# An Orthogonal Left Centralizer and Reverse Left Centralizer on Semiprime Γ-Rings

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## Abstract:

Let M be a semiprime Γ-ring . In this paper we introduce the concept of orthogonal left centralizer and reverse left centralizer on a semiprime Γ- ring and we prove the following main result:

Let M be a 2-torsion free semiprime Γ- ring, t be a left centralizer and h be a reverse left centralizer of M , such that $x\alpha z\beta y = x\beta z\alpha y$ , for all $x , y , z \in M$ , $\alpha , \beta \in \Gamma$ and t , h are commuting. Then t and h are orthogonal if and only if

$t(x)\, \Gamma\, M\, \Gamma\, h(y) + h(x)\, \Gamma\, M\, \Gamma\, t(y) = (0)$ , for all $x , y \in M$ .

**Key Words :** semiprime Γ-ring , left centralizer , reverse left centralizer , orthogonal left centralizer and reverse left centralizer .

**Mathematic Subject classification** : 16N60 , 16W25 , 42C05, 33C45.

## I-Introduction :

In 1964 [6] gave the notion of a Γ-ring . This concept is more general than the concept of a ring. In 1966 [2] generalized this concept . . The definition of prime ring and semiprime Γ-ring was introduced in [5]. The definition of 2-torsion free Γ-ring was introduced in [7]. While [8] introduced the concept of left (resp. right) centralizer and Jordan left (resp. right) centralizer of Γ-rings. The concept of higher reverse left (resp. right) centralizer and a Jordan higher reverse left (resp. right) centralizer of Γ-ring was introduce by [4] and the one important question can be answered whether there is a relation between a concepts of a higher reverse left(resp. right) centralizer and a Jordan higher reverse left(resp. right) centralizer within certain conditions .

In this paper , we define and study the concept of orthogonal left centralizer and reverse left centralizer of semiprime Γ-ring and we prove some of lemmas and theorems about orthogonally one of these theorems is :

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , where t and h are commuting .Then the following conditions are equivalent :

**(i)** t and h are orthogonal

**(ii)** t(x) Γ h(y) = (0)

**(iii)** h(x) Γ t(y) = (0)

**(iv)** t(x) Γ h(y) + h(x) Γ t(y) = (0) .

In our work we need the following Lemmas :

***<u>Lemma(1.1):</u>*** **[**1**]**

If M is a 2-torsion free semiprime Γ-ring and x , y be elements of M , then the following conditions are equivalent :

(i) xΓmΓy = (0) , for all m∈ M

(ii) yΓmΓx = (0) , for all m∈ M

(iii) xΓmΓy + yΓmΓx = (0) , for all m∈ M

If one of these conditions is fulfilled ,then xΓy = yΓx = 0 .

***<u>Lemma(1.2):</u>*** **[**3**]**

Let M be a 2-torsion free semiprime Γ-ring and x , y be elements of M if xΓmΓy + yΓmΓx = (0) , for all m∈M. Then xΓmΓy = yΓmΓx = (0) .

## II . Orthogonal Reverse Left (resp.Right) Centralizer on Semiprime Γ-Rings :

In this section we will introduce the concept of orthogonal left centralizer and a reverse left centralizer on semiprime Γ-rings.

***Definition (2.1):***

Let t be a left centralizer and h be a reverse left centralizer of a Γ-ring M . Then t and h are called **orthogonal** if $t(x)\,\Gamma\, M\,\Gamma\, h(y) = (0) = h(y)\,\Gamma\, M\,\Gamma\, t(x)$ , for all x , y ∈ M .

***Example (2.2):***

Let M be a ring of all 2 × 2 matrices of integer numbers, such that

$$M = \left\{\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}; x, y \in Z\right\} \text{ and } \Gamma = \left\{\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}; \forall n \in Z\right\}$$

. Then M is a Γ-ring .

Let $t : M \to M$ be an additive mapping of a Γ-ring M into itself , such that

$$t\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \text{ for all } \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \in M$$

$h : M \to M$ be an additive mapping of a Γ-ring M into itself , such that

$$h\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \text{ for all } \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \in M$$

Then t is a left centralizer and h is a reverse left centralizer . Then t and h are orthogonal of M .

***Example (2.3):***

Let t is a left centralizer and h is a reverse left centralizer of a Γ-ring M ,
we put $M^* = M \oplus M = \{(x,y) ; x , y \in M\}$ and $\Gamma^* = \Gamma \oplus \Gamma = \{(\alpha,\beta) ; \alpha , \beta \in \Gamma \}$,
we define $t^*$ and $h^*$ on $M^*$ by
$t^*((x,y)) = (t(x) , 0)$ and $h^*((x,y)) = (0 , h(y))$ , for all $(x,y) \in M^*$.
Then $t^*$ and $h^*$are orthogonal $M^*$ .

***Lemma (2.4):***

Let M be a semiprime Γ- ring , suppose that t be a left centralizer and h be a reverse left centralizer of M ,satisfy $t(x)\,\Gamma\, M\,\Gamma\, h(x) = (0)$ , for all x ∈ M . Then $t(x)\,\Gamma\, M\,\Gamma\, h(y) = (0)$, for all x , y ∈ M .

***Proof :***

Suppose that $t(x)\,\alpha\, z\, \beta\, h(x) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ …(1)

Replace x by x + y in (1) , we have that

$t(x + y)\,\alpha\, z\, \beta\, h(x + y) = 0$

$t(x)\,\alpha\, z\, \beta\, h(x) + t(x)\,\alpha\, z\, \beta\, h(y) + t(y)\,\alpha\, z\, \beta\, h(x) + t(y)\,\alpha\, z\, \beta\, h(y) = 0$

Therefore , by our assumption and Lemma (1.1) , we get

$t(x)\,\alpha\, z\, \beta\, h(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Thus $t(x)\,\Gamma\, M\, \Gamma\, h(y) = (0)$ , for all $x , y \in M$

***Lemma (2.5):***

Let M be a 2-torsion free semiprime Γ- ring, t be a left centralizer and h be a reverse left centralizer of M , such that $x\alpha z\beta y = x\beta z\alpha y$ , for all $x , y , z \in M$ , $\alpha , \beta \in \Gamma$ and t , h are commuting. Then t and h are orthogonal if and only if

$t(x)\,\Gamma\, M\, \Gamma\, h(y) + h(x)\,\Gamma\, M\, \Gamma\, t(y) = (0)$ , for all $x , y \in M$ .

***Proof :***

Suppose that t and h are orthogonal

T.P. $t(x)\,\Gamma\, M\, \Gamma\, h(y) + h(x)\,\Gamma\, M\, \Gamma\, t(y) = 0$ , for all $x , y \in M$

Since t and h are orthogonal , we have that

$t(x)\alpha\, z\, \beta\, h(y) = 0 = h(y)\alpha\, z\, \beta\, t(x)$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

By Lemma (1.1) , we have that

$t(x)\,\alpha\, h(y) = 0 = h(x)\,\alpha\, t(y)$ , for all $x , y \in M$ and $\alpha \in \Gamma$

$t(x)\,\alpha\, h(y) + h(x)\,\alpha\, t(y) = 0$ , for all $x , y \in M$ and $\alpha \in \Gamma$

Left multiply by $z\,\beta$ , we have that

$z\,\beta\, t(x)\,\alpha\, h(y) + z\,\beta\, h(x)\,\alpha\, t(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Since $x\alpha z\beta y = x\beta z\alpha y$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ and t and h are commuting , we have that :

$t(x)\,\alpha\, z\, \beta\, h(y) + h(x)\,\alpha\, z\, \beta\, t(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Hence $t(x)\,\Gamma\, M\, \Gamma\, h(y) + h(x)\,\Gamma\, M\, \Gamma\, t(y) = (0)$ , for all $x , y \in M$

**Conversely** , it's clear by using Lemma (1.2)

***Theorem (2.6):***

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , where t and h are commuting . Then the following conditions are equivalent :

**(i)** t and h are orthogonal

**(ii)** th = 0

**(iii)** ht = 0

**(iv)** th + ht = 0

***Proof :*** **(i) ⇔ (ii)**

Suppose that t and h are orthogonal

T.P. th = 0

Since t and h are orthogonal , we have that

$h(y)\, \alpha\, z\, \beta\, t(x) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Replace x by h(y) , we have that

$h(y)\, \alpha\, z\, \beta\, t\, t(h(y)) = 0$

$t\, (h(y)\, \alpha\, z\, \beta\, t(h(y))) = 0$

$t(h(y))\, \alpha\, z\, \beta\, t(h(y)) = 0$ , for all $y , z \in M$ and $\alpha , \beta \in \Gamma$

Since M is a semiprime Γ- ring , we have that

$t(h(y)) = 0$ , for all $y \in M \Rightarrow th = 0$

**Conversely**, suppose that th = 0

T.P. t and h are orthogonal

$h(t(x\beta y)) = 0$

$h(t(x)\, \beta\, y) = 0$

$h(y)\, \beta\, t(x) = 0$

Since t and h are commuting , we have that

$t(x)\, \beta\, h(y) = 0$

Replace x by xαz , we have that

$t(x\alpha z)\, \beta\, h(y) = 0$

$t(x)\, \alpha\, z\, \beta\, h(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ …(1)

Since t and h are commuting , we have that

$h(y)\, \alpha\, z\, \beta\, t(x) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ …(2)

Hence t and h are orthogonal .

***Proof :*** **(i) ⇔ (iii)**

By the same way in (i) ⇔ (ii) , we get (i) ⇔ (iii) .

***Proof :*** **(i) ⇔ (iv)**

Suppose that t and h are orthogonal

T.P. th + ht = 0

By (ii) and (iii) , we get the require result .

**Conversely** , suppose that th + ht = 0

T.P. t and h are orthogonal

$(th + ht)(y\beta x) = 0$

$t(h(y\beta x)) + h(t(y\beta x)) = 0$

$t(h(x)\beta y) + h(t(y)\beta x) = 0$

$t(h(x))\beta y + h(x)\beta t(y) = 0$

Replace t(h(x)) by t(x) , we have that

$t(x)\beta y + h(x)\beta t(y) = 0$

Replace β y by β h(y) , we have that

$t(x)\beta h(y) + h(x)\beta t(y) = 0$

Left multiply by zα , we have that

$z\alpha t(x)\beta h(y) + z\alpha h(x)\beta t(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Since t and h are commuting , we have that

$t(x)\alpha z\beta h(y) + h(x)\alpha z\beta t(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

That is $t(x)\Gamma M\Gamma h(y) + h(x)\Gamma M\Gamma t(y) = (0)$ , for all $x , y \in M$

By Lemma (2.5) , we get the require result .

***Theorem(2.7):***

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , where t and h are commuting .Then the following conditions are equivalent :

**(i)** t and h are orthogonal

**(ii)** $t(x)\Gamma h(y) = (0)$

**(iii)** $h(x)\Gamma t(y) = (0)$

**(iv)** $t(x)\Gamma h(y) + h(x)\Gamma t(y) = (0)$

***Proof :*** **(i) ⇔ (ii)**

Suppose that t and h are orthogonal

T.P. $t(x)\,\Gamma\,h(y) = (0)$ , for all $x , y \in M$

Since t and h are orthogonal , we have that

$t(x)\,\alpha\,z\,\beta\,h(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

By Lemma (1.1) , we have that

$t(x)\,\alpha\,h(y) = 0$

Thus , $t(x)\,\Gamma\,h(y) = (0)$, for all $x , y \in M$

**Conversely**, suppose that $t(x)\,\Gamma\,h(y) = (0)$, for all $x , y \in M$

T.P t and h are orthogonal

$t(x)\,\beta\,h(y) = 0$ , for all $x , y \in M$ and $\beta \in \Gamma$

Replace x by $x\alpha z$ , we have that

$t(x\alpha z)\,\beta\,h(y) = 0$

$t(x)\,\alpha\,z\,\beta\,h(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ …(1)

Since t and h are commuting , we have that

$h(y)\,\alpha\,z\,\beta\,t(x) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ …(2)

Hence t and h are orthogonal .

***Proof :*** **(i) ⇔ (iii)**

By the same way in (i) ⇔ (ii) , we get (i) ⇔ (iii) .

***Proof :*** **(i) ⇔ (iv)**

Suppose that t and h are orthogonal

T.P. $t(x)\,\Gamma\,h(y) + h(x)\,\Gamma\,t(y) = (0)$

By (ii) and (iii) , we get the require result .

**Conversely** , suppose that $t(x)\,\Gamma\,h(y) + h(x)\,\Gamma\,t(y) = (0)$

T.P t and h are orthogonal

By our assumption , we have that

$t(x)\,\beta\,h(y) + h(x)\,\beta\,t(y) = 0$ , for all $x , y \in M$ and $\alpha \in \Gamma$

Left multiply by $z\alpha$ , we have that

$z\,\alpha\,t(x)\,\beta\,h(y) + z\,\alpha\,h(x)\,\beta\,t(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Since t and h are commuting , we have that

$t(x)\,\alpha\,z\,\beta\,h(y) + h(x)\,\alpha\,z\,\beta\,t(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

$t(x)\ \Gamma\ M\ \Gamma\ h(y) + h(x)\ \Gamma\ M\ \Gamma\ t(y) = (0)$ , for all $x , y \in M$

By Lemma (2.5) , we get the require result .

***Corollary (2.8):***

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , where t and h are commuting . Then the following conditions are equivalent , for all $x \in M$ :

**(i)** t and h are orthogonal

**(ii)** $t(x)\ \Gamma\ h(x) = 0$

**(iii)** $h(x)\ \Gamma\ t(x) = 0$

**(iv)** $t(x)\ \Gamma\ h(x) + h(x)\ \Gamma\ t(x) = 0$

***Proof :***

Obvious

***Lemma(2.9):***

Let M be a completely prime Γ- ring , t and h are orthogonal left centralizer and reverse left centralizer resp. of M . Then either $t = 0$ or $h = 0$ .

***Proof :***

Suppose that t and h are orthogonal

T.P. $t = 0$ or $h = 0$

$t(x)\ \alpha\ z\ \beta\ h(y) = 0$ , for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

By Lemma (1.1) , we have that

$t(x)\ \alpha\ h(y) = 0$ , for all $x , y \in M$ and $\alpha \in \Gamma$

Since M is a completely prime Γ- ring , we get either

$t(x) = 0$ or $h(y) = 0$ , for all $x , y \in M$

$t = 0$ or $h = 0$

***Theorem(2.10):***

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , suppose that $t(x)\ \alpha\ t(x) = h(x)\ \alpha\ h(x)$ , for all $x \in M$ and $\alpha \in \Gamma$ .

Then t + h and t – h are orthogonal .

***<u>Proof :</u>***

( (t + h) $\alpha$ (t– h) + (t– h) $\alpha$ (t + h))(x)

= t(x) $\alpha$ t (x) – t(x) $\alpha$ h(x) + h(x) $\alpha$ t(x) – h(x) $\alpha$ h(x) + t(x) $\alpha$ t (x) + t(x) $\alpha$ h(x) – h(x) $\alpha$ t(x) – h(x) $\alpha$ h(x)

= 0

Therefore , ((t + h) $\alpha$ (t– h) + (t– h) $\alpha$ (t + h))(x) = 0 , for all x $\in$ M and $\alpha \in \Gamma$

By Corollary (2.8) (iv) $\Rightarrow$ (i) , we get the require result .

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