An Orthogonal Left Centralizer and Reverse Left Centralizer on Semiprime Γ-Rings

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Abstract:

Let M be a semiprime Γ-ring . In this paper we introduce the concept of orthogonal left centralizer and reverse left centralizer on a semiprime Γ- ring and we prove the following main result:

Let M be a 2-torsion free semiprime Γ- ring, t be a left centralizer and h be a reverse left centralizer of M , such that $x\alpha z\beta y = x\beta z\alpha y$, for all $x , y , z \in M$, $\alpha , \beta \in \Gamma$ and t , h are commuting. Then t and h are orthogonal if and only if

$t(x)\,\Gamma\, M\,\Gamma\, h(y) + h(x)\,\Gamma\, M\,\Gamma\, t(y) = (0)$, for all $x , y \in M$.

Key Words : semiprime Γ-ring , left centralizer , reverse left centralizer , orthogonal left centralizer and reverse left centralizer .

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I-Introduction :

In 1964 [6] gave the notion of a Γ-ring . This concept is more general than the concept of a ring. In 1966 [2] generalized this concept . . The definition of prime ring and semiprime Γ-ring was introduced in [5]. The definition of 2-torsion free Γ-ring was introduced in [7]. While [8] introduced the concept of left (resp. right) centralizer and Jordan left (resp. right) centralizer of Γ-rings. The concept of higher reverse left (resp. right) centralizer and a Jordan higher reverse left (resp. right) centralizer of Γ-ring was introduce by [4] and the one important question can be answered whether there is a relation between a concepts of a higher reverse left(resp. right) centralizer and a Jordan higher reverse left(resp. right) centralizer within certain conditions .

In this paper , we define and study the concept of orthogonal left centralizer and reverse left centralizer of semiprime Γ-ring and we prove some of lemmas and theorems about orthogonally one of these theorems is :

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , where t and h are commuting .Then the following conditions are equivalent :

(i) t and h are orthogonal

(ii) t(x) Γ h(y) = (0)

(iii) h(x) Γ t(y) = (0)

(iv) t(x) Γ h(y) + h(x) Γ t(y) = (0) .

In our work we need the following Lemmas :

Lemma(1.1): **[**1**]**

If M is a 2-torsion free semiprime Γ-ring and x , y be elements of M , then the following conditions are equivalent :

(i) xΓmΓy = (0) , for all m∈ M

(ii) yΓmΓx = (0) , for all m∈ M

(iii) xΓmΓy + yΓmΓx = (0) , for all m∈ M

If one of these conditions is fulfilled ,then xΓy = yΓx = 0 .

Lemma(1.2): **[**3**]**

Let M be a 2-torsion free semiprime Γ-ring and x , y be elements of M if xΓmΓy + yΓmΓx = (0) , for all m∈M. Then xΓmΓy = yΓmΓx = (0) .

II . Orthogonal Reverse Left (resp.Right) Centralizer on Semiprime Γ-Rings :

In this section we will introduce the concept of orthogonal left centralizer and a reverse left centralizer on semiprime Γ-rings.

Definition (2.1):

Let t be a left centralizer and h be a reverse left centralizer of a Γ-ring M . Then t and h are called **orthogonal** if $t(x)\,\Gamma\, M\, \Gamma\, h(y) = (0) = h(y)\,\Gamma\, M\, \Gamma\, t(x)$, for all x , y ∈ M .

Example (2.2):

Let M be a ring of all 2 × 2 matrices of integer numbers, such that

$M = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} ; x, y \in Z \right\}$ and $\Gamma = \left\{ \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} ; \forall n \in Z \right\}$. Then M is a Γ-ring .

Let t : M → M be an additive mapping of a Γ-ring M into itself , such that

$$t\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \text{, for all } \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \in M$$

h : M → M be an additive mapping of a Γ-ring M into itself , such that

$$h\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \text{, for all } \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \in M$$

Then t is a left centralizer and h is a reverse left centralizer . Then t and h are orthogonal of M .

Example (2.3):

Let t is a left centralizer and h is a reverse left centralizer of a Γ-ring M ,
we put $M^* = M \oplus M = \{(x,y) ; x , y \in M\}$ and $\Gamma^* = \Gamma \oplus \Gamma = \{(\alpha,\beta) ; \alpha , \beta \in \Gamma \}$,
we define t^* and h^* on M^* by
$t^*((x,y)) = (t(x) , 0)$ and $h^*((x,y)) = (0 , h(y))$, for all $(x,y) \in M^*$.
Then t^* and $h^*$are orthogonal M^* .

Lemma (2.4):

Let M be a semiprime Γ- ring , suppose that t be a left centralizer and h be a reverse left centralizer of M ,satisfy $t(x)\,\Gamma\, M\, \Gamma\, h(x) = (0)$, for all x ∈ M . Then $t(x)\,\Gamma\, M\, \Gamma\, h(y) = (0)$, for all x , y ∈ M .

Proof :

Suppose that $t(x)\ \alpha\ z\ \beta\ h(x) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ …(1)

Replace x by $x + y$ in (1) , we have that

$t(x + y)\ \alpha\ z\ \beta\ h(x + y) = 0$

$t(x)\ \alpha\ z\ \beta\ h(x) + t(x)\ \alpha\ z\ \beta\ h(y) + t(y)\ \alpha\ z\ \beta\ h(x) + t(y)\ \alpha\ z\ \beta\ h(y) = 0$

Therefore , by our assumption and Lemma (1.1) , we get

$t(x)\ \alpha\ z\ \beta\ h(y) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Thus $t(x)\ \Gamma\ M\ \Gamma\ h(y) = (0)$, for all $x , y \in M$

Lemma (2.5):

Let M be a 2-torsion free semiprime Γ- ring, t be a left centralizer and h be a reverse left centralizer of M , such that $x\alpha z\beta y = x\beta z\alpha y$, for all $x , y , z \in M$, $\alpha , \beta \in \Gamma$ and t , h are commuting. Then t and h are orthogonal if and only if

$t(x)\ \Gamma\ M\ \Gamma\ h(y) + h(x)\ \Gamma\ M\ \Gamma\ t(y) = (0)$, for all $x , y \in M$.

Proof :

Suppose that t and h are orthogonal

T.P. $t(x)\ \Gamma\ M\ \Gamma\ h(y) + h(x)\ \Gamma\ M\ \Gamma\ t(y) = 0$, for all $x , y \in M$

Since t and h are orthogonal , we have that

$t(x)\alpha\ z\ \beta\ h(y) = 0 = h(y)\alpha\ z\ \beta\ t(x)$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

By Lemma (1.1) , we have that

$t(x)\ \alpha\ h(y) = 0 = h(x)\ \alpha\ t(y)$, for all $x , y \in M$ and $\alpha \in \Gamma$

$t(x)\ \alpha\ h(y) + h(x)\ \alpha\ t(y) = 0$, for all $x , y \in M$ and $\alpha \in \Gamma$

Left multiply by $z\ \beta$, we have that

$z\ \beta\ t(x)\ \alpha\ h(y) + z\ \beta\ h(x)\ \alpha\ t(y) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Since $x\alpha z\beta y = x\beta z\alpha y$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ and t and h are commuting , we have that :

$t(x)\ \alpha\ z\ \beta\ h(y) + h(x)\ \alpha\ z\ \beta\ t(y) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Hence $t(x)\ \Gamma\ M\ \Gamma\ h(y) + h(x)\ \Gamma\ M\ \Gamma\ t(y) = (0)$, for all $x , y \in M$

Conversely , it's clear by using Lemma (1.2)

Theorem (2.6):

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , where t and h are commuting . Then the following conditions are equivalent :

(i) t and h are orthogonal

(ii) th = 0

(iii) ht = 0

(iv) th + ht = 0

Proof : **(i) ⇔ (ii)**

Suppose that t and h are orthogonal

T.P. th = 0

Since t and h are orthogonal , we have that

$h(y)\,\alpha\, z\, \beta\, t(x) = 0$, for all x , y , z ∈ M and α , β ∈ Γ

Replace x by h(y) , we have that

$h(y)\,\alpha\, z\, \beta\, t\, t(h(y)) = 0$

$t\,(h(y)\,\alpha\, z\, \beta\, t(h(y))) = 0$

$t(h(y))\,\alpha\, z\, \beta\, t(h(y)) = 0$, for all y , z ∈ M and α , β ∈ Γ

Since M is a semiprime Γ- ring , we have that

$t(h(y)) = 0$, for all y ∈ M ⇒ th = 0

Conversely, suppose that th = 0

T.P. t and h are orthogonal

$h(t(x\beta y)) = 0$

$h(t(x)\,\beta\, y) = 0$

$h(y)\,\beta\, t(x) = 0$

Since t and h are commuting , we have that

$t(x)\,\beta\, h(y) = 0$

Replace x by xαz , we have that

$t(x\alpha z)\,\beta\, h(y) = 0$

$t(x)\,\alpha\, z\, \beta\, h(y) = 0$, for all x , y , z ∈ M and α , β ∈ Γ …(1)

Since t and h are commuting , we have that

$h(y)\,\alpha\, z\, \beta\, t(x) = 0$, for all x , y , z ∈ M and α , β ∈ Γ …(2)

Hence t and h are orthogonal .

Proof : **(i) ⇔ (iii)**

By the same way in (i) ⇔ (ii) , we get (i) ⇔ (iii) .

Proof : **(i) ⇔ (iv)**

Suppose that t and h are orthogonal

T.P. th + ht = 0

By (ii) and (iii) , we get the require result .

Conversely , suppose that th + ht = 0

T.P. t and h are orthogonal

(th + ht) (yβx) = 0

t(h(yβx)) + h (t(yβx)) = 0

t(h(x) β y) + h (t(y) β x) = 0

t(h(x)) β y + h(x) β t(y) = 0

Replace t(h(x)) by t(x) , we have that

t(x) β y + h(x) β t(y) = 0

Replace β y by β h(y) , we have that

t(x) β h(y) + h(x) β t(y) = 0

Left multiply by zα , we have that

z α t(x) β h(y) + z α h(x) β t(y) = 0 , for all x , y , z ∈ M and α , β ∈ Γ

Since t and h are commuting , we have that

t(x) α z β h(y) + h(x) α z β t(y) = 0 , for all x , y , z ∈ M and α , β ∈ Γ

That is t(x) Γ M Γ h(y) + h(x) Γ M Γ t(y) = (0) , for all x , y ∈ M

By Lemma (2.5) , we get the require result .

Theorem(2.7):

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , where t and h are commuting .Then the following conditions are equivalent :

(i) t and h are orthogonal

(ii) t(x) Γ h(y) = (0)

(iii) h(x) Γ t(y) = (0)

(iv) t(x) Γ h(y) + h(x) Γ t(y) = (0)

Proof : **(i) ⇔ (ii)**

Suppose that t and h are orthogonal

T.P. $t(x)\ \Gamma\ h(y) = (0)$, for all $x , y \in M$

Since t and h are orthogonal , we have that

$t(x)\ \alpha\ z\ \beta\ h(y) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

By Lemma (1.1) , we have that

$t(x)\ \alpha\ h(y) = 0$

Thus , $t(x)\ \Gamma\ h(y) = (0)$, for all $x , y \in M$

Conversely, suppose that $t(x)\ \Gamma\ h(y) = (0)$, for all $x , y \in M$

T.P t and h are orthogonal

$t(x)\ \beta\ h(y) = 0$, for all $x , y \in M$ and $\beta \in \Gamma$

Replace x by $x\alpha z$, we have that

$t(x\alpha z)\ \beta\ h(y) = 0$

$t(x)\ \alpha\ z\ \beta\ h(y) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ …(1)

Since t and h are commuting , we have that

$h(y)\ \alpha\ z\ \beta\ t(x) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$ …(2)

Hence t and h are orthogonal .

Proof : **(i) ⇔ (iii)**

By the same way in (i) ⇔ (ii) , we get (i) ⇔ (iii) .

Proof : **(i) ⇔ (iv)**

Suppose that t and h are orthogonal

T.P. $t(x)\ \Gamma\ h(y) + h(x)\ \Gamma\ t(y) = (0)$

By (ii) and (iii) , we get the require result .

Conversely , suppose that $t(x)\ \Gamma\ h(y) + h(x)\ \Gamma\ t(y) = (0)$

T.P t and h are orthogonal

By our assumption , we have that

$t(x)\ \beta\ h(y) + h(x)\ \beta\ t(y) = 0$, for all $x , y \in M$ and $\alpha \in \Gamma$

Left multiply by $z\alpha$, we have that

$z\ \alpha\ t(x)\ \beta\ h(y) + z\ \alpha\ h(x)\ \beta\ t(y) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

Since t and h are commuting , we have that

$t(x)\ \alpha\ z\ \beta\ h(y) + h(x)\ \alpha\ z\ \beta\ t(y) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

$t(x)\ \Gamma\ M\ \Gamma\ h(y) + h(x)\ \Gamma\ M\ \Gamma\ t(y) = (0)$, for all $x , y \in M$

By Lemma (2.5) , we get the require result .

Corollary (2.8):

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , where t and h are commuting . Then the following conditions are equivalent , for all $x \in M$:

(i) t and h are orthogonal

(ii) $t(x)\ \Gamma\ h(x) = 0$

(iii) $h(x)\ \Gamma\ t(x) = 0$

(iv) $t(x)\ \Gamma\ h(x) + h(x)\ \Gamma\ t(x) = 0$

Proof :

Obvious

Lemma(2.9):

Let M be a completely prime Γ- ring , t and h are orthogonal left centralizer and reverse left centralizer resp. of M . Then either $t = 0$ or $h = 0$.

Proof :

Suppose that t and h are orthogonal

T.P. $t = 0$ or $h = 0$

$t(x)\ \alpha\ z\ \beta\ h(y) = 0$, for all $x , y , z \in M$ and $\alpha , \beta \in \Gamma$

By Lemma (1.1) , we have that

$t(x)\ \alpha\ h(y) = 0$, for all $x , y \in M$ and $\alpha \in \Gamma$

Since M is a completely prime Γ- ring , we get either

$t(x) = 0$ or $h(y) = 0$, for all $x , y \in M$

$t = 0$ or $h = 0$

Theorem(2.10):

Let M be a 2-torsion free semiprime Γ- ring , t be a left centralizer and h be a reverse left centralizer of M , suppose that $t(x)\ \alpha\ t(x) = h(x)\ \alpha\ h(x)$, for all $x \in M$ and $\alpha \in \Gamma$.

Then t + h and t – h are orthogonal .

Proof :

$((t + h) \alpha (t – h) + (t – h) \alpha (t + h))(x)$

$= t(x) \alpha t(x) – t(x) \alpha h(x) + h(x) \alpha t(x) – h(x) \alpha h(x) + t(x) \alpha t(x) + t(x) \alpha h(x) –$
$h(x) \alpha t(x) – h(x) \alpha h(x)$

$= 0$

Therefore , $((t + h) \alpha (t – h) + (t – h) \alpha (t + h))(x) = 0$, for all $x \in M$ and $\alpha \in \Gamma$

By Corollary (2.8) (iv) $\Rightarrow$ (i) , we get the require result .

References:

[1] M.Ashraf, and M. R. Jamal,"Orthogonal Derivations in Γ-Rings" Advanced in Algebra, 3(1), 2010 ,pp. 1-6.

[2] W . E Barnes "On The Γ-Rings of Nobusawa", Pac. J. Math, 8(3) ,1966 , pp. 411-422.

[3] S.Chakraborty and A.C. Paul, "On Jordan K-Derivations of 2-Torsion Free Prime Γ_N -rings", Punjab University J. of Math. ,Vol.40, 2008 , PP.97-101.

[4] F.R Jarullah and S.M Salih," A Jordan Higher Reverse Left (resp. right) Centralizer on Prime Γ-Rings", Iraqi Journal of Science , 61(9),2020, PP. 2341-2349 .

[5] A.H. Majeed and S.A. Hamil," **γ**- Orthogonal for K- Derivations and K- Reverse Derivations", Journal of physics,1530,2020 p.p.1-6 .

[6] N. Nobusawa," On a Generalization of the Ring Theory",Osaka Journal Math.,1,1964, p.p.81-89.

[7] M.A. Ozturk , H. Durna and T.Acet,"A note on Derivations on Prime Gamma Rings with Characteristic 2", Bulletin of the International Mathematical Virtual Institute ,10(2), 2020, p.p.201-212 .

[8] R.C.Shaheen,"On Centralizers on Some Gamma Ring",Journal of Al-qadisiyah for Pure Science ,12(2), pp.219-229, 2007.