A common fixed point theorem for two pairs of weakly compatible self-maps

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Abstract: *A common fixed point theorem for four self maps is proved through the notion of weakly compatible self-map and property EA. The obtained result generalizes the result of Brain Fisher.*

Keywords: Common fixed point, Property EA, Compatible and weakly compatible self-maps.

Introduction:

Let (X, d) be a metric space. Self-maps M and $P$are said to be commuting if $MPx = PMx$ for all $x \in X$.

Definition 1.1: According to Jungck [3], self-maps M and P on X are compatible if $\lim_{n\to\infty} d(MPx_n, PMx_n) = 0$, when ever $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} Px_n = z$ for some $z \in X$.

Definition 1.2: According to Jungck and Rhoades [4], self-maps M and P of a metric space (X, d) are weakly compatible if $Mu = Pu$ for some $u \epsilon X$ then MP$u = PMu$.

Definition 1.3: According to Aamri [1] Self maps M and Pon X satisfy property E.A. if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that$\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} Px_n = z$.

Brain Fisher [2] proved the following result:

Theorem A:*Let M be a self-map on a complete metric space X satisfying inequality*

$$d^2(Mx, My) \leq \alpha d(x, My) d(y, Mx) + \gamma d(x, Mx) d(y, My) \quad \text{for all } x, y \in X, \ldots \quad \textbf{(1)}$$

Where$0 \leq \alpha, \gamma < 1$. *Then M has a unique fixed point.*

In this paper we extend Theorem A, to four self-maps using the notion of property EA and weakly compatible maps.

Main Result:

Theorem B.*The self maps*M, N, P*and* Q *on* X *satisfying the inclusions*

$$M(X) \subset Q(X) \text{ and } N(X) \subset P(X) \quad \ldots$$

(2)*And the inequality*

$$d^2(Mx, Ny) \leq \alpha d(Px, Ny) d(Qy, Mx) + \beta d(Mx, Ny) d(Px, Qy) + \gamma d(Mx, Px) d(Ny, Qy)$$
$$\text{for all } x, y \in X. \quad \ldots \quad (3)$$

Where$0 \leq \alpha, \beta, \gamma < 1$.

Suppose that

(i) *either* (M, P) *or* (N, Q) *satisfies property EA*

(ii) *one of* $P(X)$ *and* $Q(X)$ *is complete*

(iii) (M, P) *and* (N, Q)*are weakly compatible*

Then all the four self maps will have a unique common fixed point.

Prof. Suppose that property EA satisfied by the pair of self maps(M, P). From (2) we have$M(X) \subset Q(X)$, hence there exist a sequence$\{y\}_{n=1}^{\infty}$ in X such that$Mx_n = Qy_n$ for all n so that fromDefinition 1.3 we get

$$\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qy_n = z \quad \ldots \quad (4)$$

Let $\lim_{n\to\infty} Ny_n = s$ now we prove that $s = z$.

Taking $x = x_n, y = y_n$ in (3), we have

$$d^2(Mx_{n,}, Ny_n) \leq \alpha d(Px_{n,}, Ny_n) d(Qy_n, Mx_{n,}) + \beta d(Mx_{n,}, Ny_n) d(Px_{n,}, Qy_n) + \gamma d(Mx_{n,}, Px_{n,}) d(Ny_n, Qy_n)$$

As limit $n \to \infty$, this along with (4) implies that
$d^2(z, s) \leq \alpha.0 + \beta.0 + \gamma.0 = 0$ so that $s = z$.
Thus $\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Ny_n = z$ … (5)
Similarly (5) can be obtained if the self maps (N, Q) satisfy the property EA.
Case (i): Suppose that $Q(X)$ is complete subspace of X.
Note that$\{Qy_n\}_{n=1}^{\infty}$ is Cauchy and convergent sequence in $Q(X)$. We see that$z \in Q(X)$.
i.e.$z = Qs$ for some $s \in X$. Now we Prove that $Qs = Ns$.
Writing $x = x_{n,}$ $y = s$in(3) and using (5) we get

$$d^2(Mx_{n,}, Ns) \leq \alpha d(Px_{n,}, Ns) d(Qs, Mx_{n,}) + \beta d(Mx_{n,}, Ns) d(Px_{n,}, Qs) + \gamma d(Mx_{n,}, Px_{n,}) d(Ns, Qs)$$

Appling the limit as $n \to \infty$, and using (4) we see that
$d^2(Qs, Ns) \leq \alpha.0 + \beta.0 + \gamma.0 = 0$ so that $Qs = Ns$.
From (2) we have$N(X) \subset P(X) \Longrightarrow Ns \in P(X)$ or $Ns = Pt$ for some $t \in X$.
Now taking $x = t$ and $y = s$ in (3) and using $Qs = Ns = Pt$ we get

$$d^2(Mt, Ns) \leq \alpha d(Pt, Ns) d(Qs, Mt) + \beta d(Mt, Ns) d(Pt, Qs) + \gamma d(Mt, Pt) d(Ns, Qs)$$

$d^2(Mt, Pt) \leq 0$ *or* $Mt = Pt$. *Hence* $Qs = Ns = Pt = Mt$.
That is, s is a coincidence point of Q and $N$and t is a coincidence point of $P$and M.
Case (ii): Suppose that $P(X)$ is complete subspace of X.
Since $\{Px_n\}_{n=1}^{\infty}$ is Cauchy and convergent sequence in $P(X)$. Therefore $z \in P(X)$.
i.e.$z = Pu$ for some $u \in X$. Now we Prove that $Pu = Mu$.
Writing $x = u, y = y_n$, in (3) and using (5) we get

$$d^2(Mu, Ny_n) \leq \alpha d(Pu, Ny_n) d(Qy_n, Mu) + \beta d(Mu, Ny_n) d(Pu, Qy_n) + \gamma d(Mu, Pu) d(Ny_n, Qy_n)$$

$$d^2(Mu, z) \leq \alpha d(z, z) d(z, Mu) + \beta d(Mu, z) d(z, z) + \gamma d(Mu, Pu) d(z, z)$$

$d^2(Mu, z) \leq 0$ *or* $Mu = z$. *That is* $Mu = Pu = z$.
From (2) we have $M(X) \subset Q(X) \Longrightarrow Mu \in Q(X)$ or $Mu = Qw$ for some $w \in X$.
Hence $Pu = Mu = Qw = z$. … (6)
Again writing $x = u, y = w$, in (3) and using (6) we get

$$d^2(Mu, Nw) \leq \alpha d(Pu, Nw) d(Qw, Mu) + \beta d(Mu, Nw) d(Pu, Qw) + \gamma d(Mu, Pu) d(Nw, Qw)$$

$d^2(Qw, Nw) \leq \alpha.0 + \beta.0 + \gamma.0 = 0$ or $Qw = Nw$. Thus $Pu = Mu = Qw = Nw$.
Hence w is a coincidence point of$Q$and N and u is a coincidence point of P and M.
As we know from (iii) the pairs(M, P)*and* (N, Q)*are weakly compatible, we find that*
$MPt = PMt$ *and* $NQs = QNs$.*which implies* $Mz = Pz$*and* $Nz = Qz$.
Taking $x = y = z$ in (3) we get

$$d^2(Mz, Nz) \leq \alpha d(Pz, Nz) d(Qz, Mz) + \beta d(Mz, Nz) d(Pz, Qz) + \gamma d(Mz, Pz) d(Nz, Qz)$$

$$d^2(Mz, Nz) \leq \alpha d(Mz, Nz) d(Nz, Mz) + \beta d(Mz, Nz) d(Mz, Nz) + \gamma d(Mz, Mz) d(Nz, Nz)$$

Or $(1 - \alpha - \beta) d^2(Mz, Nz) \leq 0$ *that is* $d^2(Mz, Nz) = 0$ *or*$Mz = Nz$.
Thus $Mz = Nz = Pz = Qz$. … (7)
Know to prove $Mz = z$ writing $x = z$, $y = s$ in (3) and using (7) we get

$$d^2(Mz, Ns) \leq \alpha d(Pz, Ns) d(Qs, Mz) + \beta d(Mz, Ns) d(Pz, Qs) + \gamma d(Mz, Pz) d(Ns, Qs)$$

$$d^2(Mz, z) \leq \alpha d(Mz, z) d(z, Mz) + \beta d(Mz, z) d(Mz, z) + \gamma d(Mz, Mz) d(z, z)$$

Or $(1 - \alpha - \beta) d^2(Mz, z) \leq 0$ *that is* $d^2(Mz, z) = 0$*or* $Mz = z$.
Thus $Mz = Nz = Pz = Qz = z$.
Hence z is a common fixed point of $M, N, P and\ Q$. The uniqueness of the fixed point can be easily proved.

Remark. In the Inequality (3) of Theorem B, taking $\beta = 0$, $N = M$ and $P = Q = I$, the identity map on X we get the inequality (1) as a particular case. Also weknowthe identity map commutes and hence is weakly compatible with every map. Further from the proof of Theorem A, the sequence $\{Mx_n\}_{n=1}^{\infty}$

is Cauchy for each $x \in X$. Therefore if X is complete, this converges to some $z \in X$ and its convergence is equivalent to the property EA of the pair(M, I), that is the condition (i) of Theorem B.

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