ON SOME NEW RESULT FOR MULTI-VALUED QUASI-CONTRACTION MAPS IN A V-FUZZY*b*-METRIC SPACE

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ABSTRACT: In this paper, we define V-fuzzy *b*-metric space and by using concept of a set-valued or multi-valued quasi-contraction mapping a fixed point theorem is established.**MSC:** primary 47H10; secondary 54H25.

KEYWORDS: Fuzzy metric space;fixed point;V-fuzzy *b*-metric space;

1. INTRODUCTION AND PRELIMINARIES: The initiation of a metric space has been an attractive subject that acknowledges the nearness between two marks. Over the years, innumerable mathematicians have been attempting to generalize the ideaof metric space. Two such mathematicians were Mustafa and Sims. In their paper [18], they furnish a way to interrupt distance using three points instead of two points as was the example in metric spaces. They designate their new approach a G-metric space.

***Definition* 1:** The pair (X,d) is called a G-metric space if X is a nonempty set and d is a G-metric on X . That is, $d : X^3 \to [0,\infty)$ such that for all $x_1, x_2, x_3, a \in X$, we have

(i) $G(x_1,x_2,x_3) = 0$ if and only if $x_1 = x_2 = x_3$

(ii) $G(x_1,x_1,x_2) > 0$ with $x_1 \neq x_2$

(iii) $G(x_1,x_1,x_2) \le G(x_1,x_2,x_3)$ for all with $x_2 \neq x_3$

(iv) $G(x_1,x_2,x_3) = G(x_1,x_3,x_2) = G(x_2,x_1,x_3) = G(x_2,x_3,x_1) = G(x_3,x_1,x_2) = G(x_3,x_2,x_1)$

(v) $G(x_1,x_2,x_3) \le G(x_1,a,a) + G(a,x_2,x_3)$

Sedghi et. al. in [20] modified the G-metric space and introduced their version of interpretation of distance between three points which is called the S-metric space.

***Definition* 2:** The pair (X,d) is called an S-metric space if X is a nonempty set and $d : X^3 \to [0,\infty)$ such that for all $x_1, x_2, x_3 \in X$, we have

(i) $d(x_1, x_2, x_3) \geq 0$

(ii) $d(x_1, x_2, x_3) = 0 \Leftrightarrow x_1 = x_2 = x_3$

(iii) $d(x_1, x_2, x_3) \leq d(x_1, x_1, a) + d(x_2, x_2, a) + d(x_3, x_3, a)$

In view of the work of Sedghi et. al. in [20], Abbas et. al. in [1] further generalized the S-metric space giving an interpretation of distance between n points.

***Definition* 3:** The pair (X, d) is called an A-metric space if X is a nonempty set and $d: X^n \to [0, \infty)$ such that for all $a, x_i \in X, i = 1, 2, ..., n,$ we have

(i) $d(x_1, x_2, ..., x_{n-1}, x_n) \geq 0$

(ii) $d(x_1, x_2, ..., x_{n-1}, x_n) = 0 \Leftrightarrow x_1 = x_2 = ... = x_{n-1} = x_n$

(iii) $$d(x_1, x_2, ..., x_{n-1}, x_n) \leq d(x_1, x_1, ..., x_1, x_n) + d(x_2, x_2, ..., x_2, a) + ... + d(x_{n-1}, x_{n-1}, ..., x_{n-1}, a) + d(x_n, x_n, ..., x_n, a)$$

In 2016, Gupta and Kanwar in [7] used the same approach in the fuzzy setting to extend a fuzzy metric space which uses two points to a V-fuzzy metric space which uses n points. We now outline this sequel that leads up to V-fuzzy metric spaces.

A fuzzy set was first proposed in the 1960s by an electrical engineer named Lotfi A. Zadeh in [24]. It is an extension of the classical notion of a set.It should be noted that a fuzzy set is always considered with respect to a nonempty set X called its base set. Each $x \in X$ is assigned a membership grade $0 \leq M(x) \leq 1.$ The formal definition of a fuzzy set is now given.

***Definition* 4:** The fuzzy set is a pair (X, M), where X is a nonempty set and $M: X \to [0,1].$ The value $M(x)$ is called the membership of $x \in X$. The nearer the value of $M(x)$ to unity, the higher the degree of membership of $x \in X$. Conversely, the nearer the value of $M(x)$ to zero, the lower the degree of membership of $x \in X$. A fuzzy set (X, M) is sometimes read as, M is a fuzzy set on X .

***Definition* 5:** A t-norm $*$ is a function $*: [0,1] \times [0,1] \to [0,1]$ such that for all $a, b, c, d \in [0,1],$ the following are satisfied:

(i) $a * 1 = a$ (1 acts as the identity element)

(ii) $a * b = b * a$ (symmetry)

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ (non-deceasing)

(iv) $a * (b * c) = (a * b) * c$ (associative)

Additionally, $*$ is said to be a continuous t-norm if $*$ is a t-norm and for all sequences $\{x_n\}$ and $\{y_n\}$ in $[0,1]$, where $n \in \mathbb{N}$, we have that the limit of these sequences exist and

$$\lim_{n\to\infty}(x_n * y_n) = \lim_{n\to\infty} x_n * \lim_{n\to\infty} y_n. \quad (1)$$

More specifically, $*$ is called left continuous if for each $y \in [0,1],$

$$\lim_{n\to\infty}(x_n * y) = \lim_{n\to\infty} x_n * y. \quad (2)$$

Right continuity is analogously defined.

***Definition* 6:**The 3-tuple$\left(X, M, T \right)$ is known as fuzzy metric space (shortly, FM-space) if *X* is an any set, *T* is a continuous *t*-norm, and M is a fuzzy set in $X \times X \times (0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$;

(FM-1) $M\left(x, y, t \right) > 0,$

(FM-2) $M\left(x, y, t \right) = 1$ iff $x = y,$

(FM-3) $M\left(x, y, t\right) = M\left(y, x, t\right),$

(FM-4) $T\left(M\left(x, y, t\right), M\left(y, z, s\right)\right) \le M\left(x, z, t+s\right),$

(FM-5) $M(x, y, \cdot) : [0,\infty) \to [0,1]$ is continuous.

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1,$ for all $x, y \in X$

Remark 1 (see[6]). Informally, we can think of $M(x, y, t)$ ass the degree of nearness between x and y with respect to t .

In 2010, Sun and Yang in [23] made the first step in generalizing the fuzzy metric space. They called it the Q-fuzzy metric space, and they proved several fixed point theorems in this space. We define the Q-fuzzy metric space as follows.

***Definition* 7:**The 3-tuple $\left(X, Q, T \right)$ is known as Q-fuzzy metric space if Q is a fuzzy set, *T* is a continuous *t*-norm, and Q is a fuzzy set in $X^3 \times (0,\infty)$ satisfying the following conditions for all $x, y, z, a \in X$ and $s, t > 0$;

(i) $Q\left(x, x, y, t \right) > 0,$ with $x \ne y$

(ii) $Q\left(x, x, y, t \right) \ge Q\left(x, y, z, t \right)$ with $y \ne z$

(iii) $Q\left(x, y, z, t \right) = 1 \Leftrightarrow x = y = z$

(iv) $Q\left(x, y, z, t \right) = Q\left(x, z, y, t \right) = Q\left(y, x, z, t \right)$
$= Q\left(z, x, y, t \right) = Q\left(y, z, x, t \right) = Q\left(z, y, x, t \right)$

(v) $Q\left(x, a, a, t \right) * Q\left(a, y, z, s \right) \le Q\left(x, y, z, t+s \right)$

(vi) $Q\left(x, y, z, \cdot \right) : (0,\infty) \to (0,1]$ is continuous

At this point, Gupta and Kanwar was ready to provide an interpretation of fuzzy metric space using n points. They called it V-fuzzy metric space.

***Definition* 8:**(see[7])**:-** The 3-tuple $\left(X, V, * \right)$ is called a V-fuzzy metric space if $*$ is a continuous *t*-norm, and V is a fuzzy set in $X^n \times (0,\infty)$ satisfying the following conditions for all $x_i, y, a \in X$ and $s, t > 0$;

(i) $V\left(x, x, ..., x, y, t \right) > 0,\ x \ne y$

(ii) $V\left(x_1, x_1, ..., x_1, x_2, t \right) \ge V\left(x_1, x_2, x_3, ..., x_n, t \right),\ x_2 \ne x_3 \ne ... \ne x_n$

(iii) $V\left(x_1,x_2,x_3,...,x_n,t\right)=1 \Leftrightarrow x_1=x_2=x_3=...=x_n$

(iv) $V\left(x_1,x_2,x_3,...,x_n,t\right)= V(p\left(x_1,x_2,x_3,...,x_n\right),t)$ where $p(x_1,x_2,x_3,...,x_n)$ is permutation on $x_1,x_2,x_3,...,x_n$

(v) $V(x_1,x_2,x_3,...,x_{n-1},a,t)*V(a,a,a,...,a,x_n,s)\le V(x_1,x_2,x_3,...,x_{n-1},x_n,t+s)$

(vi) $V(x_1,x_2,x_3,...,x_n,t)=1$ as $t\to\infty$

(vii) $V(x_1,x_2,x_3,...,x_n,\cdot):(0,\infty)\to(0,1]$ is continuous.

In addition to fuzzy metric spaces, there are still many extensions of metric and metric space terms. Bakhtin [2] and Czerwik [4] introduced a space where, instead of triangle inequality, a weaker condition was observed, with the aim of generalization of Banach contraction principal [3]. They called these spaces *b*-metric spaces. Relation between *b*-metric and fuzzy metric spaces is considering in [10]. On the other hand, in [21] the notion of a fuzzy *b*-metric space was introduced, where the triangle inequality is replaced by a weaker one.

***Definition* 9:**The 3-tuple $\left(X,M,T\right)$ is known as fuzzy *b*-metric space if *X* is any set, *T* is a continuous *t*-norm, and M is a fuzzy set in $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0,$ and a given real number $b\ge 1,$

(BM-1) $M\left(x,y,t\right)>0,$

(BM-2) $M\left(x,y,t\right)=1$ if and only if $x=y,$

(BM-3) $M\left(x,y,t\right)=M\left(y,x,t\right),$

(BM-4) $T(M(x,y,\tfrac{t}{b}),M(y,z,\tfrac{s}{b}))\le M(x,z,t+s),$

(BM-5) $M(x,y,\cdot):[0,\infty)\to[0,1]$ is continuous.

Now, in this paper we introduced a new space that is V-fuzzy *b*-metric space with the help of V-fuzzy metric space and *b*-Metric space and by using concept of a set-valued or multi-valued quasi-contraction mapping a fixed point theorem is established. This theorem generalizes and improves some known fixed point theorems in literature.

***Definition* 10:** (see[7]):- The 3-tuple $\left(X,V,*\right)$ is called a V-fuzzy *b*-metric space if $*$ is a continuous *t*-norm, and V is a fuzzy set in $X^n\times(0,\infty)$ satisfying the following conditions for all $x_i,y,a\in X, b\ge 1$ and $s,t>0;$

(i) $V\left(x,x,...,x,y,\tfrac{t}{b}\right)>0,\ x\ne y$

(ii) $V\left(x_1,x_1,...,x_1,x_2,\tfrac{t}{b}\right)\ge V\left(x_1,x_2,x_3,...,x_n,\tfrac{t}{b}\right),\ x_2\ne x_3\ne...\ne x_n$

(iii) $V\left(x_1,x_2,x_3,...,x_n,\tfrac{t}{b}\right)=1\Leftrightarrow x_1=x_2=x_3=...=x_n$

(iv) $V\left(x_1,x_2,x_3,...,x_n,\tfrac{t}{b}\right)= V(p\left(x_1,x_2,x_3,...,x_n\right),\tfrac{t}{b})$ where $p(x_1,x_2,x_3,...,x_n)$ is permutation on $x_1,x_2,x_3,...,x_n$

(v) $V(x_1,x_2,x_3,...,x_{n-1},a,\tfrac{t}{b})*V(a,a,a,...,a,x_n,\tfrac{t}{b})\le V(x_1,x_2,x_3,...,x_{n-1},x_n,t+s)$

(vi) $V(x_1, x_2, x_3, ..., x_n, \frac{t}{b}) = 1$ as $t \to \infty$

(vii) $V(x_1, x_2, x_3, ..., x_n, \cdot) : (0, \infty) \to (0,1]$ is continuous.

Lemma 1: (see[7]):-Let $(X, V, *)$ be a V-fuzzy *b*-metric space. Then $V(x_1, x_2, x_3, ..., x_n, \cdot)$ isnon-decreasing. That is, for some $0 < s < t,\ b \geq 1$ we have

$V(x_1, x_2, x_3, ..., x_n, \frac{s}{b}) \leq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$ (3)

*Proof:*Since $s < t,$ we have $t - s > 0.$

Now, $V(x_1, x_2, x_3, ..., x_n, \frac{s}{b}) * V(x_1, x_2, x_3, ..., x_n, \frac{t-s}{b}) \leq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$ (4)

Hence, for all $0 < s < t,\ b \geq 1$ we have

$V(x_1, x_2, x_3, ..., x_n, \frac{s}{b}) \leq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$ (5)

Lemma 2: (see[7]):-Let $(X, V, *)$ be a V-fuzzy *b*-metric space. If for all $x_1, x_2, x_3, ..., x_n \in X$ with $k \in (0,1),\ b \geq 1$ we have

$V(x_1, x_2, x_3, ..., x_n, \frac{kt}{b}) \leq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$ (6)

Then, $x_1 = x_2 = x_3 = ... = x_n.$ We now examine convergence and Cauchy sequence in V-fuzzy *b*-metric spaces.

***Definition* 11:** (see[7]):- A sequence $\{x_n\}$ in a V-fuzzy *b*-metric space $(X, V, *)$ is said to be convergent and converges to $x \in X$ if for each $t > 0, b \geq 1$ and $0 < \varepsilon < 1,$ there exist $N \in \mathbb{N}$ such that $V(x_i, x_i, x_i, ..., x_i, x, \frac{t}{b}) > 1 - \varepsilon,\ \forall i \geq N.$ (7)

That is, $V(x_i, x_i, x_i, ..., x_i, x, \frac{t}{b}) \to 1$ as $i \to \infty.$ (8)

***Definition* 12:** (see[7]):- A sequence $\{x_n\}$ in a V-fuzzy *b*-metric space $(X, V, *)$ is said to be Cauchy if for each $t > 0, b \geq 1$ and $0 < \varepsilon < 1,$ there exist $N \in \mathbb{N}$ such

$V(x_i, x_i, x_i, ..., x_i, x_j, \frac{t}{b}) > 1 - \varepsilon,\ \forall i, j \geq N.$ (9)

$V(x_i, x_i, x_i, ..., x_i, x_j, \frac{t}{b}) \to 1$ as $i, j \to \infty.$ (10)

Definition 13: (see[7]):-A V-fuzzy *b*-metric space $(X, V, *)$ is said to be complete if every Cauchy sequence in X is convergent in X.

2. FIXED-POINT THEOREM IN V-FUZZY *b*-METRIC SPACES

In this section, we extend the concept in the existing literature. We introduce the concept of multivalued quasi-contraction maps in V-fuzzy *b*-metric spaces, and we also give a fixed point theorem. We begin with following concepts in the V-fuzzy setting.

***Notation* 1:** We denote the set of all nonempty closed and bounded subsets of X in a V-fuzzy *b*-metric space by $CB_V(X)$.

***Definition* 14:** Let $A_1, A_2, ..., A_n \subseteq CB_V(X),\ b \geq 1$ and $t > 0.$ then, $V(x, A_2, A_3, ...,, A_n, t) = \inf\{V(x, a_2, a_3, ...,, a_n, \frac{t}{b}) : a_2 \in A_2, a_3 \in A_3, ..., a_n \in A_n\}.$ (11)

Lemma 3:Let $(X,V,*)$ be a V-fuzzy b-metric space. If for all $t>0, b\geq 1$ and $x\in X$ with $k\in(0,1)$ and $A\subseteq CB_V(X)$, we have

$$V(x,A,A,....,A,\tfrac{kt}{b})\geq V(x,A,A,....,A,\tfrac{t}{b}). \quad (12)$$

Hen, $x\in A.$

Proof: Assume for a contradiction that $x\neq A.$ (13)

Let $y\in A$. Then, $V(x,y,....,y,\frac{kt}{b})\geq V(x,y,....,y,\frac{t}{b})$. This implies $x=y\in A$ by Lemma 2.

This contradicts equation (13). Therefore, our original assumption is false.

Hence, $x\in A.$

***Definition* 15:** Let $A_1,A_2,....,A_n\subseteq CB_V(X)$, $b\geq 1$ and $t>0$. the Hausdorff V-fuzzy b-metric or Hausdorff V-fuzzy b-metric distance is denoted by $V_H(A_1,A_2,....,A_n,\frac{t}{b})$ and is defined by

$$V_H(A_1,A_2,....,A_n,\tfrac{t}{b})=\max\left\{\begin{array}{l}\sup\limits_{x\in A_1}V\left(x,A_2,...,A_n,\tfrac{t}{b}\right),\\ \sup\limits_{x\in A_1}V\left(A_1,x,A_3...,A_n,\tfrac{t}{b}\right),\\ \qquad\vdots\\ \sup\limits_{x\in A_n}V\left(A_1,A_2,...,x,\tfrac{t}{b}\right),\end{array}\right\} \quad (14)$$

where,

$$\begin{array}{l}V\left(x,A_2,A_3,...,A_n,\tfrac{t}{b}\right)=\inf\{V\left(x,a_2,a_3,...,a_n,\tfrac{t}{b}\right):a_2\in A_2,a_3\in A_3,...,a_n\in A_n\},\\ V\left(A_1,x,A_3,...,A_n,\tfrac{t}{b}\right)=\inf\{V\left(a_1,x,a_3,...,a_n,\tfrac{t}{b}\right):a_1\in A_1,a_3\in A_3,...,a_n\in A_n\},\\ \vdots\\ V\left(A_1,...,A_{n-1},x,\tfrac{t}{b}\right)=\inf\{V\left(a_1,...,a_{n-1},x,\tfrac{t}{b}\right):a_1\in A_1,...,a_{n-1}\in A_{n-1}\}.\end{array} \quad (15)$$

Remark 2: Informally, we can think of the Hausdorff V-fuzzy b-metric as the greatest degree of nearness from a point in one set to the closest point in the other sets with respect to t.

***Definition* 16.**Let $(X,V,*)$ be a V-fuzzy b-metric space. The mapping $T:X\to CB_V(X)$ is said to be a q-multivalued quasi-contraction if there exist $0\leq q<1$ such that for all $a_i\in X, i=1,2,...,n,\ b\geq 1$

$$V_H(Ta_1,Ta_2,....,Ta_n,\tfrac{t}{b}) \le q.\max\begin{Bmatrix} V\left(a_1,a_2,...,a_n,\tfrac{t}{b}\right), \\ V\left(a_1,Ta_1,Ta_1,...,Ta_1,\tfrac{t}{b}\right), \\ V\left(a_1,Ta_2,Ta_3,...,Ta_n,\tfrac{t}{b}\right), \\ V\left(a_2,Ta_2,Ta_2,...,Ta_2,\tfrac{t}{b}\right), \\ V\left(a_2,Ta_1,Ta_3,...,Ta_n,\tfrac{t}{b}\right), \\ \vdots \\ V\left(a_n,Ta_n,Ta_n,...,Ta_n,\tfrac{t}{b}\right), \\ V\left(a_n,Ta_1,Ta_2,...,Ta_{n-1},\tfrac{t}{b}\right) \end{Bmatrix}. \tag{16}$$

3. MAIN RESULT

We now proceed to give a fixed point theorem which is our main result.

Theorem 1: *Let $(X,V,*)$ be a complete V-fuzzy b-metric space. If the mapping $T:X\to CB_V(X)$ is a q-multi-valued quasi-contraction, then T has a fixed point in X. That is, there exists $u\in X$ such that $u\in Tu$.*

Proof: T is given to be q-multi-valued quasi-contraction. This implies that there exists $0\le q<1$ such that for all $a_i\in X, i=1,2,,...,n,\ \ b\ge 1$

$$V_H(Ta_1,Ta_2,....,Ta_n,\tfrac{t}{b}) \le q.\max\begin{Bmatrix} V\left(a_1,a_2,...,a_n,\tfrac{t}{b}\right), \\ V\left(a_1,Ta_1,Ta_1,...,Ta_1,\tfrac{t}{b}\right), \\ V\left(a_1,Ta_2,Ta_3,...,Ta_n,\tfrac{t}{b}\right), \\ V\left(a_2,Ta_2,Ta_2,...,Ta_2,\tfrac{t}{b}\right), \\ V\left(a_2,Ta_1,Ta_3,...,Ta_n,\tfrac{t}{b}\right), \\ \vdots, \\ V\left(a_n,Ta_n,Ta_n,...,Ta_n,\tfrac{t}{b}\right), \\ V\left(a_n,Ta_1,Ta_2,...,Ta_{n-1},\tfrac{t}{b}\right) \end{Bmatrix}. \tag{17}$$

It is clear that for some $a_1\in A_1$, with $a_2\in A_2, a_3\in A_3,...,a_n\in A_n$, we have $V\left(a_1,a_2,...,a_n,\tfrac{t}{b}\right)\le V_H(A_1,A_2,....,A_n,\tfrac{t}{b})$. Using this fact and setting $a_1\in Ta_0$, there exists $a_2\in Ta_1,...,a_n\in Ta_{n-1}$, Inequality 2 becomes

$$V\left(a_1,a_2,...,a_n,\tfrac{t}{b}\right) \le V_H(Ta_0,Ta_1,....,Ta_{n-1},\tfrac{t}{b}) \le q.\max\left\{\begin{array}{l} V\left(a_0,a_1,a_2,...,a_{n-1},\tfrac{t}{b}\right), \\ V\left(a_0,Ta_0,Ta_0,...,Ta_0,\tfrac{t}{b}\right), \\ V\left(a_0,Ta_1,Ta_2,...,Ta_{n-1},\tfrac{t}{b}\right), \\ V\left(a_1,Ta_1,Ta_1,...,Ta_1,\tfrac{t}{b}\right), \\ V\left(a_1,Ta_0,Ta_2,...,Ta_{n-1},\tfrac{t}{b}\right), \\ \qquad \vdots, \\ V\left(a_{n-1},Ta_{n-1},Ta_{n-1},...,Ta_{n-1},\tfrac{t}{b}\right), \\ V\left(a_{n-1},Ta_0,Ta_1,...,Ta_{n-2},\tfrac{t}{b}\right) \end{array}\right\}. \quad (18)$$

Similarly setting $a_2 \in Ta_1$, there exists $a_3 \in Ta_2,...,a_{n+1} \in Ta_n$, Inequality 2 becomes

$$V\left(a_2,a_3,...,a_{n+1},\tfrac{t}{b}\right) \le V_H(Ta_1,Ta_2,....,Ta_n,\tfrac{t}{b}) \le q.\max\left\{\begin{array}{l} V\left(a_1,a_2,a_3,...,a_n,\tfrac{t}{b}\right), \\ V\left(a_1,Ta_1,Ta_1,...,Ta_1,\tfrac{t}{b}\right), \\ V\left(a_1,Ta_2,Ta_3,...,Ta_n,\tfrac{t}{b}\right), \\ V\left(a_2,Ta_2,Ta_2,...,Ta_2,\tfrac{t}{b}\right), \\ V\left(a_2,Ta_1,Ta_3,...,Ta_n,\tfrac{t}{b}\right), \\ \qquad \vdots, \\ V\left(a_n,Ta_n,Ta_n,...,Ta_n,\tfrac{t}{b}\right), \\ V\left(a_n,Ta_1,Ta_1,...,Ta_{n-1},\tfrac{t}{b}\right) \end{array}\right\}. \quad (19)$$

Continuing in this manner by induction, we obtain a sequence $\{a_k\}_{k=0}^{\infty}$ such that

$$V\left(a_k,a_{k+1},...,a_{k+n-1},\tfrac{t}{b}\right) \le V_H(Ta_{k-1},Ta_k,....,Ta_{k+n-2},\tfrac{t}{b}) \le q.\max\left\{\begin{array}{l} V\left(a_{k-1},a_k,a_{k+1},...,a_{k+n-2},\tfrac{t}{b}\right), \\ V\left(a_{k-1},Ta_{k-1},Ta_{k-1},...,Ta_{k-1},\tfrac{t}{b}\right), \\ V\left(a_{k-1},Ta_k,Ta_{k+1},...,Ta_{k+n-2},\tfrac{t}{b}\right), \\ V\left(a_k,Ta_k,Ta_k,...,Ta_k,\tfrac{t}{b}\right), \\ V\left(a_k,Ta_{k-1},Ta_{k+1},...,Ta_{k+n-2},\tfrac{t}{b}\right), \\ \qquad \vdots, \\ V\left(a_{k+n-2},Ta_{k+n-2},Ta_{k+n-2},...,Ta_{k+n-2},\tfrac{t}{b}\right), \\ V\left(a_{k+n-2},Ta_{k-1},Ta_k,...,Ta_{k+n-3},\tfrac{t}{b}\right) \end{array}\right\}. \quad (20)$$

We now show that $\{a_k\}_{k=0}^{\infty}$ is Cauchy. If $i = j,$ we get $V(a_i,a_i,...,a_i,a_j,\tfrac{t}{b}) = 1 > 1-\varepsilon,$ where $\varepsilon \in (0,1),\ b \ge 1$ and therefore $\{a_k\}$ will be trivially Cauchy. Thus, without loss of generality assume $i < j$ and $i \ne j.$ We have

$$V\left(a_i,a_i,...,a_i,a_j,\tfrac{t}{b}\right) \le V_H(Ta_{i-1},Ta_{i-1},....,Ta_{i-1},Ta_{j-1},\tfrac{t}{b}) \le q.\max\begin{Bmatrix} V\left(a_{i-1},a_{i-1},...,a_{i-1},a_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right), \\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right), \\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{j-1},\tfrac{t}{b}\right), \\ \vdots \\ V\left(a_{j-1},Ta_{j-1},...,Ta_{j-1},Ta_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{j-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right) \end{Bmatrix}.$$

$$= q.\max\begin{Bmatrix} V\left(a_{i-1},a_{i-1},...,a_{i-1},a_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right), \\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{j-1},Ta_{j-1},...,Ta_{j-1},Ta_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{j-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right) \end{Bmatrix}. \quad (21)$$

Now, we consider the five cases.
Case 1: If

$$\max\begin{Bmatrix} V\left(a_{i-1},a_{i-1},...,a_{i-1},a_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right), \\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{j-1},Ta_{j-1},...,Ta_{j-1},Ta_{j-1},\tfrac{t}{b}\right), \\ V\left(a_{j-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right) \end{Bmatrix}. \quad (22)$$

$$= V\left(a_{i-1},a_{i-1},...,a_{i-1},a_{j-1},\tfrac{t}{b}\right).$$

Then, as $i, j \to \infty$ and using the fact that $q \in (0,1)$, we have

$$\begin{aligned} 1 \ge V\left(a_{i-1},a_{i-1},...,a_{i-1},a_{j-1},\tfrac{t}{b}\right) &\ge \frac{1}{q} V\left(a_i,a_i,...,a_i,a_j,\tfrac{t}{b}\right) \\ &\ge \frac{1}{q^2} V\left(a_{i+1},a_{i+1},...,a_{i+1},a_{j+1},\tfrac{t}{b}\right) \\ &\ge ... \\ &\ge \frac{1}{q^{s+1}} V\left(a_{i+s},a_{i+s},...,a_{i+s},a_{j+s},\tfrac{t}{b}\right), \quad s \in \mathbb{N} \\ &\ge ... \\ &\ge 1. \end{aligned} \quad (23)$$

This implies that $V\left(a_i,a_i,...,a_i,a_j,\tfrac{t}{b}\right)\to 1$ as $i,j\to\infty$. Therefore, $\{a_k\}_{k=0}^{\infty}$ is Cauchy.

Case 2: If

$$\max\left\{\begin{array}{l} V\left(a_{i-1},a_{i-1},...,a_{i-1},a_{j-1},\tfrac{t}{b}\right),\\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right),\\ V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{j-1},\tfrac{t}{b}\right),\\ V\left(a_{j-1},Ta_{j-1},...,Ta_{j-1},Ta_{j-1},\tfrac{t}{b}\right),\\ V\left(a_{j-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right)\end{array}\right\}. \tag{24}$$

$$=V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right),$$

then, as $i,j\to\infty$ and using the fact that $q\in(0,1)$, we have

$$\begin{aligned} 1\geq V\left(a_{i-1},Ta_{i-1},...,Ta_{i-1},Ta_{i-1},\tfrac{t}{b}\right)&\geq \frac{1}{q}V\left(a_i,a_i,...,a_i,a_j,\tfrac{t}{b}\right)\\ &\geq \frac{1}{q^2}V\left(a_{i+1},a_{i+1},...,a_{i+1},a_{j+1},\tfrac{t}{b}\right)\\ &\geq ...\\ &\geq \frac{1}{q^{s+1}}V\left(a_{i+s},a_{i+s},...,a_{i+s},a_{j+s},\tfrac{t}{b}\right), \quad s\in\mathbb{N}\\ &\geq ...\\ &\geq 1. \end{aligned} \tag{25}$$

This implies that $V\left(a_i,a_i,...,a_i,a_j,\tfrac{t}{b}\right)\to 1$ as $i,j\to\infty$. Therefore, $\{a_k\}_{k=0}^{\infty}$ is Cauchy.

The other three cases can be done in a similar manner.

Since $(X,V,*)$ is complete, there exists $u\in X$ such that

$$V(a_n,a_n,...,a_n,u,\tfrac{t}{b})\to 1 \text{ as } n\to\infty. \tag{26}$$

That is, $a_n\to u$ as $n\to\infty$. Now, let $p\in(0,1)$. Then,

$$\begin{aligned} V(u,Tu,...,Tu,\tfrac{t}{b})\leq 1 = V(u,u,...,u,\tfrac{pt}{b})&\\ =\lim_{n\to\infty}V(a_n,a_n,...,a_n,\tfrac{pt}{b}) &\qquad \text{Since } a_n\in Ta_{n-1}.\\ \leq \lim_{n\to\infty}V(a_n,Ta_{n-1},...,Ta_{n-1},\tfrac{pt}{b})&\\ =V(u,Tu,...,Tu,\tfrac{pt}{b}).& \end{aligned} \tag{27}$$

Therefore, by Lemma 3, we have $u\in Tu$. Hence, u is a fixed point of T in X .

Remark 3: Let $n=2$ and $b=1$ in Definition 10, we get the fuzzy metric space definition.

Let $n=2$ and $b=1$ in Definition 16 and also let $CB(X)$ be the set of closed and bounded subsets of X we get the following definition and corollary.

***Definition* 17:** Let $(X,M,*)$ be a fuzzy metric space. The multivalued map $T:X\to CB(X)$ is said to be a q-multivalued quasi-contraction, if there exist $q\in[0,1)$ such that for all $a_i\in X, i=1,2,$

$$d_H(Ta_1, Ta_2) \le q.\max\left\{\begin{matrix} d(a_1,a_2), d(a_1,Ta_1), d(a_2,Ta_2) \\ d(a_1,Ta_2)d(a_2,Ta_1) \end{matrix}\right\}. \quad (38)$$

Corollary 1: Let $(X, M, *)$ be a complete fuzzy metric space. If the mapping $T : X \to CB(X)$ is a q-multivalued quasi-contraction where $0 \le q < 1,$ then T has a fixed point in $X.$ That is, there exist $u \in X$ such that $u \in Tu.$

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