Some Classes of Schwarzian Functions and its Coefficient Inequality that is Sharp

Gurmeet Singh
Department of Mathematics
Khalsa College, Patiala-147002,Punjab,India
E-mail: meetgur111@gmail.com

Misha Rani
Research Fellow,Department of Mathematics
Punjabi University,Patiala-147002, Punjab, India
E-mail: mishagargsamana@gmail.com

Abstract - In our present work, we defined an inequality called Fekete – Szegö Inequality for the functions $f(z)$ associated with the class of starlike functions, along with their extremal functions and some subclasses.

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1. Introduction - As we are taking into consideration the concepts of geometric function theory and in this theory , there is a theorem called Riemann Mapping theorem, which we can say , is the main pillar of this theory , proved by Koebe [7]. From this theorem, a conjecture called Bieberbach conjecture ([5],[12], [13]) was produced. Basically, this conjecture was introduced by L.Bieberbach [1] but finally proved by Louis De Branges[2]. When mathematicians were tackling with this conjecture, then an inequality originates called Fekete – Szegö Inequality, which was given by M. Fekete and G. Szegö [4]. Till now many researchers have solved this inequality for different classes and subclasses of starlike functions, convex functions, close - to - convex functions and for many other functions. But here, we establish this inequality for different subclasses of starlike functions.

Firstly , we consider some fundamental classes and results:-

A consists of the family of analytic functions f with the normalization $f(0) = 0$, $f'(0) = 1$ and having functions of the type $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; S consists of the family of functions f normalized by $f(0) = 0$, $f'(0) = 1$ where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; univalent in the open disk $E = \{ z \in C: |z| < 1\}$ and $S^*(\phi)$ be the class of functions in $f \in S$, introduced by Ma and Minda [9] , for which $\frac{z f'(z)}{f(z)} \prec \phi(z)$.

In this equation, "$\prec$ " denotes subordination [which states that if $f(z)$ and $g(z)$ are two analytic functions , then there exists a Schwarzian function $w(z)$ (which is analytic in E) in such a way that $|w(z)| < 1$, $w(0) = 0$ and $f(z) = g\{w(z)\}$; $z \in E$, then the function $f(z)$ is subordinate to $g(z)$ and we write it as $f(z) \prec g(z)$.

The concept of subordination was given by Lindelof [8] . Here, $\phi(z)$ is an analytic function with positive real part on *E*; which maps the unit disk *E* onto a region starlike with respect to 1 as well as symmetric with respect to real axis, satisfying conditions ϕ (0) = 0 and $\phi'(0) > 0$ and Schwarzian function is an analytic function of the type $w(z)= \sum_{n=1}^{\infty} c_n z^n$ with the conditions $w(0) = 0$ and $|w(z)| < 1$.

Miller et. al. [10] proved the conditions

$$|c_1| \leq 1,\ |c_2| \leq 1-|c_1|^2$$

for the above defined bounded analytic functions.

The class introduced below, denoted by $TS^*[\alpha,\beta]$, is a subclass of S^* , so satisfying the condition

$$\left\{(1-\alpha)\frac{f(z)}{z} + \alpha\ \frac{z f'(z)+\beta z^2 f''(z)}{f(z)}\right\} \prec \phi\ (z) \qquad \ldots(1.1)$$

The necessary and sufficient condition for any function to be starlike is

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0; z \in E$$

This condition was introduced by Duren [3] and the result $|a_n| \leq n$ for univalent starlike functions was given by Nevanlinna [11] .

In this paper, we define the class $TS^*[\alpha,\beta]$ with new subclass denoted by $TS^*[\alpha,\beta,\delta]$.

My work is related to the work of Keogh and Merkes [6] with parameters $\alpha = 1$, $\beta = 0$, $\delta = 1$.

2. Main Results:-

THEOREM 1:- Let $f(z) \in TS^*[\alpha,\beta]$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$; *w(z)* is a *Schwarzian function* , *then* $|a_3 - \mu\ a_2^2| \leq$

$$\begin{cases} \frac{2\,(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)}{(1+2\alpha\beta)^2\{1+(1+6\beta)\alpha\}} - \frac{4\mu}{(1+2\alpha\beta)^2}\ ;\ \mu \leq \frac{\alpha(1+2\beta)}{1+(1+6\beta)\alpha}\ ; \\ \frac{2}{1+(1+6\beta)\alpha} \quad ; \frac{\alpha(1+2\beta)}{1+(1+6\beta)\alpha} \leq \mu \leq \frac{\alpha(1+2\beta)+(1+2\alpha\beta)^2}{[1+(1+6\beta)\alpha]}\ ; \\ \frac{4\mu}{(1+2\alpha\beta)^2} - \frac{2\,(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)}{(1+2\alpha\beta)^2[1+(1+6\beta)\alpha]}\ ;\ \mu \geq \frac{\alpha(1+2\beta)+(1+2\alpha\beta)^2.}{[1+(1+6\beta)\alpha]} \end{cases}$$

The result is sharp.

PROOF:- By definition of $TS^*[\alpha,\beta]$, given by (1.1)

and using $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots$

and $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$.

we get

$1 + (1 + 2\alpha\beta)\, a_2 z + [1 + \alpha(1 + 6\beta)\, a_3] z^2 - \alpha(1 + 2\beta)\, a_2^2 z^2 + \ \ldots$

$$= 1 + 2c_1 z + 2(c_2 + c_1^2) z^2 + \cdots \quad .$$

After comparing like coefficients , we can easily obtain

$$a_2 = \frac{2\, c_1}{1 + 2\alpha\beta} \quad \text{and} \quad a_3 = \frac{2\left[c_2 + \left(1 + \frac{2\alpha(1+2\beta)}{(1+2\alpha\beta)^2}\right) c_1^2\right]}{1 + (1+6\beta)\alpha} \quad .$$

Using these values, we can construct

$$a_3 - \mu\, a_2^2 = \frac{2}{1+(1+6\beta)\alpha}\, c_2 + \left(\frac{2\left(1 + \frac{2\alpha(1+2\beta)}{(1+2\alpha\beta)^2}\right)}{1+(1+6\beta)\alpha} - \frac{4\mu}{(1+2\alpha\beta)^2} \right) c_1^2.$$

By applying mode on both sides , we can easily find out that

$$|a_3 - \mu\, a_2^2| \leq \frac{2}{1+(1+6\beta)\alpha}\, |c_2| + \left| \frac{2\left(1 + \frac{2\alpha(1+2\beta)}{(1+2\alpha\beta)^2}\right)}{1+(1+6\beta)\alpha} - \frac{4\mu}{(1+2\alpha\beta)^2} \right| |c_1|^2 \ .$$

Now, by using $|c_2| \leq 1 - |c_1|^2$, we get

$$|a_3 - \mu\, a_2^2| \leq \frac{2}{1+(1+6\beta)\alpha} + \left\{ \left| \frac{2\left(1 + \frac{2\alpha(1+2\beta)}{(1+2\alpha\beta)^2}\right)}{1+(1+6\beta)\alpha} - \frac{4\mu}{(1+2\alpha\beta)^2} \right| - \frac{2}{1+(1+6\beta)\alpha} \right\} |c_1|^2 \ .$$

Case 1:- If $\mu \leq \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)}{2\{1+(1+6\beta)\alpha\}}$.

Then, $|a_3 - \mu\, a_2^2| \leq \ \frac{2}{1+(1+6\beta)\alpha} + \left\{ \frac{4\alpha(1+2\beta)}{(1+2\alpha\beta)^2\{1+(1+6\beta)\alpha\}} - \frac{4\mu}{(1+2\alpha\beta)^2} \right\} |c_1|^2.$

Subcase 1(a) :- When $\mu \leq \frac{\alpha(1+2\beta)}{1+(1+6\beta)\alpha}$.

Then, by using $|c_1| \leq 1$, we get

(1.2) $$|a_3 - \mu\, a_2^2| \ \leq \frac{2\,(1 + 2\alpha + 4\beta^2\alpha^2 + 8\alpha\beta)}{(1+2\alpha\beta)^2\{1+(1+6\beta)\alpha\}} - \frac{4\mu}{(1+2\alpha\beta)^2} \quad .$$

Subcase 1(b) :- When $\mu \geq \frac{\alpha(1+2\beta)}{1+(1+6\beta)\alpha}$.

Then,

(1.3) $$|a_3 - \mu\, a_2^2| \ \leq \ \frac{2}{1+(1+6\beta)\alpha} \ .$$

Case 2 :- If $\mu \geq \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)}{2\{1+(1+6\beta)\alpha\}}$.

Then, $|a_3 - \mu a_2^2| \leq \frac{2}{1+(1+6\beta)\alpha} + \left\{\frac{4\mu}{(1+2\alpha\beta)^2} - \frac{2\left(2+\frac{2\alpha(1+2\beta)}{(1+2\alpha\beta)^2}\right)}{1+(1+6\beta)\alpha}\right\}|c_1|^2$.

Subcase 2(a):- When $\mu \geq \frac{\alpha(1+2\beta)+(1+2\alpha\beta)^2}{[1+(1+6\beta)\alpha]}$.

Then,

(1.4) $$|a_3 - \mu a_2^2| \leq \frac{4\mu}{(1+2\alpha\beta)^2} - \frac{2(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)}{(1+2\alpha\beta)^2[1+(1+6\beta)\alpha]} .$$

Subcase 2(b) :- When $\mu \leq \frac{\alpha(1+2\beta)+(1+2\alpha\beta)^2}{[1+(1+6\beta)\alpha]}$.

Then,

(1.5) $$|a_3 - \mu a_2^2| \leq \frac{2}{1+(1+6\beta)\alpha} .$$

Combining (1.2), (1.3), (1.4) and (1.5) , we get the required result.

FOR EXTREMAL:- Extremal functions of the above defined function, are given by

$$f(z) = z\left(1 - \frac{2\alpha(1+2\beta+4\beta^2\alpha)}{[1+(1+6\beta)\alpha](1+2\alpha\beta)}z\right)^{-\frac{1+(1+6\beta)\alpha}{\alpha(1+2\beta+4\beta^2\alpha)}}$$

and $$f(z) = z(1+2z^2)^{\frac{1}{1+(1+6\beta)\alpha}} .$$

COROLLARY 1.1:- $TS^*[1,0] = S^*$, as by substituting $\alpha = 1$ and $\beta = 0$ the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu, if\ \mu \leq \frac{1}{2}; \\ 1, if \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, if\ \mu \geq 1. \end{cases}$$

which is the required result for the class S^* given by Keogh and Merkes [6].

THEOREM 1.2:- Let $f(z) \in TS^*[\alpha,\beta,\delta]$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$ *; w(z)* is a *Schwarzian function* , then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)\delta^2}{(1+2\alpha\beta)^2[1+\alpha(1+6\beta)]} - \frac{4\mu\delta^2}{(1+2\alpha\beta)^2}; \mu \leq \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)\delta^2-\delta(1+2\alpha\beta)^2}{2\delta^2(1+\alpha(1+6\beta)]}; \\ \frac{2\delta}{1+\alpha(1+6\beta)}; \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)\delta^2-\delta(1+2\alpha\beta)^2}{2\delta^2[1+\alpha(1+6\beta)]} \leq \mu \leq \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)\delta^2+\delta(1+2\alpha\beta)^2}{2\delta^2[1+\alpha(1+6\beta)]}; \\ \frac{4\mu\delta^2}{(1+2\alpha\beta)^2} - \frac{2(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)\delta^2}{(1+2\alpha\beta)^2[1+\alpha(1+6\beta)]}; \mu \geq \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)\delta^2+\delta(1+2\alpha\beta)^2}{2\delta^2[1+\alpha(1+6\beta)]}. \end{cases}$$

The result is sharp.

PROOF:- If $f(z) \in T\mathcal{S}^*[\alpha,\beta,\delta]$, then

(1.6) $$(1-\alpha)\frac{f(z)}{z} + \alpha\,\frac{z f'(z)+\beta z^2 f''(z)}{f(z)} = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta} .$$

By putting the values of *f (z)* and *w(z)* in (1.6) , we get

$$1 + (1+2\alpha\beta)\, a_2 z + [\{1+\alpha(1+6\square)\}\, a_3 - \alpha(1+2\beta)\, a_2^2]\, z^2 + \ldots$$

$$= 1 + 2\delta c_1 z + 2\left(\delta c_2 + \delta^2 c_1^2\right) z^2 + \ldots \; .$$

By comparing, we can obtain

$$a_2 = \frac{2\delta c_1}{(1+2\alpha\beta)} \quad \text{and} \quad a_3 = \frac{2[\delta c_2 + \frac{\left(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\right)\delta^2 c_1^2}{(1+2\alpha\beta)^2}]}{1+\alpha(1+6\beta)} .$$

After putting the values of a_2 and a_3 and taking absolute values , we get

$$|a_3 - \mu\, a_2^2| \le \frac{2\delta}{1+\alpha(1+6\beta)} + \left\{\left|\frac{2\left(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\right)\delta^2}{(1+2\alpha\beta)^2\{1+\alpha(1+6\beta)\}} - \frac{4\mu\,\delta^2}{(1+2\alpha\beta)^2}\right| - \frac{2\delta}{1+\alpha(1+6\beta)}\right\} |c_1|^2.$$

Case 1:- If $\mu \le \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)}{2\{1+\alpha(1+6\beta)\}}$.

Then, $$|a_3 - \mu\, a_2^2| \le \frac{2\delta}{1+\alpha(1+6\beta)} + \left\{\frac{2[\left(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\right)\delta^2 - \delta(1+2\alpha\beta)^2]}{(1+2\alpha\beta)^2\{1+\alpha(1+6\beta)\}} - \frac{4\mu\,\delta^2}{(1+2\alpha\beta)^2}\right\} |c_1|^2 .$$

Subcase 1(a) :- When $\mu \le \frac{\left(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\right)\delta^2 - \delta(1+2\alpha\beta)^2}{2\delta^2\{1+\alpha(1+6\beta)\}}$.

Then, using $|c_1| \le 1$, we get

(1.7) $$|a_3 - \mu\, a_2^2| \le \frac{2\,(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)\,\delta^2}{(1+2\alpha\beta)^2\{1+\alpha(1+6\beta)\}} - \frac{4\mu\,\delta^2}{(1+2\beta\alpha)^2} .$$

Subcase 1(b) :- When $\mu \ge \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)\delta^2 - \delta(1+2\alpha\beta)^2}{2\delta^2\{1+\alpha(1+6\beta)\}}$.

Then ,

(1.8) $$|a_3 - \mu\, a_2^2| \le \frac{2\delta}{1+\alpha(1+6\beta)} .$$

Case 2 :- If $\mu \ge \frac{(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta)}{2\{1+\alpha(1+6\beta)\}}$. Then , $|a_3 - \mu\, a_2^2|$
$$\le \frac{2\delta}{1+\alpha(1+6\beta)} + \left\{\frac{4\mu\,\delta^2}{(1+2\alpha\beta)^2} - \frac{2[\left(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\right)\delta^2 + \delta(1+2\alpha\beta)^2]}{(1+2\alpha\beta)^2\{1+\alpha(1+6\beta)\}}\right\} |c_1|^2 .$$

Subcase 2 (a) :- When $\mu \ge \frac{\left(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\right)\delta^2 + \delta(1+2\alpha\beta)^2}{2\delta^2\{1+\alpha(1+6\beta)\}}$.

Then ,

(1.9) $$|a_3 - \mu\, a_2^2| \leq \frac{4\mu\,\delta^2}{(1+2\alpha\beta\,)^2} - \frac{2\,(\,1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\,)\delta^2}{(1\,+2\alpha\beta\,)^2\{1+\alpha\,(1+6\beta)\}} \quad .$$

Subcase 2(b) :- When $\mu \leq \frac{\left(1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\,\right)\delta^2+\delta(1+2\alpha\beta\,)^2}{2\delta^2\{1+\alpha(1+6\beta)\}}$.

Then,

(1.10) $$|a_3 - \mu\, a_2^2| \leq \frac{2\delta}{1+\alpha\,(1+6\beta)} \quad .$$

Combining (1.7), (1.8), (1.9) and (1.10), we get the required result of the class $T\mathcal{S}^*[\alpha,\beta,\delta]$.

EXTREMAL :- Extremal functions are given by

$$f\,(z) = z\left(1-\frac{2\alpha\delta\,\,(1+2\beta+4\beta^2\alpha)}{[1+(1+6\beta)\alpha](1+2\alpha\beta\,)}z\right)^{-\frac{1+(1+6\beta)\alpha}{\alpha(1+2\beta+4\beta^2\alpha)}}$$

and $$f\,(z) = z\left(1+2\delta z^2\right)^{\frac{1}{1+\alpha\,(1+6\beta)}} \quad .$$

COROLLARY 1.2 :- $T\mathcal{S}^*[\,\alpha,\beta,1] = T\mathcal{S}^*[\alpha,\beta]$, as by substituting δ =1, the result becomes

$$|a_3 - \mu\, a_2^2| \leq \begin{cases} \frac{2\,(\,1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\,)}{(1+2\alpha\beta\,)^2\{1+(1+6\beta)\alpha\}} - \frac{4\mu}{(1+2\alpha\beta\,)^2}\,;\ \mu \leq \frac{\alpha(1+2\beta)}{1+(1+6\beta)\alpha}\,; \\ \frac{2}{1+(1+6\beta)\alpha} \quad ;\frac{\alpha(1+2\beta)}{1+(1+6\beta)\alpha} \leq \mu \leq \frac{\alpha\,(1+2\beta)+(1+2\alpha\beta\,)^2}{[1+(1+6\beta)\alpha]}\,; \\ \frac{4\mu}{(1+2\alpha\beta\,)^2} - \frac{2\,(\,1+2\alpha+4\beta^2\alpha^2+8\alpha\beta\,)}{(1+2\alpha\beta\,)^2[1+(1+6\beta)\alpha]}\,;\ \mu \geq \frac{\alpha\,(1+2\beta)+(1+2\alpha\beta\,)^2.}{[1+(1+6\beta)\alpha]} \end{cases}$$

which is the required result for the class $T\mathcal{S}^*[\alpha,\beta]$.

COROLLARY 1.3 :- $T\mathcal{S}^*[1,0,1] = \mathcal{S}^*$, as by substituting α = 1, β = 0 and δ =1 the result becomes

$$|a_3 - \mu\, a_2^2| \leq \begin{cases} 3-4\mu\;, if \quad \mu \leq \frac{1}{2}; \\ 1\;, if \;\; \frac{1}{2} \leq \mu \leq 1; \\ 4\mu-3\;, if \quad \mu \geq 1. \end{cases}$$

which is the required result for the class $\mathcal{S}^*$ given by Keogh and Merkes [6].

3.Conclusion In this paper, we defined Fekete – Szegö inequality for functions $f(z)$ in class $\mathcal{S}^*$, which is a subclass of class S, having different parameters. Here we defined and explained this inequality for class $T\mathcal{S}^*$ with parameters α, β and δ along with their extremal functions and corollaries that are related to the latest researches of this field. The method used here is the method of subordination, which is the most important technique of this field.

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