Research Article

A Study on the Commutator Subgroups of Commutator in the Group in n Terms of Elements and it's Properties

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Abstract: This paper aims at treating a study on the commutator in the group in n terms of elements for different algebraic structures as groups; subgroup and cyclic group in real numbers. After that we discuss the commutator and commutator subgroups of groups in n terms of elements which will give us a practical knowledge to see the applications. If G is a finite group and a , $b \in G$, then C = {ab $a^{-1}b^{-1}$: a,b $\in G$}is commutator subgroup and ab $a^{-1}b^{-1}$ is the commutator of a and b. Finally, we find out the commutator in the group in n terms of elements in different types of group.

Keywords: Multiplication Composition, Solvable Group, Commutator and commutator subgroup.

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1. Introduction

A group is a particular type of an algebraic system .The first thing that our forefather must have learnt in the solvable and commutator of the mathematics must has been "Group Sense", a sense of distinguishing between one or two objects. The study of solvable and commutator started but soon it become an abstract discipline much beyond the needs that had arisen. Let a,b $\in G$ be arbit rary elements. The elemenst ab $a^{-1}b^{-1}$ is the commutator of a and b taken in this order. We often denoted by [a, b] in place of ab $a^{-1}b^{-1}$ i.e. [a, b] = ab $a^{-1}b^{-1}$.The subgroups, generated by the complex consisting of the commutators of all ordered pairs of elements belonging to the group G [1]. Symbolically C = {ab $a^{-1}b^{-1}$: a,b $\in G$}, the subgroup G', generated by C is commutator subgroup of G i. e. [2]. Then G' is the smallest subgroup of G containing C.It can be shown that the inverse of a commutator is a commutator .i.e. [3] and the product of two commutators is not necessarily a commutator [4]. While the purpose of study in some cases may be concrete structures, it is always easy to abstract structures and idealize the structure. After all apply them to concrete situations. In concrete situations, solvable and commutator was found that a given situation satisfies the basic axioms of the structure and having known as the properties of that structure (see.i.e.[5], [6]) . Then the commutator is easy to the properties and solution of the situation. Then we find the commutator in the group in n terms of elements in different types of group (see .i. e. [7], [8]). Hence by the complex, consisting of all commutators of ordered pairs of elements of G, may or may not be a subgroup of G.

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2. Significance of Solvable and Commutator of a Group

We begin this section related to definition with the following significance of solvable and commutator of a group.

Solvable Group: A group G is said to be solvable if there exist a finite chain of subgroups.

$G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \ldots . \supseteq H_K = \{e\}$

Such that H_i is normal subgroup of H_{i-1} and each quotient group H_{i-1} / H_i is abelian.

Solvable Series: A group G is said to be solvable if there exist a finite chain of subgroup

$G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \ldots . \supseteq H_K = \{e\}$

Such that H_i is normal subgroup of H_{i-1} and each quotient group H_{i-1} / H_i is abelian.

Therefore each series of subgroup G is called a solvable series of G.

Normal series: A group G is said to be solvable if there exist a finite chain of subgroups.

$G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \ldots . \supseteq H_K = \{e\}$

Such that H_i is normal subgroup of H_{i-1} and each quotient group H_{i-1} / H_i is abelian. Then the series is called a normal series of G.

Example: i. The symmetric group S_4 of degree 4 is solvable. ii. Every abelian group is solvable.

Commutator: Let G be a finite group and a,b $\in G$. Then the element ab $a^{-1}b^{-1}$ is said to the commutator of (a,b).

Commutator subgroup: Let G be a finite group and a,b $\in G$. Then the smallest subgroup containing $\{ab\, a^{-1}b^{-1} : a,b \in G\}$ is said to the commutator subgroup of G.

Theorem- 1:

Show that a group G is solvable if and only if $G^{(k)} = \{e\}$ for some integer k.

Solution: Let $G^{(k)}=\{e\}$, for some integer k. Now we will prove that G is solvable.

We observe that $G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \ldots\ldots \supseteq G^{(k)} = e$ is a solvable series for G.

Each $G^{(i)}=[G^{(i-1)}]^1$ is a normal subgroup of $G^{(i-1)}$ and quotient group $G^{(i-1)}/[G^{(i-1)}]^1$ is an abelian group for each i. Hence G is solvable.

Conversely, Let G be a solvable group. Now we will prove that $G^{(k)} = \{e\}$

Let $G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots\ldots \supseteq N_k = \{e\}$

Each N_i is a normal subgroup of N_{i-1} and that N_{i-1} / N_i is an abelian.

Consequently $N_{i-1}{}^1$ must be contained in N_i

$\therefore N_1 \supseteq N_0{}^1 = G^1$

$N_2 \supseteq N_1{}^1 \supseteq (N_0{}^1)^1 = (G^1)^1 = G^{(2)}$

$N_3 \supseteq N_2{}^1 = (G^{(2)})^1 = G^{(3)}$

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$N_k \supseteq G^{(k)} \quad \therefore G^{(k)} \subseteq N_k = \{e\}$ (1)

But $G^{(k)}$ being a group. $\therefore \{e\} \subseteq G^{(k)}$ (2)

From (1) and (2) then we get, $G^{(k)} = \{e\}$.

Hence a group G is solvable if and only if $G^{(k)} = \{e\}$ for some integer K.

Theorem-2: Let H be a normal subgroup of a group G. If both H and G/H are solvable then show that G is solvable.

Proof: Let H be a normal subgroup of a group G. Then the identity element of G/H is H.

First suppose that G/H is a solvable. Now we will prove that G is solvable.

Since H is solvable ⇒

$H = H_0 \supset H_1 \supset H_2 \supset \ldots\ \ldots\ \ldots\ \supset H_n = \{e\}$ (3)

And G/H is solvable ⇒

G/H = G_0/H ⊃ G_1/H ⊃ … … … ⊃ G_m / H = {H} (4)

Here each G_{i+1}/H is a normal subgroup of G_i/H. Hence G_{i+1} is a normal subgroup of G_i.

From (4), then we get,

G_m/H = {H}

⇒ G_m = H (5)

From (2), then we get

G = $G_0 \supset G_1 \supset G_2 \supset \ldots\ \ldots\ \ldots\ \supset G_{m-1} \supset G_m$

⇒ G=$G_0 \supset G_1 \supset G_2 \supset \ldots\ \ldots\ \ldots\ \supset G_{m-1} \supset G_m$ = H [By using (5)]

⇒ G = $G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_{m-1} \supset H = H_0 \supset H_1 \supset H_2 \supset \ldots \supset H_n = \{e\}$ [By using (3)]

⇒G = $G_0 \supset G_1 \supset G_2 \supset \ldots\ \ldots\ \ldots\ \supset H_0 \supset H_1 \supset H_2 \supset \ldots\ \ldots\ \ldots\ \supset H_n = \{e\}$

Hence G is solvable. (Proved)

Theorem-3 [9]:

Let $G^{/}$ be a commutator subgroup of a group G. Then a quotient group G/H is abelian if and only

if H contains the commutator subgroup $G^{/}$ of G.

Proof: Let K ={ a b $a^{-1}b^{-1}$: a,b ∈ G}

Let $G^{/}$ be the commutator subgroup of G. Then $G^{/}$ is the smallest subgroup of G containing K.

i.e. K ⊆ $G^{/}$

Let G/H be abelian.

Since K ={ ab $a^{-1}b^{-1}$: a,b ∈ G} ∴ ab $a^{-1}b^{-1}$∈ K

As G/H is an abelian.

∴ (Ha)(Hb) = (Hb)(Ha) ⇒ Hab = Hba ⇒ ab $(ba)^{-1}$ ∈ H

⇒ ab $a^{-1}b^{-1}$ ∈ H

i.e. every element of K is in H. ∴ K ⊆ H

Now, $G^{/}$ being the smallest subgroup containing K. so $G^{/}$⊆ H.

Hence H contains G.

Conversely, Let $G^{/}$⊆ H. then k ⊆ H [∵ K ⊆ $G^{/}$]

But k ⊆ H ⇒ ab $a^{-1}b^{-1}$∈ k ⇒ ab $a^{-1}b^{-1}$∈ H

⇒ ab $(ba)^{-1}$∈ H ⇒ Hab = Hba ⇒ (Ha)(Hb) = (Hb) (Ha)

∴ G/H is an abelian.

Problem:

Let $G^{/}$ be a commutator subgroup of G . Show that $G^{/}$ = 1 if and only if G is abelian.

Solution: Let $G^{/}$ be a commutator subgroup of G.

Let K = {$a^{-1}b^{-1}$ ab : a,b ∈ G }. Then $G^{/}$ is the smallest subgroup of G containing K

i.e K ⊆ $G^{/}$ (6)

Let $G^{/}$= 1. Now we will prove that G is abelian.

We have, $a^{-1}b^{-1}$ ab ∈ K ⇒ $a^{-1}b^{-1}$ ab ∈ $G^{/}$ [By using (6)]

⇒ $(ba)^{-1}$ab =1

⇒ a b = b a

⇒ G is abelian.

Conversely, Let G be abelian. Now we will prove that $G^{/}$=1.

We have, K = $a^{-1}b^{-1}$ab for some a,b ∈ G.

$G^{/}$ = $a^{-1}b^{-1}$ab [By using (6)]

= $a^{-1}(b^{-1}a)$b [By Associative Law]

= $a^{-1}(ab^{-1})$b [By commutative Law]

= $(a^{-1}a)$ $(b^{-1}b)$ [By Associative Law]

= e. e

⇒$G^{/}$= e⇒ $G^{/}$=1 (Showed)

3. Result and Discussion with Property

Here, we discuss the result with property of the commutator in the group in n terms of elements for different algebraic structures as groups and related theories.

Theorem-1 [10]: If G is a group and $a_1, a_2, a_3, a_4 \ldots \ldots \ldots \in G$. Then the Commutator for n elements $a_1, a_2 \ldots\ldots\ldots a_n$ will be of the form$(a_1, a_2 \ldots\ldots\ldots a_n)^{-1} = {a_1}^{-1}{a_2}^{-1} \ldots\ldots\ldots {a_n}^{-1}a_1a_2a_3 \ldots\ldots\ldots a_n$

Proof:

Let G be a group. If a and b are of a group of G.Then ab=bac for some $c \epsilon G$. If a and b commute, Then ofcourse c=e. In general ,$c \neq e$ and c= $a^{-1}b^{-1}ab$. An element of this form is called a commutator and is usually denoted by (a,b).That is

$$(a, b)=a^{-1}b^{-1}ab$$

Let G be a group. Let us denoted by G' be subgroup generated by the set of all commutators (a, b)=$a^{-1}b^{-1}ab$ of G for all $a, b \epsilon G$. Then $G'$is called commutator subgroup of G.

We also define commutator of higher order by recursive roles:

$$(x_1,x_2, \ldots\ldots\ldots x_{n-1}, x_n) = (x_1,x_2, \ldots\ldots\ldots x_{n-1}), x_n$$

$$= (X, x_n) \quad \text{Where } X = (x_1,x_2, \ldots\ldots\ldots x_{n-1})$$

$$= X^{-1}{x_n}^{-1}Xx_n$$

These are called simple commutators.

The set of elements which can be obtained by successive commutation are call complex commutators.

We shall represent conjugate by a exponent $a^x = x^{-1}ax$

Where x is fixed in G and for all $a \in G.$

Form of commutator for more than two elements:

To find the commutators for more than two elements, at first we consider the definition for two elements.

It says if ab=bac for some $c \in G$ $then\ c\ is\ called\ commutator\ for\ a, b\ where$ ∀ $a , b \in G.$

If G is a commutative group then there is nothing to prove because ab=ba.

Then c=e=$aa^{-1}bb^{-1} = a^{-1}b^{-1}ab$

Which the form of a commutator for two elements is defined as,

If non abelion group then $ab \neq ba$

If c is a commutator for (a, b) then ab=bac

That means commutator $c \epsilon G$ is such an element that, if taken operation with ba in the right hand side then it becomes equal to ab as if they commuted.

Now bac=ab

⇒ ac = $b^{-1}ab$ [taking b^{-1} in both sides]

⇒ c = $a^{-1}b^{-1}ab$ [taking a^{-1} in both sides]

Now if three elements a_1,$a_2, a_3 \in G$ their operation together will be a_1,a_2, a_3

If we take this operation in reverse order it looks like this $a_3a_2a_1$

If c be a commutator for three elements a_1,a_2, a_3 i.e. (a_1,a_2, a_3)=c

Then

a_1,$a_2, a_3 = a_3a_2a_1$ c

⇒ $a_3a_2a_1\ c = a_1a_2a_3$

⇒ $a_2a_1\ c = {a_3}^{-1}a_1a_2a_3$

⇒ $a_1c = {a_2}^{-1}{a_3}^{-1}a_1a_2a_3$

⇒ c = ${a_1}^{-1}{a_2}^{-1}{a_3}^{-1}a_1a_2a_3$

So for three elements the form of the commutator will be

($a_1a_2a_3$)=${a_1}^{-1}{a_2}^{-1}{a_3}^{-1}a_1a_2a_3$ c

For n elements a_1,$a_2, a_3, a_4\ \ldots \ldots \ldots a_n \in G$

If c is the commutator for those elements

Then we can write

$(a_1,a_2, \ldots \ldots \ldots a_{n-1}, a_n) = a_na_{n-1} \ldots \ldots \ldots a_2a_1\ c$

⇒ $a_na_{n-1} \ldots \ldots \ldots a_2a_1\ c = a_1,a_2, \ldots \ldots \ldots a_{n-1}a_n$

⇒ $a_{n-1} \ldots \ldots \ldots a_2a_1\ c = {a_n}^{-1}a_1a_2, \ldots \ldots \ldots a_{n-1}a_n$

⇒ $a_{n-2} \ldots \ldots \ldots a_2a_1\ c = {a_{n-1}}^{-1}{a_n}^{-1}a_1a_2, \ldots \ldots \ldots a_{n-1}a_n$

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⇒ $a_1\ c = {a_2}^{-1} \ldots \ldots\ \ldots\ \ldots\ {a_{n-1}}^{-1}{a_n}^{-1}a_1a_2 \ldots \ldots \ldots a_{n-1}a_n$

⇒ $c = {a_1}^{-1}{a_2}^{-1} \ldots \ldots\ \ldots\ \ldots\ {a_{n-1}}^{-1}{a_n}^{-1}a_1a_2 \ldots \ldots \ldots a_{n-1}a_n$

$\therefore (a_1 a_2 \ldots\ldots\ldots a_{n-1} a_n) = a_1{}^{-1} a_2{}^{-1} \ldots\ldots\ldots a_n{}^{-1} a_1\, a_2\, a_3 \ldots\ldots\ldots a_n$

Which is commutator for n elements.

We can clearly see the similarity between the form for two and n elements.

Now we will prove that commutators for more than two elements also follow properties and theorems just like the commutator for two elements.

Property [11]:

An important property of commutator is if $x, y \epsilon G$ where G is a group then$(y, x) = (x, y)^{-1}$

Proof: At first we prove this property $(y, x) = y^{-1}x^{-1}yx$

$=(x^{-1}y^{-1}yx)^{-1}$

$=(x, y)^{-1}$

Hence $(y, x) = (x, y)^{-1}$

Then for three elementsa_1,$a_2, a_3, \in G$

The property would be $(a_1$,$a_2, a_3) = (a_1, a_2, a_3)^{-1}$

We know $(a_1$,$a_2, a_3) = a_1{}^{-1}a_2{}^{-1}a_3{}^{-1}a_1a_2a_3$

$=(a_3{}^{-1}a_2{}^{-1}a_1{}^{-1}a_3a_2a_1)^{-1}$

$\therefore (a_1, a_2, a_3) = (a_3, a_2, a_1)^{-1}$

Now for n elements a_1,$a_2, a_3, a_4, \ldots \ldots \ldots a_n \in G$

$(a_1$,$a_2, a_3, \ldots\ldots\ldots, a_n) = a_1{}^{-1}a_2{}^{-1}a_3{}^{-1} \ldots\ldots\ldots a_n{}^{-1}a_1a_2a_3 \ldots\ldots\ldots\ldots a_n$

$=a_n{}^{-1}a_{n-1}{}^{-1}a_{n-2}{}^{-1}a_na_{n-1} \ldots\ldots\ldots a_1)^{-1}$

$\therefore (a_1a_2a_3 \ldots\ldots\ldots\ldots a_n) = (a_n\, a_{n-1} \ldots\ldots\ldots a_1)^{-1}$

It proves that commutators for more than two or n elements follows the properly of simple commutator i.e. commutator for two elements.

Theorem-1[12]: Let G' be a commutator subgroup of a group G. Then G is abelian if and only if $G' = \{e\} = E$

Proof: At first we prove this theorem for the commutator of two elements.

Let G be abelian so that $xy = yx$ for all $x, y \in G$

Now $x^{-1}y^{-1}xy$=$x^{-1}xy^{-1}y$= ee= e for all $x, y \in G$

Hence, G' consist of only one element. But $\{e\}$ is the smallest subgroup of G containing$\{e\} = E$.

Hence by definition $G' = \{e\} = E$

Conversely, let $G' = \{e\} = E$

Let x, y be any two elements of G. So that $x^{-1}y^{-1}xy \in G'$.

But $G' = \{e\} = E$

So $x^{-1}y^{-1}xy = E$

Or, $(yx)^{-1}xy = e$

Or, $xy = ((yx)^{-1})^{-1} = xy$

Hence G is abelian.

Now we prove that the theorem also true for the commutator of n elements.

Let $a_1, a_2, a_3, a_4 \ldots \ldots \ldots a_n \in G$ if G is an abelian group

Then $a_1a_2a_3 \ldots \ldots \ldots .. a_n = a_n\, a_{n-1} \ldots \ldots \ldots a_1$

Now $a_1^{-1}a_2^{-1}a_3^{-1} \ldots \ldots \ldots a_n^{-1}a_1a_2a_3 \ldots \ldots \ldots .. a_n = a_1^{-1}a_1a_2^{-1}a_2a_3^{-1}a_3 \ldots \ldots \ldots a_n^{-1}a_n$

$= ee \ldots\ldots. e = e$

Hence G' consist of only one element e. But $\{e\}$ is smallest subgroup of G

containing $\{e\} = E, Hence\ by\ definition\ G' = \{e\} = E$

Conversely, Let $G' = \{e\} = E$

Let $a_1a_2a_3 \ldots \ldots \ldots .. a_n$ be any n elements of G. So that

$a_1^{-1}a_2^{-1}a_3^{-1} \ldots \ldots \ldots a_n^{-1}a_1a_2a_3 \ldots \ldots \ldots .. a_n \epsilon G'$

But $G' = \{e\} = E$

So $a_1^{-1}a_2^{-1}a_3^{-1} \ldots \ldots \ldots a_n^{-1}a_1a_2a_3 \ldots \ldots \ldots .. a_n = e$

Or, $(a_n\, a_{n-1} \ldots \ldots \ldots a_1)^{-1}(a_1a_2a_3 \ldots \ldots \ldots .. a_n) = e$

Or, $(a_1a_2a_3 \ldots \ldots \ldots .. a_n) = [(a_na_{n-1} \ldots \ldots \ldots a_1)^{-1}]^{-1}$

$= a_na_{n-1} \ldots \ldots \ldots a_1$

Hence G is abelian.

Theorem-2 [13]: Let G' be a commutator subgroup of a group G. Then G' is normal subgroup of G.

Proof:

At first we prove this theorem for the commutator of two elements.

Let K ={ a b $a^{-1}b^{-1}$: a,b $\in G$}

Let $G^{/}$ be the commutator subgroup of G. Then $G^{/}$ is the smallest subgroup of G containing K.

i.e. K ⊆ $G^{/}$ (7)

Let x ∈ G and c ∈ $G^{/}$

∴ $xcx^{-1} = cc^{-1}(xcx^{-1})$ [∵ $cc^{-1} = e$]

$= c[c^{-1}(xcx^{-1})]$ [By Associative Law]

$= c[c^{-1}x\,(cx^{-1})]$

$= c[c^{-1}x (c^{-1})^{-1} x^{-1}]$

Since c ∈ $G^{/}$ and $G^{/}$ is a commutator subgroup of G.

i.e. $c^{-1}x (c^{-1})^{-1} x^{-1} \in G^{/}$ $\Rightarrow xcx^{-1} \in G^{/}$

Hence $G^{/}$ is a normal subgroup in G.

Now we prove that the theorem also true for the commutator of n elements.

Let $a_1, a_2, a_3, a_4 \ldots \ldots \ldots, a_n \epsilon\, G'$, then $a_n a_{n-1} \ldots \ldots \ldots\, a_1 \in G'$

Also $a_1 a_2 \ldots \ldots \ldots\, a_n \epsilon\, G'$ so G' will be normal subgroup of G if $x \in G$ implies

$xa_1a_2a_3 \ldots \ldots \ldots\, a_n x^{-1} \in G'$.

Now $xa_1a_2a_3 \ldots \ldots \ldots\, a_n x^{-1}$

$= xa_1, a_2, a_3, a_4 \ldots \ldots \ldots\, a_n x^{-1}(a_n a_{n-1} a_1)^{-1}(a_n a_{n-1} \ldots \ldots \ldots\, a_1)$

$=(xa_1a_2a_3 \ldots \ldots \ldots\, a_n x^{-1} a_1{}^{-1} a_2{}^{-1} a_3{}^{-1} \ldots \ldots \ldots a_n{}^{-1})(a_n a_{n-1} \ldots \ldots \ldots\, a_1)$

$=(x, a_1, a_2, a_3, a_4 \ldots \ldots \ldots, a_n)(a_n a_{n-1} \ldots \ldots \ldots\, a_1)$

Since $a_n a_{n-1} \ldots \ldots \ldots\, a_1 \epsilon G'$

Also $x, a_1, a_2, a_3, a_4 \ldots \ldots \ldots, a_n \epsilon\, G'$

$\therefore xa_1a_2a_3 \ldots \ldots \ldots\, a_n x^{-1} = (x, a_1, a_2, a_3, a_4 \ldots \ldots \ldots, a_n)(a_n a_{n-1} \ldots \ldots \ldots\, a_1) \in G'$

So G' is a normal subgroup of G.

Theorem-3 [14]: Let G' be a commutator subgroup of a group G. Then the quotient group G/$G^{/}$ is an abelian.

Proof:

At first we prove this theorem for the commutator of two elements.

Let $G^{/}$a and $G^{/}$b be any two elements of G/ $G^{/}$.

Since K ={ a b $a^{-1}b^{-1}$: a,b ∈ G}

$\therefore$ a b $a^{-1}b^{-1} \in$ K ⇒ a b $a^{-1}b^{-1} \in G^{/}$ [∵ K ⊆ $G^{/}$]

⇒ a b $(ba)^{-1} \in G^{/}$ [∵ $(ba)^{-1} = a^{-1}b^{-1}$]

⇒ $G^{/}$ab = $G^{/}$ba

⇒ ($G^{/}$a)($G^{/}$ b) = ($G^{/}$b) ($G^{/}$a)

⇒ G / $G^{/}$ is an abelian.

Now we prove the theorem for commutator of n elements.

Let G' be a commutator subgroup of a group G. then the quotient group G / $G^{/}$ is abelian.

Let $a_1, a_2, a_3, a_4, \ldots \ldots \ldots, a_n \epsilon\, G$

Then $G'a_1, G'a_2, G'a_3, \ldots \ldots \ldots\, G'a_n$ are n elements of the quotient group G / $G^{/}$.

Now we will prove that G / $G^{/}$ is abelian.

$a_1a_2a_3 \ldots \ldots \ldots\, a_n a_1{}^{-1} a_2{}^{-1} a_3{}^{-1} \ldots \ldots \ldots a_n{}^{-1} \in G'$.

Or, $(a_1a_2a_3 \ldots \ldots \ldots .. a_n)(a_n\, a_{n-1} \ldots \ldots \ldots\, a_1)^{-1} \in G'$

Or, $G'(a_1a_2a_3 \ldots\ldots\ldots..a_n) = G'(a_na_{n-1} \ldots\ldots\ldots\ a_1)$

Or, $G'a_1G'a_2 \ldots\ldots\ldots..G'a_n = G'a_nG'a_{n-1} \ldots\ldots\ldots\ G'a_1$

Hence G/G' is abelian.

From the very basic definition of commutator for two elements we have derived the form of commutator for n elements. So, form of commutator for n elements follows the important properties and theories of commutator for two elements.

So, form of commutator for n elements $a_1, a_2, a_3, a_4 \ldots\ldots\ \ldots\ a_n \epsilon G$ is denoted and defined by

$(a_1a_2a_3 \ldots\ldots\ldots\ a_n) = a_1{}^{-1}a_2{}^{-1}a_3{}^{-1} \ldots\ldots\ldots a_n{}^{-1}a_1a_2a_3 \ldots\ldots\ldots\ a_n$

4. Conclusion

We hope that this work will be useful for group theory related to solvable and commutator of a group. We have also shown that our derived form of commutator for n elements follows the important properties and theories of commutator for two elements. Then the complex, consisting of all commutators of ordered pairs of elements of G, may or may not be a subgroup of G. This result is the the commutator in the group in n terms of elements in different types of group. Then all expected results in this paper will help us to understand better solution of complicated to solvable and commutator of a group.

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