

On Finsler Geometry: Review of applications in Solid Mechanics and Air craft motion

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Abstract

In Finsler geometry, each point of a base manifold can be endowed with coordinates describing its position as well as a set of one or more vectors describing directions. Finsler geometry encompasses Riemannian, Euclidean, and Minkowskian geometries as special cases, and thus it affords great generality for describing a number of phenomena in physics. This paper is involving application of Finsler geometry in solid mechanics is reviewed. With the help of finsler Gauge transformation it has been shown the geometrization of Electromagnetism in Finsler spaces.

Keywords: Finsler Space, Riemannian Geometry, Geodesics

Introduction

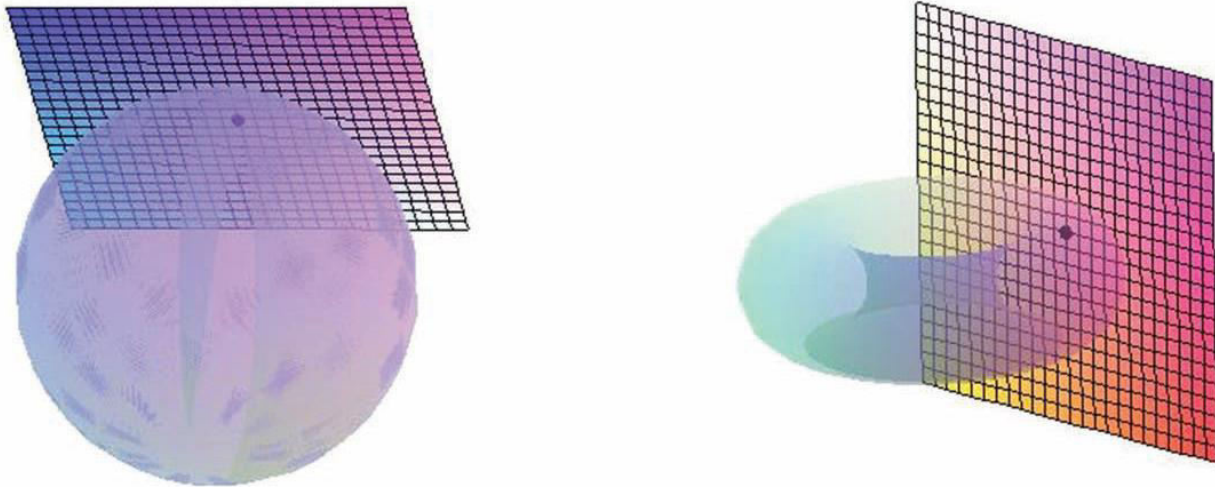
The main application of Finsler geometry is the field of Electromagnetism. The approach was firstly by Randers [5], but in his work Finsler geometry was not mentioned, although it was used. He produces ageodesic equation which is identical with Lorentz equation for a charged particle. The unified theory of gravitation and electrodynamics is developed from a Finslerian tangent space gauge transformation [22, 25]. The transformation physically interpreted as containing physical fields and the resulting metric is similar to the metric introduced by Kaluza-Klein, but has different physical interpretations in the scheme of Finsler geometry [10, 23, 24, 26, 27]

Homogeneous isotropic elastic solids can be described their mechanical behavior by constitutive models that depend only on some metric or strain tensor that may generally vary with position in a body. In conventional continuum mechanics approaches, such models typically assign one or more time- and positiondependent vector(s) or higher-order tensor(s), in addition to total deformation or strain, that describe physical mechanisms associated with evolving internal structure.

Secondly, discussed the application of Finsler Geometry in Navigation field also.

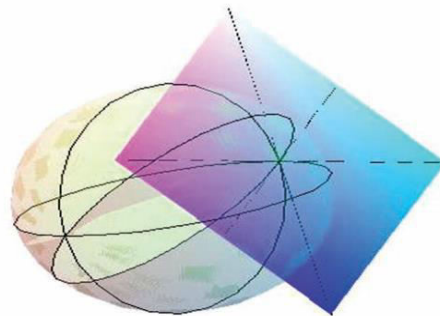
The Euclidean vector space is a vector space with a symmetric and positively definite bilinear form. This bilinear form induces the norm (length) of vectors by the formula

$$|v| = \sqrt{g(v, v)}$$



Using the length of curves, we can define the distance of two points P and Q of M as the infimum of lengths of all curves connecting P and Q . For any two points P and Q in a Riemannian manifold, the distance from P to Q is the same as the distance from Q to P . Geodesics in the Riemannian manifold (M, g) are curves which are extremals of the length functional and they are the best possible analogues of lines in the Euclidean space. The important property of a geodesic γ is that for any two points on γ which are close enough, γ is the shortest curve connecting

these two points. Hence, the distance of these two points is realized by this geodesic γ . Geodesics are obtained as solutions of a system of differential equations. Another important property is that, given a point P in M and a tangent vector v in the tangent space of M at P , there exist a unique local geodesic $\gamma(t)$ such that $\gamma(0) = P$ and $\gamma'(0) = v$. In other words, the position at the beginning is P and the velocity at the beginning is v . If the Riemannian manifold (M, g) is complete, then for any P of M there is a neighbourhood U of P such that any point Q of U can be joined with P by the unique minimizing geodesic lying in U . In the Euclidean space with the standard norm, geodesics are the straight lines. In the sphere with the natural Riemannian metric, geodesics are the main circles.



NAVIGATION DATA AND RELATED RANDERSMETRIC

Let us discussed the real problem of the ship on the sea. Let us consider the surface of the sea to be the two-dimensional space E^2 equipped with the standard norm. In the more general case, the sea can be modelled by the two-dimensional sphere with the natural Riemannian metric. However, in full mathematical generality the sea can be any Riemannian manifold of arbitrary dimension. The possible tracks of the ship are curves on the manifold and it is natural to aim at the destination in the shortest possible time. To measure the length of particular trajectories, or the time necessary for travelling these particular trajectories, respectively, we use the Riemannian metric h on M and the integral above. Naturally, the curves of interest are geodesics with respect to this metric

In 1931, E. Zermelo studied the following problem ([4][5]):

Suppose that a ship sails on the open sea in calm waters and a mild breeze comes up. How must the ship be steered in order to reach a given destination in the shortest time?

The problem was solved by Zermelo himself for the Euclidean flat plane and by Z. Shen ([9][10]) in the case when the sea is a Riemannian manifold (M, h) under the assumption that the wind W is a time-independent mild breeze, i.e. $h(x, W) < 1$. Recently, Zermelo navigation problems on Finsler manifolds have been discussed and applied widely by many scholars (e.g. [4][8][6]).

Essentially, Zermelo navigation problem is tightly related to the geometry of indicatrix on a Finsler manifold. Let us consider a Finsler manifold (M, Φ) . For each $x \in M$, the indicatrix S_Φ of x is a closed hypersurface of $T_x M$ around the origin x defined by

$$S_\Phi = \{y \in T_x M \mid \Phi(x, y) = 1\}$$

Let $W = W(x)$ be a vector field on M . Consider the parallel shift $S_{\Phi+\{W\}}$ of S_Φ along W . It is easy to see that $y \in S_{\Phi+\{W\}}$ if and only if $\Phi(x, y - W_x) = 1$. Further, when $\Phi(x, -W_x) < 1$, $S_{\Phi+\{W\}}$ contains the origin x of $T_x M$. In this case, it is easy to see that, for any $y \in T_x M \setminus \{0\}$, there is a unique positive number $F = F(x, y) > 0$ such that

$$\frac{y}{F(x, y)} \in S_{\Phi+\{W\}}$$

that is, $F = F(x, y)$ satisfies the following

$$\Phi\left(x, \frac{y}{F(x, y)} - W_x\right) = 1$$

It is not difficult to show that $F = F(x, y)$ is a regular Finsler metric. On the other hand, when $\Phi(x, -W_x) = 1$, $S_{\Phi+\{W\}}$ must pass through the origin x of $T_x M$. In this case, for any $y \in A_x := \{y \in T_x M \mid g_W(y, W) > 0\}$, there is still a unique positive number $F = F(x, y)$ such that $F = F(x, y)$ satisfies (1.1), equivalently, $F = F(x, y)$ satisfies (1.2). Obviously, such $F = F(x, y)$ is a Finsler metric with singularity and not regular on whole TM . We call $F = F(x, y)$ a conic Finsler metric (for the definition and fundamental properties of conic Finsler metrics, see [7]). Actually, the conic Finsler metric $F = F(x, y)$ is regular on conic domain $A := \bigcup_{x \in M} A_x \subset TM$. Inevitably, it is natural to meet conic Finsler metrics in studying natural sciences.

In general, Finsler metric $F = F(x, y)$ obtained from (1.2) is called a solution of the Zermelo navigation problem with navigation data (Φ, W) ([8]).

Modeling of the aircraft motion

In order to define the place of an aircraft in space, the coordinates of the center of mass of the aircraft and the position of the connected coordinate system with respect to a stationary coordinate system (for example that one can be the coordinate system connected to the ground) have to be given. To describe aircraft dynamics can be defined six ordinary differential equations (ODEs). The first three of them are the so called Euler ODEs

$$\frac{d\gamma}{dt} = \omega_x - (\omega_y \cos\gamma - \omega_z \sin\gamma) \tan v$$

$$\frac{dv}{dt} = \omega_y \sin\gamma + \omega_z \cos\gamma$$

$$\frac{d\psi}{dt} = (\omega_y \cos\gamma - \omega_z \sin\gamma) \sec v$$

where γ denotes roll angle, v the pitch angle, ψ the yaw angle and $\omega_x, \omega_y, \omega_z$ are the angular velocities.

The other three kinematic equations give a relation between the derivatives of the coordinates of the center of mass in normal ground coordinate system $OX_g Y_g Z_g$ with projection of the linear and angular velocities.

$$\frac{dx_g}{dt} = V_x \cos\psi \cos v + V_z (\sin\psi \sin\gamma - \sin v \cos\psi \cos\gamma) - V_y (\sin v \cos\psi \sin\gamma + \sin\psi \cos\gamma)$$

$$\frac{dy_g}{dt} = V_x \cos v \sin\psi - V_z (\cos\psi \sin\gamma + \sin v \sin\psi \cos\gamma) - V_y (\cos\psi \cos\gamma - \sin v \sin\psi \sin\gamma)$$

$$\frac{dz_g}{dt} = V_x \sin v + V_z \cos v \cos\gamma + V_y \cos v \sin\gamma$$

Many of the components of the above stated equations are not a matter of interest of the present problem, since it concerns the airplane dynamics as a moving mass point and there is no need to be considered the characteristics of the body shape ($\gamma = 0$, the yaw angle- $\psi = \theta$ -heading angle, the pitch angle- $v = \alpha$ -angle of attack). Then the system of ODEs transforms into:

$$\frac{dx_g}{dt} = V \cos\theta \cos\alpha$$

$$\frac{dy_g}{dt} = V \cos\alpha \sin\theta$$

$$\frac{dz_g}{dt} = V \sin\alpha$$

Passing through a plane inclined at $\alpha = \pi$:

$$\frac{dx_g}{dt} = -V \cos\theta$$

$$\frac{dy_g}{dt} = -V \sin\theta$$

During the flight of an aircraft as a mass point, except the coordinates, a change of mass is observed, caused by fuel consumption. For a short range flight this change of mass is negligibly small and the assumption as $w = \text{const}$, $\frac{dw}{dt} = 0$ is a very suitable one. But for long range flights the change of mass is very significant and must be taken into consideration.

$\frac{dm}{dt} = \frac{-C_s}{g}$, where C_s -fuel flow rate and g - gravitational acceleration.

Therefore, assuming $g = \text{const}$, and putting $f = \frac{-C_s}{g}$, the expression can be simplified.

$$\frac{dw}{dt} = -f$$

In the process of model preparation one more assumption have to be analyzed, namely that the main part of the passenger aircraft flight is the so called cruise flight. Cruise flight is an airplane flight in a horizontal plane with cruising velocity. The characteristics of this velocity are:

- balance of the acting forces, which in horizontal plane is given by $T_h = D$, where T_h - thrust and D -drag
- very slight dependence of fuel flow rate on the velocity of the airplane, and
- Minimum fuel burn per kilometer (maximum range).

As a result of that overview, a system of three ODEs is achieved, such that the optimal trajectory, should solve them under some additional conditions.

$$\dot{x} = -V \cos\theta$$

$$\dot{y} = -V \sin\theta$$

$$\dot{w} = -f$$

Statement of the Optimal Control Problem

The purpose of the current work is to determine an aircraft trajectory, from an initial point to a final destination point and more precisely to determine a trajectory of the center of mass of an aircraft in a 2-D horizontal coordinate system. In here the approach of [2,3](Sridhar, Chen, Ng and Linke) is followed. It is required by the authors that to establish such kind of trajectory travelling time between the two points, taking into consideration the presence of wind, while avoiding regions of airspace, which are forbidden to be used must be minimized. These regions can be no-go-areas or parts of the airspace, which represent any kind of danger for the aircraft or facilitate persistent contrail formation. First the wind is introduced in the ODEs, characterizing the motion of a mass point in horizontal plane.

$$\dot{x} = -V \cos \theta + u(x, y)$$

$$\dot{y} = -V \sin \theta + v(x, y)$$

$$\dot{w} = -f$$

where $u(x, y)$ and $v(x, y)$ are the wind velocity components along the x- and y - axis, which can be functions of both x and y coordinates and also sometimes of the time t. The horizontal plane trajectory is optimized as finding the control variable - heading angle θ , for which cost functional attains minimum. The undesirable regions are formulated as soft state constraints. As a result a constrained optimization problem needs to be solved. To do that there are numerous approaches as: the Lagrange multiplier method, the augmented Lagrange multiplier method, penalty function method and so on. Considering the nature of the particular problem a penalty function method was chosen.

Conclusion

The Mathematical formulation of the Zermelo Navigation problem using Riemannian and Finsler geometry was introduced. Some special situations with the complete solution of the problem were mentioned. The research in other special situations and other possible ways of the description of the problem were indicated. The potential for the further study is huge because the problem is too complex to be solved explicitly in full generality.

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