,\ldots,N-1$ such that $G(s) < 0$. Since $G(-1) \geq 0$, there exists s such that $G(s) < 0$ and $G(s-1) \geq 0$. Hence, $\mu_{s+1} G(s) < \mu_s G(s-1)$.

we have

$$\sum_{i=1}^{I} \lambda_i\,(z_{i.s}+1)\,G(s+1) = \sum_{i=1}^{I} \lambda_i\,(z_{i.s}+1)\,z_{i.s}+1 - g - h_{s+1} + \mu_{s-1} G(s)$$

$$= \sum_{i=1}^{I} \lambda_i\,(z_{i.s}+1)\,(z_{i.s}+1 - G(s) + \lambda_i((z_{i.s}+1)G(s) + \mu_{s-1} G(s) - g - h_{s+1}$$

$$< \sum_{i=1}^{I} \lambda_i\,(z_{i.s}+1)\,(z_{i.s}+1 - G(s) + \lambda_i((z_{i.s}+1)G(s) + \mu_s G(s-1) - g - h_s$$

$$< \sum_{i=1}^{I} \lambda_i\,(z_{i.s}+1)G(s).$$

If $\sum_{i=1}^{I} \lambda_i\,(z_{i.s}+1) = 0$ then $G(s) = (g + h_{s+1})/\mu_{s+1} \geq 0$. Which is impossible

Therefore, $G(s+1) < G(S) < 0$. Since $\mu_{s+2} \geq \mu_{s+1}$.

We have $\mu_{s+2}\,G(s+1) < \mu_{s+1}\,G(s) < 0$. Consequently, we can repeat the argument above until we reach state N -1.for which $G(N-1) < 0$.

But $G(N-1) = (g + h_N) / \mu_N \geq 0$.which yields a contradiction[6] .Therefore for all s =-1 ,….,N - 1, $0 \leq G(s) \leq (g + h_{s+1}) / \mu_{s+1}$.

Main Proof of Theorem 3.1. The existence of a canonical triplet (g, G(˙), **z**) to (3) – (5) is a direct consequence of Lemma 3.1. Since the state space is finite, by a result (g.G (˙),Z is an optimal solution. Therefore $g_N^* = g$ and $z^* = z$.

It remains to show that $z_{i,s}^* = \inf\{ z : r_i(z)(z - G(s)) \geq 1\}$ and that it is the unique optimal stationary policy . For s = 0,….,N-1 and i = 1 …… I. Let

$$v_{i,s}(z) = \lambda_i(z)(z - G(s)),$$

$$r'_{i,s}(z) = (1 - F_i(z)) - f_i(z)(z - G(s)) \quad \text{almost everywhere (a.e.) on } [\alpha_i, \beta_i].$$

Also $v'_{i,s}(z) > 0$ is equivalent to $r_i(z)(z - G(s)) < 1$ and $v'_{i,s}(z) < 0$ is equivalent to $r_i(z)(z - G(s)) \geq 1$. The IGHR assumption implies that $r_i(z)\ (z - G(s)) \geq 1$ a.e. on $\inf\{z : r_i(z)(z - G(s)) \geq 1\}, \beta_I$.

Therefore, $v'_{i,s}(\cdot) > 0$ a.e. on $(\alpha_i, \inf\ \{z : r_i(z)(z - G(s)) \geq 1\})$ and

$$v'_{i,s}(\cdot) < 0 \text{ a.e. on } \inf\{z : r_i(z)\ (z - G(s)) \geq 1\}, \beta_i .$$

Thus, $v_{i,s}(\cdot)$ is strictly unimodal and $z_{i,s}^* = \inf\{ z : r_i(z)(z - G(s)) \geq 1\}$ is its unique maximizer on $[\alpha_i, \beta_i]$.

We still need to show that z^* is the unique optimal stationary policy [8] .Under IGHR, $z_{i,s}^*$ is the unique maximizer of $\sup\{\lambda_i(z)(z - G(s))\}$. So,

$$g_N^* > \sum_{i=1}^{I} \lambda_i(z_i)(z_i - G(s)) + \mu_s G(s-1) - h_s \text{ for all } z_i \neq z_{i,s}^*$$

Since we have a unichain model, by a result we say that the uniqueness of the optimal

stationary policy, z^*

We are now able to characterize an optimal stationary policy explicitly[10]. It might be possible that $z_{i,s}^* = \beta_i$ for some state s. In this case it is optimal for the service provider not to accept customers of class i when in state s. However, this can only occur if the class-i customers' willingness-to-pay distribution has finite support. Indeed, if F_i has infinite support then, for all $s = 0, \dots, N - 1$, $\sup\{\lambda_i(z)(z - G(s))\} > 0$ and $z_{i,s}^* < \infty = \beta_i$. Moreover,

$$z_{i,s}^* = \inf\{z : (z - G(s))r_i(z) \geq 1\} \geq \inf\{z : z r_i(z) \geq 1\}.$$

Since $\inf\ \{z : z r_i(z) \geq 1\}$is the optimal price to charge when the demand function is $1 - F_i(z)$, we observe that holding costs and capacity limitations force the service provider to charge higher prices[12] .Therefore, one can understand G(s) as a price premium charged by the service provider to account for the additional congestion created by a customer's admission into state s.

4. Conclusion

We characterized optimal pricing policies that maximize the long-run average profit per unit time. In systems with finite capacity and in systems with infinite capacity under uniform asymptotic holding cost and service rate, we found an exact solution to the Average –cost optimality equations (ACOE) that cor- responds to an optimal stationary pricing policy. In systems with infinite capacity and more general holding cost and service rate structure, we showed that an optimal stationary pricing policy exists as the limiting pricing solution for finite capacity systems whose size grows to ∞. Moreover, we proved that the optimal stationary prices are nondecreasing with the state index and perform a congestion control that prevents high holding costs in congested states.

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