Amalgamative Application of Derivative Operator and Differential Sub-Ordination in Developing Theorems

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ABSTRACT:The authors have recently introduced a new generalized derivatives operator $\mu_{\lambda_1 \lambda_2}{}^{n,m}$, which generalized many well-known operators studied earlier by many different authors. The trend of finding new operators such as the differential or the integral operators has attached many researchers and in fact, has become necessary to some of the researchers. The aim of this paper is to the equality relation.

$(1+n)\mu_{\lambda_1 \lambda_2}^{n+1,m}f(z) = z\left(\mu_{\lambda_1 \lambda_2}^{n,m}f(z)\right)' + n\left(\mu_{\lambda_1 \lambda_2}^{n+1,m}f(z)\right)$, to discuss some interesting results using the technique of differential subordination. The results include the subordination and the indlusion. Whenever $n = 0, \lambda_2 = 0$ we have results given by Oros [10].

1. INTRODUCTION AND DEFINITIONS

Assume that A is denoting the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, a_k$ (1.1)
is complex number which are analytic in the open unit disc $U = \{z: z \in \mathbb{C}: |z| < 1\}$ on the complex plane $\mathbb{C}$. Assume that $S, S^*(\alpha), C(\alpha)(0 \leq \alpha \leq 1)$ indicates the subclasses of A consisting of functions that are univalent, starlike of order α and convex of order $\alpha in\ U$, respectively. In particular, the classes $S^*(0) = S^* and\ C(0) = C$ are the familiar classes of starlike and convex functions in U, respectively.

And a function$f \in C(\alpha) if\ Re[1 + \frac{zf''}{f'} > \alpha$. Furthermore a function f analytic in U

is said to be convex if it is univalent and $f(U)$ is convex. Assume that $H(U)$ is the class of holomorphic function in unit disc $U == \{z: z \in \mathbb{C}: |z| < 1\}$.

Recognize $A_n = \{f \in H(U): f(z) = z + a_{n+1}z^{n+1} + \cdots, (z \in U)\}, with\ A_1 = A.$ For $a \in \mathbb{C}\ and\ n \in \mathbb{N} = \{1,2,3, \dots\}$

$assume\ that\ H[a,n] = \{f \in H(U): f(z) = z + a_n z^n + a_{n+1}z^{n+1}+, \dots, (z \in U)\}.$
Let be given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, a_k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, analytic in the open unit disc $U = \{z: z \in \mathbb{C}: |z| < 1\}$. Then the Hadamard product (or convolution) $f * g$ of two functions $f, g\ is\ defined\ by\ f(z) * g(z) = (f * g)(z) = z + a_k b_k z^k$. Next, we state basic ideas on subordination. If f $and\ g$ are analytic in U, then the function f is said to be subordinate to g, and can be written as $f \prec g\ or\ f(z) \prec g(z)\ (z \in U)$, if and only if there exists the Schwarz function $w, analytic\ in\ U, w(0) = 0\ and\ |w(z)| < 1$ such that
$f(z) = g(w(z))(z \in U)$, Furthermore if g is univalent in$U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.
Suppose $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U, if P is analytic in U satisfies the (second order) differential subordination $\psi[p(z), zp'(z), z^2 p''(z); z] \prec h(z), \quad (z \in U), \qquad (1.2)$
Then p is known as solution of the differential subordination.
The univalent function q is known as dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q\ for\ all\ p\ satisfying\ (1.2) dominant\ \tilde{q}\ that\ satisfies$
$\tilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant of (1.2) . (Note that the best dominant is unique up to a rotation of U).
Now, $(x)_k$ *denotes the Pochhammer symbol* (*or the shifted factorial*) *defined by*
$(x)_k = \begin{cases} \{1\ for\ k = 0, x \in \mathbb{C}\{0\}, \\ x(x+1)(x+2) \dots (x+k-1) for\ k \in \mathbb{N} = \{1,2,3, \dots, \} and\ x \in \mathbb{C}. \end{cases}$ In [1] the authors introduced and studied the generalization derivative operator $\mu_{\lambda_1 \lambda_2}{}^{n,m}$ which given as the following:

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Definition 1. Suppose for $f \in A$ the generalized derivative operator $\mu_{\lambda_1\lambda_2}{}^{n,m}$ is defined by $\mu_{\lambda_1\lambda_2}{}^{n,m}: A \to A$
$\mu_{\lambda_1\lambda_2}{}^{n,m} f(z) = \varphi_{\lambda_1\lambda_2}{}^m(z) * R^n f(z), \ (z \in U),$

$R^n f(z)$ denotes the Ruscheweyh derivative operator and $\varphi^m_{\lambda_1\lambda_2}(z)$ analytic function defined by

$$\varphi^m_{\lambda_1\lambda_2} f(z) = z + \sum_{n=2}^{\infty} \frac{(1 + \lambda_1(k-1))^{m-1} z^k}{(1 + \lambda_2(k-1))^m}, \qquad (z \in U),$$

$$\mu_{\lambda_1\lambda_2}{}^{n,m} f(z) = z + \sum_{n=2}^{\infty} \frac{(1 + \lambda_1(k-1))^{m-1} z^k c(n,k) a_k z^k}{(1 + \lambda_2(k-1))^m}, (z \in U),$$

where $n, m \in \mathbb{N}_0 = \{0,1,2,,...\}, \lambda_2 \geq \lambda_1 \geq 0 \ and \ c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}$. Special cases of this operator include the Ruscheweyh derivative operator in the cases is as follows:
$\mu^{n,1}_{\lambda_1,0} \equiv \mu^{n,m}_{0,0} \equiv \mu^{n,m}_{0,\lambda_2} \equiv R^n$ [14], the Salagean derivative operator $\mu^{0,m+1}_{1,0} \equiv S^n$ [15], the generalized Ruscheweyh derivative operator $\mu^{n,2}_{\lambda_1,0} \equiv R^n_\lambda$ [3], the generalized Salagean derivative operator introduced by A1-Oboudi $\mu^{0,m+1}_{\lambda_1,0} \equiv S^n_\beta$ [2], and the generalized A1-Shaqsi and Darus derivative operator $\mu^{0,m+1}_{\lambda_1,0} \equiv D^n_{\lambda,\beta}$ [4]. Now, Let us remind the well known Carlson-Shaffer operator $L(a,c)$ [5] associated with the incomplete beta function $\varphi(a,c;z)$, defined as follows:

$$L(z,c) = A \to A, L(a,c)f(z) := \varphi(a,c;z) * f(z)(z \in U),$$
$$where \ \varphi(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \ ,$$

It is easily seen that $\mu^{0,1}_{\lambda_1,0} f(z) = \mu^{0,m}_{0,0} f(z) = \mu^{0,0}_{0,\lambda_2} f(z) = f(z)$ and $\mu^{1,1}_{\lambda_1,0} f(z) = \mu^{1,m}_{0,0} f(z) = \mu^{0,0}_{0,\lambda_2} f(z) = zf'(z)$ and also $\mu^{\alpha-1,0}_{\lambda_1,0} f(z) = \mu^{\alpha-1,m}_{0,0} f(z) = \mu^{\alpha-1,0}_{0,\lambda_2} f(z)$ where $\alpha = 1,2,3,,....$

To prove our results, we need the following equality:

$$(1+n)\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z) = z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' + n\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right), (z \in U). \qquad (1.3)$$

where $n, m \in \mathbb{N}_0 = \{0,1,2,...\}, \lambda_2 \geq \lambda_1 \geq 0$. Next, we give another definition as follows:

Definition 2. For $n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, and \ 0 \leq \alpha \leq 1, we \ let \ R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$ denote the class of functions $f \in A$ which satisfy the condition $Re\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' > \alpha, (z \in U).$ (1.4)
It is clear that the classes $R^{0,1}_{\lambda_1,0}(\alpha) \equiv R(\lambda_1, \alpha)$ the class of function $f \in A$ satisfying
$Re(\lambda_1 z f^n(z) + f'(z) > \alpha, (z \in U). studied \ by \ Ponnusamy$ [13] $and \ many \ others.$
In the present paper, we shall use the method of differential subordination to derive certain properties of the generalization derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2} f(z)$. Note that, differential subordination has been studied by various authors, and here we follow similar work done by Oros [9] and Oros and Oros [11].

2. MAIN RESULTS
In proving our main results, we need the following Lemmas.

Lemma 1. ([6], p. 71). Let h be analytic, univalent, convex in U, with $h(0) = \alpha, \gamma \neq 0$ and $Re\gamma \geq 0. If \ p \in H\{a,n\}$ and $p(z) + \frac{zp'(z)}{\gamma} \prec h(z), (z \in U), \ p(z) \prec q(z) \prec h(z), \ (z \in U)$, where $q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\left(\frac{\gamma}{n}\right)-1} dt$, $(z \in U)$. The function q is convex and is the best $(a,n) - dominant.$

Lemma 2 ([7]). Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, where $\alpha > 0$ and n is a positive integer. $if \ p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, (z \in U)$, is holomorphic in U and $p(z) + \alpha z p'(z) \prec h(z), (z \in U), then \ p(z) \prec g(z).$

Lemma 3 ([8]). Let $f \in A, Re\left[1 + \frac{zf^n(z)}{f'(z)}\right] > -\frac{1}{2}, then \ 2/z \int_0^z f(t)dt, (z \in U \ and \ z \neq 0)$, belongs to the convex functions. Now we begin with the first result as the following.

Theorem 1. Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}, (z \in U)$, be convex in U, with $h(0) = 1$ and $0 \leq \alpha < 1$. If $n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0$, and the differential sub-ordination.

$$\left(\left(\mu_{\lambda_1,\lambda_2}^{n+1,m}f(z)\right)' \prec h(z), (z \in U), \tag{2.1}$$

$$\left(\left(\mu_{\lambda_1,\lambda_2}^{n+1,m}f(z)\right)' \prec q(z) = 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}} \text{ where } \sigma \text{ is given by}$$

$$\sigma(x) = \int_0^z \frac{t^x}{1+t} dt, (z \in u). \tag{2.2}$$

The function q is convex and is the best dominant.
Proof. By differentiating (1.3) , with respect to z, we obtain

$$\left(\mu_{\lambda_1,\lambda_2}^{n+1,m}f(z)\right)' = \frac{(1+n)\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)\right)' + z\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)''\right)}{(1+n)} \tag{2.3}$$

Using (2.3) in (2.1), differential subordination (2.1) becomes

$$\frac{(1+n)\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)\right)' + z\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)''\right)}{(1+n)} \prec h(z) = \frac{1+(2\alpha-1)z}{1+z} \tag{2.4}$$

$$p(z) = \left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)\right)' = [z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)^{n-1}c(n,k)a_k z^k}{(1+\lambda_2(k-1))^m}]$$

$$= 1 + p_1 z + p_2 z^2 +, \dots, (p \in H[1,1], z \in U). \tag{2.5}$$

Using (2.5) in (2.4),the differential sub-ordination becomes:

$$p(z) + \frac{zp'(z)}{1+n} \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}. \text{ By using Lemma 1, we have}$$

$$p(z) \prec q(z) = \frac{(n+1)\int_0^z h(t)t^n dt}{z^{n+1}} = \frac{(n+1)\int_0^z \left[\frac{1+(2\alpha-1)t}{1+t}\right]t^n dt}{z^{n+1}}$$

$$= \frac{(n+1)}{z^{n+1}}[\sigma(n) + (2\alpha-1)\int_0^z \frac{t^{n+1}}{1+t} dt]$$

$$= (2\alpha - 1) + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}} \text{ where } \sigma \text{ is given by (2.2)}, so \text{ we get}$$

$\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)\right)' \prec q(z) = (2\alpha - 1) + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}$ the functions q is convex and is the best dominant. The proof is complete.

Theorem 2. If $n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, then we have $R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha) \subset R_{\lambda_1,\lambda_2}^{n+1,m}(\delta)$ where $\delta = (2\alpha - 1) + 2(n+1)(1-\alpha)\sigma(n)$. And σ is given by (2.2).

Proof. Let $f \in R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha)$, then from (1.4) we have $Re\left(\mu_{\lambda_1,\lambda_2}^{n+1,m}f(z)\right)' > \alpha, (z \in U)$. which equivalent is to $\left(\mu_{\lambda_1,\lambda_2}^{n+1,m}f(z)\right)' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}$ using Theorem 1, we have $\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)\right)' \prec q(z) = 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}$ since q is convex and $q(U)$ is Symmetric with respect to the real axis, we deduce $Re\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)\right)' \prec Req(1) = \delta = \delta(\alpha, \lambda_1) = 2\alpha - 1 + 2(n+1)(1-\alpha)\sigma(n)$.
For which we deduce $R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha) \subset R_{\lambda_1,\lambda_2}^{n,m}(\delta)$. This completes the proof of the Theorem 2.

Theorem 3. Let q be a convex function in U, with $q(0) = 1$ and let $h(z) = q(z) + \lambda_1 zq'(z), (z \in U)$ (2.6) then $\left(\mu_{\lambda_1,\lambda_2}^{n+1,m}f(z)\right)' \prec h(z), (z \in U)$ then $\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)\right)' \prec q(z)$. and this result is sharp.

Proof. Let $p(z) = \left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)'$ using (2.3), $differential\ sub-ordination$

(2.6)$becomes\ p(z) + \frac{zp'(z)}{1+n} \prec h(z) = q(z) + \lambda_1 z q'(z), (z \in U). Hence \left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' \prec q(z),$

$(z \in U)\ And\ the\ result\ is\ sharp. This\ completes\ the\ proof\ of\ the\ theorem. We\ give\ a$ simple application for

Theorem 3.

Example 1. For $n = 1, m = 0, \lambda_2 \geq \lambda_1 \geq 0, q(z) = \frac{1+z}{1-z}, f \in A\ and\ z \in U\ and\ applying$ Theorem 3, we have

$$h(z) = \frac{1+z}{1-z} + \lambda_1 z \left[\frac{1+z}{1-z}\right]' = \frac{1+2\lambda_1 z - z^2}{(1-z)^2}\ After\ that\ we\ find$$

$$\left(\mu_{\lambda_1,\lambda_2}^{1,0} f(z)\right) = z + \sum_{k=2}^{\infty} \frac{c(1,k)a_k z^k}{1 + \lambda_1(k-1)}, \left(\mu_{\lambda_1,\lambda_2}^{1,0} f(z)\right)'$$

$$= 1 + \sum_{k=2}^{\infty} \frac{k^2 a_k z^k}{1 + \lambda_1(k-1)} = f'(z) * \left(\varphi_{\lambda_1,\lambda_2}^{0} f(z)\right)'\ And\ similarly\ we\ shall\ find$$

$$\left(\mu_{\lambda_1,\lambda_2}^{2,0} f(z)\right) = z + \sum_{k=2}^{\infty} \frac{c(2,k)a_k z^k}{1 + \lambda_1(k-1)} = z + \sum_{k=2}^{\infty} \frac{k(k+1)a_k z^k}{2(1 + \lambda_1(k-1))}\ Then$$

$\left(\mu_{\lambda_1,\lambda_2}^{2,0} f(z)\right)' = 1 + \sum_{k=2}^{\infty} \frac{k^2(k+1)a_k z^{k-1}}{2(1+\lambda_1(k-1))} = f'(z) * \left(\varphi_{\lambda_1,\lambda_2}^{0} f(z)\right)' * [1 + \sum_{k=2}^{\infty} \frac{(k+1)z^{k-1}}{2}]$ From Theorem 3 we

deduce $f'(z) * \left(\varphi_{\lambda_1,\lambda_2}^{0} f(z)\right)' * [1 + \sum_{k=2}^{\infty} \frac{(k+1)z^{k-1}}{2}] \prec \frac{1+2\lambda_1 z - z^2}{(1-z)^2}, implies$

$$f'(z) * \left(\varphi_{\lambda_1,\lambda_2}^{0} f(z)\right)' \prec \frac{1+z}{1-z}, (z \in U).$$

Theorem 4. $Let\ q\ be\ a\ convex\ function\ in\ U, with\ q(0) = 1\ and\ let\ h(z) = q(z) + zq'(z),$
   $(z \in U). If\ n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, f \in A\ and\ satisfies\ the\ differential\ sub-ordination$
   $\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' \prec h(z), \frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z), (z \in U). And\ the\ result\ is\ sharp.$

Proof. $Let p(z) = \frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} = \frac{z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1}c(n,k)}{(1+\lambda_2(k-1))^m} a_k z^k}{z}$ (2.11)

$= 1 + p_1 z^1 + p_2 z^2 + \cdots (p \in H[1,1],$
$(z \in U). Differentiating\ , with\ respect\ to\ z, we\ obtain$
$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' = p(z) + zp'(z), (z \in U).$ Using   (2.12),

 differential sub-ordination (2.10), becomes $p(z) + zp'(z) \prec h(z) = q(z) + zq'(z)$,using Lemma 2,   we deduce

$p(z) \prec q(z), (z \in U), amd\ using\ (2.11), we\ have\ \frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z), (z \in U)$, This proves Theorem 4. We give a simple application for Theorem 4.

Example 2. For $n = 1, m = 0, \lambda_2 \geq \lambda_1 \geq 0, q(z) = \frac{1}{1-z}, f \in A\ and\ z \in U\ and\ applying\ Theorem\ 4$, we have

$$h(z) = \frac{1}{1-z} + z \left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2},$$

After that we find $\mu_{\lambda_1,\lambda_2}^{1,0} f(z) = z + \sum_{k=2}^{\infty} \frac{ka_k z^k}{1 + \lambda_1(k-1)}, f(z) * \emptyset_{\lambda_1,\lambda_2}^{0}(z) * (z + \sum_{k=2}^{\infty} kz^k).$

And from example 1, we have $\left(\mu_{\lambda_1,\lambda_2}^{1,0} f(z)\right)' = f'(z) * \left(\emptyset_{\lambda_1,\lambda_2}^{0}(z)\right)'.$

Implies $\frac{f(z) * \emptyset_{\lambda_1,\lambda_2}^{0}(z) * (z + \sum_{k=2}^{\infty} kz^k)}{z} \prec \frac{1}{(1-z)}.$

Theorem 5. Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$, $\qquad (z \in U)$.

Be convex in U, with $h(0) = 1$ and $0 \le \alpha < 1$. if $n, m \in \mathbb{N}_0, \lambda_2 \ge \lambda_1 \ge 0, f \in A$ and the differential subordination

$$(\mu_{\lambda_1,\lambda_2}^{n,m} f(z))' \prec h(z), \tag{2.13}$$

Then $\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)In(1+z)}{z}$. The function q is convex and is the best dominant.

Proof. Let $p(z) = \frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z}$,

$$= \frac{z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1} c(n,k)}{(1+\lambda_2(k-1))^m} a_k z^k}{z}, \tag{2.14}$$

$$= 1 + p_1 z + p_2 z^2 + \cdots (p \in H[1,1], z \in U).$$

Differentiating (2.14), with respect to z, we obtain

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' = p(z) + zp'(z), \qquad (z \in U). \tag{2.15}$$

Using (2.15), differential subordination (2.13), becomes

$$p(z) = zp'(z) \prec h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \qquad (z \in U).$$

From Lemma 1, we deduce $\quad p(z) \prec q(z) = \frac{1}{z} \int_0^z h(t)dt = \frac{1}{z} \int_0^z \left(\frac{1+(2\alpha-1)t}{1+t}\right) dt$

$$= \frac{1}{z}\left[\int_0^z \frac{1}{1+t} dt + (2\alpha - 1) \int_0^z \frac{t}{1+t} dt\right] = 2\alpha - 1 + \frac{2(1-\alpha)In(1+z)}{z}.$$

Using (2.14), we have $\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)In(1+z)}{z}$.

The proof is complete. From Theorem 5, we deduce the following corollary:

Corollary: if $f \in R_{\lambda_1,\lambda_2}^{n,m}(\alpha)$ then $\operatorname{Re}\left(\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z}\right) > 2\alpha - 1 + 2(1-\alpha)In\,2, \quad (z \in U)$.

Proof. Since $f \in R_{\lambda_1,\lambda_2}^{n,m}(\alpha)$, from Definition 2 $Re\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' > \alpha, \qquad (z \in U)$, Which is equivalent to $\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}$. Using Theorem 5, we have

$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)In(1+z)}{z}$ Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce $\operatorname{Re}\left(\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z}\right) > Req(1) = 2\alpha - 1 + 2(1-\alpha)In\,2, \qquad (z \in U)$.

Theorem 6. Let, $h \in H(U)$, with $h(0) = 1, h'(0) \ne 0$ which satisfy the inequality

$$Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \qquad (z \in U).$$

if $n, m \in \mathbb{N}_0, \lambda_2 \ge \lambda_1 \ge 0, f \in A$ and satisfies the differential subordination

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' \prec h(z), \qquad (z \in U). \tag{2.16}$$

Then $\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt$.

Proof. Let $p(z) = \frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} =$

$$\frac{z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^{m-1} c(n,k)}{(1+\lambda_2(k-1))^m} a_k z^k}{z}, \tag{2.17}$$

$$= 1 + p_1 z + p_2 z^2 + \cdots, (p \in H[1,1], z \in U).$$

Differentiating (2.17), with respect to z we have

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' = p(z) + zp'(z), \qquad (z \in U). \tag{2.18}$$

Using (2.18), the differential subordination (2.16) becomes $(z) = zp'(z) \prec h(z), \qquad (z \in U)$

From Lemma 1, we deduce $p(z) \prec q(z) = \frac{1}{z} \int_0^z h(t)dt$. With (2.17), we obtain

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt.$$

From Lemma 3, we have that the function q is convex, and from Lemma 1, q is the best dominant for subordination (2.16). This completes the proof of Theorem 6.

3. CONCLUSION

We remark that several subclasses of analytic univalent functions can be derived using the operator $\mu_{\lambda_1,\lambda_2}^{n,m}$ and studied their properties.

References

Al-Abbadi, M. H. and M. Darus, 2009. Differential subordination for new generalised derivative operator , (submitted).

Al-Oboudi, F.M, 2004. On univalent functions defined by a generalised Salagean Operator, Into, J. Math. Math. Sci. 27, 1429-1436.

Al-Shaqsi, K. and M. Darus, 2009. On univalent functions with respect to k-summetric points defined by a generalization ruscheweyh derivative operators, Journal of Analysis and Application, 7(1), 53-61.

Al-Shaqsi, K. and M. Darus, 2008. Differential Subordination with generalised derivative operator, Int. J. Comp. Math. Sci, 2(2), 75-78.

Carlson, B. C. and D.B. Shaffer, 1984. Starlike and prestarlike hypergeometric function, SIAM J. Math. Anal. 15, 737-745.

Miller, S. S. and P.T. Mocanu, 2000. Differential subordinations: Theory and applications, in: Series on monographs and textbooks in pure and applied mathematics, 225, Marcel Dekker, New York.

Miller, S. S. and P. T. Mocanu, 1985. On some classes of first-order differential subordinations, Michigan Math. J. 32, 185-195.

Mocanu, P. T. and T. Bulboaca, Gr. St. Salagean, 1999. Teoria geometrica a functiilor univalente, Casa Cartii de Stiinta, Cluj-Napoca.

Oros, G.I, 2005. A class of holomorphic functions defined using a differential operator, General Mathematics, 13(4), 13-18.

Oros, G. I, 2003. On a class of holomorphic functions defined by the Ruscheweyh derivative, Int. J. Math. Math. Sci. 65, 4139-4144.

Oros, G. and G. I. Oros, 2004. Differential superordination defined by Salagean operator, General Mathematics, 12(4), 3-10.

Padmanabhan, K. S. and R. Manjini, 1986. Certain applications of differential subordination. Publ. Inst. Math. (N.S.), Tome 39(53), 107-118.

Ponnusamy, S, 1992. Differential subordination and starlike functions, Complex Variables Theory Appl, 19(3), 185-195.

Ruscheweyh, St, 1975. New criteria for univalent functions, Proc. Amer. Math. Soci. Vol. 49, pp. 109-115.

Salagean, G. S, 1983. Subclasses of univalent functions, Lecture Notes in Math. (SpringerVerlag), 1013, 362-372.