Research Article

ON MULTIDIMENSIONAL FIXED POINT THEOREMS IN ORDERED V-FUZZY b-METRIC SPACES

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ABSTRACT: In this article, we establish some coincidence point and common fixed pointtheorems in the recently introduced notion of partially ordered V-fuzzy b-metric spaces for ϕ-construction. Using the results, suitable conditions are framed to make sure the existence of multidimensional coincidence point and common fixed point results, which generalize and improve fixed point results of Gupta and Kanwar [13].

MSC: primary 47H10; secondary 54H25

KEYWORDS: Common fixed point; Fuzzy metric spaces; V-fuzzy metric spaces; V-fuzzy b-metric spaces.

INTRODUCTION:

Fixed point theorems have been studied in many contexts, one of which is the fuzzy setting. The concepts of fuzzy sets were initially introduced by Zadeh [34] in 1965. To use this concept in topology and analysis, the theory of fuzzy sets and its applications have been developed by eminent authors. It is well known that a fuzzy metric space is an important generalization of a metric space. Sedghi et al. [30] generalized the definition of metric space and introduced the notion of S-metric space. Soon after, Abbas et al. [1] generalized definition of S-metric space and introduced the notion of A-metric space. On the other hand, Sun and Yang [31] generalized the definition of fuzzy metric space (see, [10], [23]) and introduced the notion of G-fuzzy metric space. In the process of generalization, Gupta and Kanwar [13] generalized the definition of G-fuzzy metric space and introduced the notion of V-fuzzy metric space and established the coupled fixed point results in a partially ordered V-fuzzy metric space.In 1987, Guo and Lakshmikhantham [12] introduced an important notion of coupled fixedpoint for some continuous and discontinuous operators. Thereafter, many authors introducedcoupled, tripled and higher dimensional fixed point and coincidence point results in different spaces. Sedgi et al. [29] introduced coupled fixed point results in fuzzy metric spaces. Recently, Roldan et al.[27] established multidimensional coincidence results in partially ordered fuzzy metric spaces for compatible mappings.Many authors, obtained fixed point results under the assumptions $(a):\sum_{n=1}^{\infty}\phi^{n}(t)=1<\infty$ forall t > 0 and some other conditions (see, [8] [14] [33] [15]). Ciric [7], weakened thecondition (a) and introduce the condition (CBW). Jachymski [20] introduced condition: $0 < \phi(t) < t$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for all t> 0. Recently, Fang [9] gave the weaker conditionof $\phi-$contraction in the context of Menger probabilistic metric spaces and fuzzy metricspaces.

In addition to fuzzy metric spaces, there are still many extensions of metric and metric space terms. Bakhtin [2] and Czerwik [6] introduced a space where, instead of triangle inequality, a weaker condition was observed, with the aim of generalization of Banach contraction principal [3]. They called these spaces b-metric spaces. Relation between b-metric and fuzzy metric spaces is considering in [29]. On the other hand, in [33] the notion of a fuzzy b-metric space was introduced, where the triangle inequality is replaced by a weaker one.

Our aim in this article is to present some multidimensional coincidence and common fixed point theorems for ϕ-contraction in partially ordered V-fuzzy b-metric spaces with the help of some coincidence point and common fixed point results for a pair of mappings.

PRELIMINARIES:

In order to state and prove out results, we will use the following notions. These notions can be found in [26]. Throughout in this article, I = [0, 1], m is any Natural number, X^m will denote the Cartesian product of m copies of X where X is a nonempty set, n and p will denote non-negative integers. $i, j \in \{1,2,......,m\}$ and for any $Y,U \in X^m$ will mean, $Y = (x_1, x_2,......, x_m)$ and $U = (u_1,u_2,......,u_m)$ respectively. For brevity $g(x)$ will be denoted as gx.

Henceforth, let a fix partition $\{A, B\}$ of $A_m = \{1,2,......., m\}$, i.e., $A \cup B = \Lambda_m$ and $A \cap B = \phi$ such that A and B are non-empty sets. We will denote :

$\Omega_{A,B} = \{\sigma : \Lambda_m \to \Lambda : \sigma(A) \subseteq A$ and $\sigma(B) \subseteq B\}$ and

$\Omega'_{A,B} = \{\sigma : \Lambda m \to \Lambda : \sigma(A) \subseteq B$ and $\sigma(B) \subseteq A\}$

Let $(X,\preceq)$ be a partially ordered space, $x,u \in$ X and $i \in \Lambda_m$, we will use the following notation:

$$x \preceq_i v \Leftrightarrow \begin{cases} x \preceq u & \text{if } i \in A \\ x \succeq u & \text{if } i \in B \end{cases}$$

Consider on X^m the following partial order: for $Y,U \in X^m$, $Y \preceq_m U \Leftrightarrow x_i \preceq_i v_i$ for all $i \in (1,2,.....,m)$. If $Y \preceq_m U$ or $Y \succeq_m U$, then Y and U are comparable (we will denote $(Y \approx U)$)

Definition 2.1: [26] A binary operation $*:[0,1]\times[0,1] \to [0,1]$ is said to be continuoustriangular norm (t-norm) if * satisfies the following conditions:

(i) a * b = b and a * (b * c) = (a * b) * a, $\forall\ a,b,c \in [0,1]$;

(ii) * is continuous;

(iii) 1 * a = a $\forall a \in [0,1]$

(iv) $a * b \le c * d$, whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$

then $*_{i=1}^{m} = a_1 * a_2 * ...... * a_m$.

Definition 2.2: [13] A triple (X, V, *) is said to be V-fuzzy metric space (V-FMS), where X is a nonempty set, * is a continuous t-norm, and V is a fuzzy set on $X^n \times (0,\infty)$ satisfyingthe following conditions for all t,s> 0:

(V-FMS1) $V(x,x,x,......,x,y,t) > 0$ for all $x,y \in X, x \neq y$;

(V-FMS2) $V(x_1,x_1,.....,x_1,x_2,t) \geq V(x_1,x_2,.....,x_n,t)$ for all $x_1,x_2,.....,x_m \in X$ with $x_1 \neq x_2 \neq .... \neq x_n$;

(V-FMS3) $V(x_1,x_2,......,x_n,t) = 1$ if and only if $x_1 = x_2 = ... = x_n$;

(V-FMS4) $V(x_1,x_2,.....,x_n,t) \geq V(p(x_1,x_2,....,x_n t))$, where p is a permutation function;

(V-FMS5) $V(x_1,x_2,.....,x_n,t+s) \geq V(x_1,x_2,.....,x_{n-1},l,t) * V(l,l,l,....,l,x_n,s)$;

(V-FMS6) $\lim_{t\to\infty} V(x_1,x_2,....,x_n,t) = 1$;

(V-FMS7) $V(x_1,x_2,.....,x_n,....) : (0,\infty) \to [0,1]$ is continuous.

Example 2.3: Define a continuous t-norm as $*b = ab$a and let $X = \mathbf{R}$ and (X, A) be A-metric space. Define $V : X^n \times (0,\infty) \to I$ such that,

$V(x_1,x_2,....,x_n,t = \left[\exp\left(\frac{A(x_1x_2,......,x_n)}{t}\right)\right]^{-1}$ for all $x_1,x_2,......,x_n \in X$ and t> 0. Then $(X,V,*)$ is a V-fuzzy metric space.

Definition 2.4: [13] Let triple $(X,V,*)$ be a V-fuzzy metric space. A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$ x if $\lim_{n\to\infty} V(x_n,x_n,.....,x_n,x,t) \to 1$

for all t > 0, i.e., for each $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$, such that $V(x_n,x_n,.....,x_n,x,t) > 1-\varepsilon$ for all $n \geq n_0$ and for all t > 0.

Definition 2.5: [13] Let triple $(X,V,*)$ be a V-fuzzy metric space. A sequence $\{x_n\}$in X is said to be Cauchy sequence if for any $V(x_n,x_n,.......,x_n,x_m,t) \to 1$ as $n,m \to \infty$, for all t > 0 and $n,m \in \mathbf{N}$, i.e., for any $\varepsilon > 0$, there is $n_0 \in \mathbf{N}$, such that $V(x_n,x_n,......,x_n,x_m,t) > 1-\varepsilon$ for all $n,m \geq n_o$ and for all t > 0.

Definition 2.6: [13] A V-fuzzy metric space $(X,V,*)$ is said to be complete if every Cauchy sequence in X is convergent sequence.

Definition 2.7: [32] Let $(Y^r, \preceq_r)$ be a partially ordered set and S and T two self maps of Y^r We say that S is T-isotone map, if for any $X_1, X_2 \in Y^r$ $T(X_1) \preceq_r T(X_2) \Rightarrow S(X_1) \preceq_r S(X_2)$.

Definition 2.8: [26] Let $(X, \preceq)$ be a partially ordered space and $F : X^m \to X$ and $g : X \to X$ two mappings. We say that F has the mixed g-monotone property (w.r.t {A, B}) if F is g-monotone non-decreasing in arguments of A and g-monotone non-decreasing in arguments of B, i.e. for all $x_1,......,x_m,y,z,\in X$,

$gy \preceq gz \Rightarrow F(x_1,.....,x_{i-1},y,x_{i+1},.....,x_m) \preceq_i F(x_1,.......,x_{i-1},z,x_{i+1},......,x_m)$ for all i.

Definition 2.9: Two self maps T and G in X are said to be weakly compatible if $TGx = GTx$ for all $x \in X$ such that $Tx = Gx$.

Definition 2.10: [18] Let $\Phi = (\sigma_1, \sigma_2, ..., \sigma_m)$ be an m-tuple mappings from $\{1,2,....., m\}$ into it. The mappings $F : X^m \to X$ and $g : X \to X$ are said to be Φ-weakly compatible if $gF\left(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ..., x_{\sigma_i(m)}\right) = F\left(gx_{\sigma_i(1)}, gx_{\sigma_i(2)}, ..., gx_{\sigma_i(m)}\right)$,
whenever $gx_i = F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ..., x_{\sigma_i(m)})$ for all i and some $(x_1, x_2, ..., x_m) \in X^m$

Definition 2.11: Given $T, G : X \to X$ we will say a point $x \in X$ is said to be

- fixed point if $T(x) = x$.
- common fixed point if $T(x) = G(x) = x$
- coincidence point if $T(x) = G(x)$.

(see, [11]) Given $T : X^2 \to X$, a point $(x, y) \in X^2$ is said to be

- coupled coincidence point of T and g if $T(x, y) = gx$ and $T(y, x) = gy$.
- coupled common fixed point of T and g if $T(x, y) = gx = x$ and $T(y, x) = gy = y$.

  Given $F : X^m \to X$ and $g : X \to X$ a point is said to be
- (see, [26]) Φ-coincidence point if
  $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ..., x_{\sigma_i(m)}) = gx_i$
  for all $i \in (1,2,....., m)$ and $(\sigma_1, \sigma_2, ..., \sigma_m)$ is an m-tuple mappings from {1,2,.., m} into itself.

Definition 2.12: [9] Let ϕ_w denote the family of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying the condition, for each $t > 0$ there exist $r \geq t$ such that $\lim_{m \to \infty} \varphi^m(r) = 0$.

Lemma 2.13: [13] In a V-fuzzy metric space $V(x_1, x_2, ..., x_n, t)$ is non-decreasing with respect to t.

Lemma 2.14: [9] Let $\phi \in \Phi_w$, then for every $t > 0$ $\exists$ $r \geq t$ such that $\phi(r) < t$.

Proposition 2.15: [25] If $Y \preceq_m U$, it follow that

$$\left(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(m)}\right) \preceq_m \left(\mu_{\sigma(1)}, \mu_{\sigma(2)}, ..., \mu_{\sigma(m)}\right) \text{ if } \sigma \in \Omega_{A,B'}$$
$$\left(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(m)}\right) \succeq_m \left(\mu_{\sigma(1)}, \mu_{\sigma(2)}, ..., \mu_{\sigma(m)}\right) \text{ if } \sigma \in \Omega'_{A,B'}$$

Definition 2.16:[16]The 3-tuple (X, M, T) is known as fuzzy b-metric space if X is any set, T is a continuous t-norm, and M is a fuzzy set in $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$, and a given real number $b \geq 1$,

(i) $M(x, y, t) > 0$,

(ii) $M(x, y, t) = 1$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $T(M(x, y, \frac{t}{b}), M(y, z, \frac{s}{b})) \leq M(x, z, t + s)$,

(v) $M(x, y, \cdot) : [0, \infty) \to [0,1]$ is continuous.

Definition 2.17: **[**18]The 3-tuple $\left(X, V, * \right)$ is called a V-fuzzy b-metric space if $*$ is a continuous t-norm, and V is a fuzzy set in $X^{n}\times(0,\infty)$ satisfying the following conditions for all $x_{i}, y, a\in X, b\geq 1$ and $s, t>0$;

(i) $V\left(x,x,...,x,y,\frac{t}{b} \right)>0,\ x\neq y$

(ii) $V\left(x_1,x_1,...,x_1,x_2,\frac{t}{b} \right)\geq V\left(x_1,x_2,x_3,...,x_n,\frac{t}{b} \right),\ x_2\neq x_3\neq ...\neq x_n$

(iii) $V\left(x_1,x_2,x_3,...,x_n,\frac{t}{b} \right)=1 \Leftrightarrow x_1=x_2=x_3=...=x_n$

(iv) $V\left(x_1,x_2,x_3,...,x_n,\frac{t}{b}\right)= V(p\left(x_1,x_2,x_3,...,x_n\right),\frac{t}{b})$ where $p(x_1,x_2,x_3,...,x_n)$ is permutation on $x_1,x_2,x_3,...,x_n$

(v) $V(x_1,x_2,x_3,...,x_{n-1},a,\frac{t}{b})*V(a,a,a,...,a,x_n,\frac{t}{b})\leq V(x_1,x_2,x_3,...,x_{n-1},x_n,t+s)$

(vi) $V(x_1,x_2,x_3,...,x_n,\frac{t}{b})=1$ as $t\to\infty$

(vii) $V(x_1,x_2,x_3,...,x_{n-1},\cdot):(0,\infty)\to(0,1]$ is continuous.

Lemma 2.18: [18]Let $\left(X,V,*\right)$ be a V-fuzzy b-metric space. Then $V(x_1,x_2,x_3,...,x_n,\frac{t}{b})$ is non-decreasing with respect to t.

Proof: Since $t>0, b\geq 1$ and $t+s>t$ for $s>0,$ by letting $l=x_n$ in condition (v) of V-fuzzy b-metric space we get, $V(x_1,x_2,x_3,...,x_n,\frac{t+s}{b})\geq V(x_1,x_2,x_3,...,x_n,\frac{t}{b})*V(x_n,x_n,x_n,...,x_n,\frac{s}{b}).$ This implies that $V(x_1,x_2,x_3,...,x_n,\frac{t+s}{b})\geq V(x_1,x_2,x_3,...,x_n,\frac{t}{b}).$ So, $V(x_1,x_2,x_3,...,x_n,\frac{t}{b})$ is non-decreasing with respect to t.

Lemma 2.19:[18] Let $(X,V,*)$ be a V-fuzzy metric space such that $V(x_1,x_2,x_3,...,x_n,\frac{kt}{b})\geq V(x_1,x_2,x_3,...,x_n,\frac{t}{b})$ with $b\geq 1,\ k\in(0,1).$ Then $x_1=x_2=x_3=...=x_n.$

Proof: By assumption $V(x_1,x_2,x_3,...,x_n,\frac{kt}{b})\geq V(x_1,x_2,x_3,...,x_n,\frac{t}{b})$ (1)For $t>0, b\geq 1,$ since $\frac{kt}{b}<\frac{t}{b},$ by lemma 2.1 we have $V(x_1,x_2,x_3,...,x_n,\frac{kt}{b})\leq V(x_1,x_2,x_3,...,x_n,\frac{t}{b}).$ (2)From (1) and (2), and the definition V-fuzzy metric space we get $x_1=x_2=x_3=...=x_n.$

Definition 2.20: [18] Let $(X,V,*)$ is said to be V-Fuzzy b-metric space. A sequence $\{x_r\}$ is said to converge to a point $x\in X$ if $V(x_r,x_r,x_r,...,x_r,x,\frac{t}{b})\to 1$ *as* $r\to\infty$ *for all* $t>0, b\geq 1,$ that is, for each $\varepsilon>0,$ there exist $n\in N$ such that for all $r\geq N,$ we have $V(x_r,x_r,x_r,...,x_r,x,\frac{t}{b})\to 1-\varepsilon,$ and we write $\lim_{r\to\infty} x_r = x.$

Definition 2.21: [18] Let $(X,V,*)$ is said to be V-Fuzzy b-metric space. A sequence $\{x_r\}$ is said to Cauchy sequence if $V(x_r,x_r,x_r,...,x_r,x_q,\frac{t}{b})\to 1$ *as* $r,q\to\infty$ *for all* $t>0, b\geq 1,$ that is, for each $\varepsilon>0,$ there exist $n_0\in N$ such that for all $r,q\geq N,$ we have $V(x_r,x_r,x_r,...,x_r,x_q,\frac{t}{b})\to 1-\varepsilon.$

Definition 2.22:[18] The V-Fuzzy b-metric space $(X,V,*)$ is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.23:[18] The mappings $P: X \times X \to X \; and \; Q: X \to X$ are said to be compatible on V-Fuzzy b-metric space if $\lim_{r\to\infty} V(QP(x_r, y_r), QP(x_r, y_r), ..., QP(x_r, y_r), P(Qx_r, Qy_r), \frac{t}{b}) = 1$
And $\lim_{r\to\infty} V(QP(y_r, x_r), QP(y_r, x_r), ..., QP(y_r, x_r), P(Qy_r, Qx_r), \frac{t}{b}) = 1$, whenever $\{x_r\}$ and $\{y_r\}$ are sequences in X such that $\lim_{r\to\infty} Q(x_r) = \lim_{r\to\infty} P(x_r, y_r) = x \; and \; \lim_{r\to\infty} Q(y_r) = \lim_{r\to\infty} P(y_r, x_r) = y$ for all $x, y \in X \, and \, t > 0, b \geq 1$.

MAIN RESULTS

In this section, we establish our main results and utilize these results to acquire the multidimensional results in partially ordered V-fuzzy metric spaces.

Definition 3.1: Let $(X, V, *)$ be a V-fuzzy b-metric space and T, S two self maps on X. T and S are said to be compatible if and only if

$$\lim_{m\to\infty} V\left(T(S(x_m)), T(S(X_m)), ..., T(S(X_m)), S(T(x_m)), \frac{t}{b} \right) = 1$$

for all $t >$ 0 & $b \geq 1$, whenever $\{x_m\} \in X$ such that $\lim_{m\to\infty} T(x_m) = \lim_{x\to\infty} S(x_m) = x$ for some $x \in X$.

Lemma 3.2: Let $(X, V, *)$ be a V-Fuzzy b-Metric Space and $\{V_m\}$ a sequence in $(X, V, *)$. If there exists afunction $\phi \in \Phi_w$ such that

(i) $\phi(t) > 0$, for all t > 0;

(ii) $V(v_m, v_m, ..., v_m, v_{m+1}, \varphi(\frac{t}{b})) \geq V(v_{m-1}, v_{m-1}, ..., v_{m-1}, v_m, \frac{t}{b})$
for every $m \in \mathbf{N}$ and t > 0 and $b \geq 1$, then $\{v_m\}$ is a Cauchy sequence.

Proof: From (V-Fuzzy b-Metric Space (vi)) of definition of V-fuzzy b-metric space, we have $\lim_{t\to\infty} V(v_1, v_2, ..., v_{n-1}, v_n, \frac{t}{b}) = 1$ Which implies that for any $\varepsilon > 0$, there is $t_0 > 0$ such that $\lim_{t\to\infty} V(v_0, v_0, ..., v_0, v_1, \frac{t_0}{b}) = 1 - \varepsilon$

Now we have $\phi \in \Phi_w$, then there exists $t_1 \geq t_0$ such that $\lim_{t\to\infty} \phi^m(t_1) = 0$

Therefore, for t> 0, there is $m_0 \in \mathbf{N}$, such that $\lim_{m\to\infty} \phi^m(t_1) \leq t$ for all $m \geq m_0$, from condition (i), $\phi^m(t) > 0$, for all $m \in \mathbf{N}$ and t > 0. It follows by induction and condition (ii) that $V(v_m, v_m, ..., v_m, v_{m+1}, \varphi^m(\frac{t}{b})) \geq V(v_0, v_0, ..., v_0, v_1, \frac{t}{b})$ for all $m \in \mathbf{N}$ and t > 0. Using Lemma 2.13, we have $V(v_m, v_m, ..., v_m, v_{m+1}, \frac{t}{b}) \geq V(v_m, v_m, ..., v_m, v_{m+1}, \varphi^m(\frac{t_1}{b}))$

$$\geq V(v_0, v_0, ..., v_0, v_1, \tfrac{t_1}{b}) \geq V(v_0, v_0, ..., v_0, v_1, \tfrac{t_0}{b}) \geq 1 - \varepsilon$$

Hence, we can observe that $m \to \infty$, $V(v_m, v_m, ..., v_m, v_{m+1}, \frac{t}{b}) \to 1$ for any $\varepsilon > 0$ and t > 0.
Now, from (V-Fuzzy b-metric space (v)) of definition of V-Fuzzy b-Metric Space we know that for $p \in \mathbf{N}$ and t > 0 we have

$$V(v_m, v_m, ..., v_m, v_{m+p}, \tfrac{t}{b}) \geq V\left(v_m, v_m, ..., v_m, v_{m+1}, \frac{t}{pb} \right)$$

$$*V\left(v_{m+1} v_{m+1}, ..., v_{m+1} v_{m+2}, \frac{t}{pb} \right) * .... p\text{-times}......$$

$$*V\left(v_{m+p-1}, v_{m+p-1}, ..., v_{m+p-1}, v_{m+p}, \frac{t}{pb}\right)$$

Letting, $m \to \infty$, we get

$$V(v_m, v_m, ..., v_m, v_{m+p}, \tfrac{t}{b}) \geq 1*1*....p \text{-times}....*1*1.$$

Hence, we can conclude that sequence $\{v_m\}$ is a Cauchy sequence.

Theorem 3.3: Let $(X,V,*)$ be a complete V-Fuzzy b-Metric Space, and $(X,\preceq)$ a partially ordered set. Let $T: X \to X$ and $G: X \to X$ be two maps such that

(i) $TX \subseteq G(X)$.

(ii) T is a G-isotone mapping.

(iii)Assume that there exists a function $\phi \in \Phi_w$, such that

$$V\big(T(x),T(x).....,T(x),T(y),\varphi(\tfrac{t}{b})\big) \geq V\big(G(x),G(x),....,G(x),G(y),\tfrac{t}{b}\big) \qquad ....(1)$$

$\forall x, y \in X, t > 0$ and $G(x) \preceq G(y)$.

Suppose that either

(a_1) T, G are continuous and compatible maps; or

(a_2) X has the following property

(a) If x_m x is a non-decreasing sequence such that $x_m \to x$ then $x_m \leq x \forall m \in \mathbf{N}$.

(b) If x_m is a non-increasing sequence such that $x_m \to x$ then $x_m \geq x \forall m \in \mathbf{N}$

and also suppose that either

$(a_{2.1})$ G is continuous and T, G are compatible maps; or

$(a_{2.2})$ G(X) is closed.

If there exists $x_0 \in X$ such that $G(x_0 \approx T(x_0))$ then T and G have a coincidence point.

Proof: Let $x_0 \in X$ x be a point such that $G(x_0) \approx T(x_0)$. Given that $T(X) \subseteq G(X)$. So, we choose $x_1 \in X$ such that $G(x_1) = T(x_0)$. Continuing in this way, we construct a sequence $\{x_m\} \in X$ for $m \in \mathbb{N} \cup \{0\}$ such that $G(x_{m+1}) = T(x_m)$.

Since $G(x_0) \approx T(x_0)$, we suppose that $G(x_0) \preceq T(x_0)$ (the case $G(x_0 \succeq T(x_0))$ treated as same). Assume that $G(x_{m-1}) \preceq G(x_m)$ and we have T is G-isotone mapping which implies $T(x_{m-1}) \preceq T(x_m)$ .We set $G(x_0) = v_0 \preceq T(x_0) = v_1$ and $T(x_{m-1}) = V_m \preceq T(x_m) = V_{m+1}$

Indeed, the sequence $\{v_m\}$ is a non-decreasing sequence. From (1), we get

$$\begin{aligned} V(v_m, v_m, ..., v_m, v_{m+1}, \varphi(\tfrac{t}{b})) &= V(T_{m-1}, T_{m-1}, ..., T_{m-1}, T_m, \varphi(\tfrac{t}{b})) \\ &\geq V(G(x_{m-1}), G(x_{m-1}), ..., G(x_{m-1}), G(x_m), \tfrac{t}{b}) \\ &= V(v_{m-1}, v_{m-1}, ..., v_{m-1}, v_m, \tfrac{t}{b}) \qquad ....(2) \end{aligned}$$

for all $m \in \mathbb{N} \cup \{0\}$ and t > 0, $b \geq 1$.Obviously from condition (iii) we observe that $\phi(t) > 0$ for all t > 0.

From Lemma 3.2, we conclude that $\{v_m\}$ is a Cauchy sequence. Since $(X,V,*)$ is complete V-Fuzzy b-Metric Space, there exists a point $v \in X$ such that , i.e.,

$$\lim_{m\to\infty} T(x_m) = \lim_{m\to\infty} G(x_m) = v \qquad ....(3)$$

By considering the condition (a_1), T and G are compatible, that is

$$\lim_{m\to\infty} V\left(G(T(x_m)), G(T(x_m)), ..., G(T(x_m)), T(G(x_m)), \tfrac{t}{b}\right) = 1,$$

for all t > 0. Since T and G both are continuous maps, we get

$$V(G(v), G(v), ..., G(v), T(v), \tfrac{t}{b}) = 1$$

for all t > 0, which implies that $G(v) = T(v)$. Thus, v is a coincidence point of T and G in X.
Now, suppose that conditions (a_2) and $(a_{2.1})$ hold. Since G is continuous and T, G are compatible maps, we have

$$\lim_{m\to\infty} T(G(x_m)) = \lim_{m\to\infty} G(T(x_m)) = \lim_{m\to\infty} G(G(x_m)) = G(v) \qquad ....(4)$$

By Lemma 2.14, we have for every t > 0 there exists $r \geq t$ such that $t - \phi(r) > 0$. Using Lemma 2.14, (1), (4), (V-Fuzzy b-Metric Space (iii)), (V-Fuzzy b-Metric Space (v)) and considering $m \to \infty$, we get

$$\begin{aligned}
& V\left(G(v), G(v), ..., G(v), T(v), \tfrac{t}{b}\right) \\
& \geq V\left(G(v), G(v), ..., G(v), G(G(x_{m+1}) \tfrac{t}{b} - \varphi(r)\right) \\
& * V\left(G(G(x_{m+1}), G(G(x_{m+1}), ..., G(G(x_{m+1}), T(v), \phi(r)\right) \\
& = V(G(v), G(v), ..., G(v), G(T(x_m), \tfrac{t}{b} - \varphi(r)) \\
& * V(G(T(x_m)), G(T(x_m)), ..., G(T(x_m)), T(v), \phi(r)) \\
& \geq V(G(T(x_m)), G(T(x_m)), ..., G(T(x_m)), T(v), \phi(r)) \\
& = V(T(G(x_m)), T(G(x_m)), ..., T(G(x_m)), T(v), \phi(r)) \\
& \geq V(G(G(x_m)), G(G(x_m)), ..., G(G(x_m)), G(v), r) = 1
\end{aligned}$$

Hence, $G(v) = T(v)$, v is coincidence point of T and G.
Now, considering conditions (a_2) and $(a_{2.2})$. Since $(X, V, *)$ is complete V-Fuzzy b-Metric Space and $G(X)$ is closed, there exists ${}_0 \in X$ v such that

$$\lim_{m\to\infty} T(x_m) = \lim_{m\to\infty} G(x_m) = G(v_0) = v \qquad .....(5)$$

$G(x_m)$ is a non-decreasing sequence. So, $G(x_m) \leq G(v_0)$ for all $m \in \mathbf{N}$. Using (1) and in the view of Lemma 2.13 and Lemma 2.14, we get

$$\begin{aligned}
& V(T(x_m), T(x_m), ..., T(x_m), T(v_0), \tfrac{t}{b}) \\
& \geq V(T(x_m), T(x_m), ..., T(x_m), T(v_0), \phi(r)) \\
& \geq V(G(x_m), G(x_m), ..., G(x_m), G(v_0), r) \\
& \geq V(G(x_m), G(x_m), ..., G(x_m), G(v_0), t) \qquad .....(6)
\end{aligned}$$

for all t > 0, $b \geq 1$, and $m \in \mathbf{N}$, taking $m \to \infty$ in (6) we get $T(x_m) \to T(v_0)$ and by uniqueness of the limit we conclude that $G(v_0) = T(v_0)$. Hence, v_0 is a coincidence point of T and G.

Theorem 3.4: In addition to hypothesis of Theorem 3.3, suppose that X is a totally ordered set then T and G have a unique coincidence point. Moreover, if G is weakly compatible with T then T and G have unique common fixed point.

Proof: Assume that $x, v \in X$ are coincidence points of T and G. Since we have for all coincidence points $x, v \in X$, there exists a point $u \in X$ such that $G(u)$ is comparable to $G(x)$ and $G(v)$. Let $u_0 = u$ then define a sequence $G(u_m)$. The sequence $G(u_m)$ and its limit defined similar as in Theorem 3.3, so we have and $G(u_o) = T(u_1)$.
By the (V-Fuzzy b-Metric Space (vi)), we have

$$\lim_{t\to\infty} V(G(u), G(u), ..., G(u), G(x), \tfrac{t}{b}) = 1$$

which implies that for any $\varepsilon \in (0,1)$ there exists t_1 such that

$$V(G(u_0), G(u_0), ..., G(u_0), G(x), \tfrac{t_1}{b}) > 1-\varepsilon. \qquad .....(7)$$

As $\phi \in \Phi_w$, so there exists $r \geq t_1$ such that $\lim_{m\to\infty} \phi^m(r) \to 0$, which implies, there exists $m_0 \in \mathbb{N}$ such that $\varphi^m(r) < \frac{t}{b}$ for all $m \geq m_0$ and t > 0.

By (1) and Lemma 2.13, we get

$$\begin{aligned}
& V(G(u_m), G(u_m), ..., G(u_m), G(x), \tfrac{t}{b}) \\
& \geq V(G(u_m), G(u_m), ..., G(u_m), G(x), \phi^m(r)) \\
& = V\big(T(u_{m-1}), T(u_{m-1}), ..., T(u_{m-1}), G(x), \phi^m(r)\big) \\
& \geq V\big(G(u_{m-1}), G(u_{m-1}), ..., G(u_{m-1}), G(x), \phi^{m-1}(r)\big) \\
& \geq .... \\
& \geq V(G(u_0), G(u_0), ..., G(u_0), G(x), r) \\
& \geq V(G(u_0), G(u_0), ..., G(u_0), G(x), t_1) \qquad ......(8) \\
& > 1-\varepsilon
\end{aligned}$$

for all $m \geq m_0$ and t > 0.

Hence, from (8) we can conclude that $\lim_{m\to\infty} G(u_m) = G(x)$ and similarly we can easily show that $\lim_{m\to\infty} G(u_m) = G(v)$ and from Uniqueness of limit we get $G(x) = G(v)$.

Now, let $e = T(v) = G(v)$ and T, G are weakly compatible mappings i.e., $T(e) = T(G(v)) = G(T(v)) = G(e)$. So, e is a coincidence point which implies that $G(e) = G(v) = e$. Hence, e is a coincidence and common fixed point of T and G. Now suppose that there exists $e^*(\neq e) \in X$ such that $T(e^*) = G(e^*) = e^*$. Then $e = G(e) = G(e^*) = e^*$ which shows uniqueness of common fixed point of T and G.

Corollary 3.5: Let $(X, V, *)$ be a complete V-Fuzzy b-Metric Space and $(X, \preceq)$ a partially ordered set. X has the following property:

(a) If $\{x_m\}$ is a non-decreasing sequence such that $x_m \to x$ the $x_m \leq x \forall m \in \mathbf{N}$.

(b) If $\{x_m\}$ is a non-increasing sequence such that $x_m \to x$ then $x_m \geq x \forall m \in \mathbf{N}$.

Suppose that $T: X \to X$ be a mapping such that T is non-decreasing and continuous mapping. Let there exists $\phi \in \Phi_w$ such that for all t > 0, $b \geq 1$, and $x, v \in X$ with $x \leq v$ we have $V(T(x), T(x), ..., T(x), T(v), \varphi(\tfrac{t}{b})) \geq V(x, x, ..., x, v, \tfrac{t}{b})$

Also, assume that there exists $y_0 \in X$ such that $y_o \approx T(y_0)$ then T has a fixed point. Furthermore, If X is a totally ordered set then T has a unique fixed point.

Lemma 3.6: Let $(X, V, *)$ be a V-Fuzzy b-Metric Space such that * is a continuous t-norm. Define $V^m : X^m \times X^m \times .....\text{n-times}\ldots \times X^m \times \mathbf{R}^+ \to I$ such that

$$V^m(A_1, A_2, ..., A_n, \tfrac{t}{b}) = *_{i=1}^m V(a_{1i}, a_{2i}, ..., a_{ni}, \tfrac{t}{b})$$

for all $A_j = (a_{j1}, a_{j2}, ..., a_{jm}) \in X^m$, and for all t > 0, $b \geq 1$, where $j \in (1, 2, ..., n)$.

Then, the following properties hold.

(i) $(X^m, V^m, *)$ is also a V-Fuzzy b-Metric Space.

(ii) Let $\{A_r = (a_r^1, a_r^2, ..., a_r^m)\}$ be a sequence on X^m and a point $A = (a_1, a_2, ..., a_m) \in X^m$, Then $\{A_r\} \to A$ implies and implies by $\{a_r^i\} \to a_i$ for $i \in (1,2,....,m)$.

(iii) If (X, V, *) is complete then $(X^m, V^m, *)$ is also complete.

Proof: (i) As, * is a continuous mapping and $(X, V,)$ be a V-Fuzzy b-Metric Space, so that all properties of definition of V-Fuzzy b-Metric Space are trivially satisfied by $(X^m, V^m, *)$. Hence, $(X^m, V^m, *)$ is also a V-Fuzzy b-Metric Space.

(ii) If $\{A_r\} \to A$ as $r \to \infty$, i.e., for every $\varepsilon \in (0,1)$, there exists $n_0 \in N$ such that for all $r \geq n_0$, we have

$$V^m(A_r, A_r, ..., A_r, A, \tfrac{t}{b}) \geq 1 - \varepsilon \text{ for all t > 0.}$$

Since we have

$$\begin{aligned} V^m(A_r, A_r, ..., A_r, A, \tfrac{t}{b}) &= *_{i=1}^m V(a_r^i, a_r^i, ..., a_r^i, a_i, \tfrac{t}{b}) \\ &= V(a_r^1, a_r^1, ..., a_r^1, a_1, \tfrac{t}{b}) * .... \\ &* V(a_r^j, a_r^j, ..., a_r^j, a_j, \tfrac{t}{b}) * .... * \\ &V(a_r^m, a_r^m, ..., a_r^m, a_m, \tfrac{t}{b}) \end{aligned} \quad .....(9)$$

and we have

$$\begin{aligned} &\min_{1 \leq j \leq m} V\left(a_r^j, a_r^j, ..., a_r^j, a_j, \tfrac{t}{b}\right) \\ &V\left(a_r^1, a_r^1, ..., a_r^1, a_1, \tfrac{t}{b}\right) \\ &* .... V\left(a_r^j, a_r^j, ..., a_r^j, a_j, \tfrac{t}{b}\right) * .... \\ &* V\left(a_r^m, a_r^m, ..., a_r^m, a_m, \tfrac{t}{b}\right) \\ &< 1 - \varepsilon \end{aligned} \quad ....(10)$$

for all $r \geq n_0$ and t> 0, $b \geq 1$.

Thus, (9) and (10) implies that, for all $r \geq n_0, V(a_r^i, a_r^i, ..., a_r^i, a_i, \tfrac{t}{b}) \geq 1 - \varepsilon$ for every $i \in (1,2,....,m)$.Hence, $\{a_r^i\} \to \{a_i\}$ as $r \to \infty$.

Conversely, suppose that $\{a_r^i\} \to \{a_i\}$ as $r \to \infty$ for all $i \in (1,2,.....,m)$ and * as continuous mapping, from definition of V^m we get

$$\begin{aligned} \lim_{r \to \infty} V^m(A_r, A_r, ..., A, \tfrac{t}{b}) &= \lim_{r \to \infty} *_{i=1}^m V(a_r^i, a_r^i, ..., a_r^i, a_i, \tfrac{t}{b}) \\ &= *_{i=1}^m \lim_{r \to \infty} V(a_r^i, a_r^i, ..., a_r^i, a_i, \tfrac{t}{b}) = 1 \end{aligned}$$

for all t > 0. Hence, $\{A_r\} \to A$ as $r \to \infty$.

(iii) Suppose that sequence $\{A_n\}$ is a Cauchy in $(X^m, V^m, *)$ i.e., for every $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbf{N}$ such that $V^m(A_r, A_r, ..., A_r, A_n, \tfrac{t}{b}) \geq 1 - \varepsilon$ for all $r, n \geq n_0$ and t> 0, $b \geq 1$. We have

$$V^m(A_r, A_r, ..., A_r, A_n, \tfrac{t}{b}) = *_{i=1}^m V(a_r^i, a_r^i, ..., a_r^i, a_n^i, \tfrac{t}{b}) \quad .....(11)$$

and $\min_{1 \leq j \leq m} V(a_r^i, a_r^i, ..., a_r^i, a_n^i, \tfrac{t}{b}) \geq$

$$*_{i=1}^m V(a_r^i, a_r^i, ..., a_r^i, a_n^i, \tfrac{t}{b}) > 1 - \varepsilon \quad ....(12)$$

for all $r, n \geq n_0$ and t>0,, $b \geq 1$.

By (11) and (12), we can conclude that $\{a_r^i\}$ is a Cauchy sequence in $X,V,*$ for all $i \in (1,2,....,m)$.
Now, let $(X,V,*)$ complete V-Fuzzy b-Metric Space i.e., $\{a_r^i\} \to a_i$ for every $i \in (1,2,....,m)$ and $a_i \in X$ which implies that sequence $\{A_n\}$ converge to a point A on X^m. Hence $(X^m,V^m,*)$ is also a complete V-Fuzzy b-Metric Space.

Definition 3.7: Let $(X,V,*)$ be a V-fuzzy b-metric space. Let $(X,\preceq)$ be a partially ordered set and $\Phi = (\sigma_1,\sigma_2,....,\sigma_m)$ an m-tuple mappings from (1,m,…..,m) to itself. The mappings $F: X^m \to X$ and $g: X \to X$ are said to be $\Phi-$compatible in V-FMS if

$$\exists \lim_{r\to\infty} gx_r^i = \lim_{r\to\infty} F(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},...., x_r^{\sigma_i(m)})$$

for all monotonic sequences $\{x_r^1\},\{x_r^2\},....,\{x_r^m\}$ in X. We have

$$\lim_{r\to\infty} V\Big(gF\big(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},...., x_r^{\sigma_i(m)}\big), \big(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},...., x_r^{\sigma_i(m)}\big),$$
$$......, gF\left(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},...., x_r^{\sigma_i(m)}\right), F\left(gx_r^{\sigma_i(1)}, gx_r^{\sigma_i(2)},...., gx_r^{\sigma_i(m)}\right), \tfrac{t}{b}\Big) = 1.$$

for all $t > 0,\ b \geq 1.$ and $i \in (1,2,.....,m)$

Remark 3.8: If m = 2 in Definition 3.6 (see, [13]), then $F: X \times X \to X$ and $g: X \to X$ are compatible mappings on V-fuzzy b-metric spaces.

Lemma 3.9: Let $(X,V,*)$ be a V-fuzzy b-metric space such that * is a continuous t-norm. If $F: X^m \to X$ and $g: X \to X$ are Φ-compatible mappings on $(X,V,*)$ then $T: X^m \to X^m$ and $G: X^m \to X^m$ are also compatible in $(X^m,V^m,*)$ where T and G defined as

$$T(Y) = \Big(F\big(x_{\sigma_1(1)}, x_{\sigma_1(2)},...., x_{\sigma_1(m)}\big),....,F\big(x_{\sigma_i(1)}, x_{\sigma_i(2)},...., x_{\sigma_i(m)}\big),$$
$$....., F\big(x_{\sigma_m(1)}, x_{\sigma_m(2)},...., x_{\sigma_m(m)}\big)\Big)$$

And $G(Y) = (gx_1, gx_2, gx_3......, gx_m)$
For all $Y \in X^m$ and $x_i \in X, i \in (1,2,3,....,m)$

Proof: Let $\{x_r^1\},\{x_r^2\},......,\{x_r^m\}$ are monotonic sequences in X such that

$$\lim_{r\to\infty} gx_r^i = \lim_{r\to\infty} F\big(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},...., x_r^{\sigma_i(m)}\big)$$

for all $x_i \in X$ and $i \in (1,2,.....,m)$
By Lemma 3.7, $\lim_{r\to\infty} G(Y_r) = \lim_{r\to\infty} T(Y_r)$, where $Y_r = \big(\{x_r^1\},\{x_r^2\},....,\{x_r^m\}\big) \in X^m$. Since F and g are Φ-compatible and * is continuous mapping, we get

$$\lim_{r\to\infty} V^m(GT(Y_r), GT(Y_r),.....,GT(Y_r), TG(Y_r), \tfrac{t}{b})$$
$$= \lim_{r\to\infty} *_{i=1}^m V\Big(gF\big(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},....., x_r^{\sigma_i(m)}\big), gF\big(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},....., x_r^{\sigma_i(m)}\big)$$
$$...,\left(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},....., x_r^{\sigma_i(m)}\right), F\left(gx_r^{\sigma_i(1)}, gx_r^{\sigma_i(2)},....., gx_r^{\sigma_i(m)}\right), \tfrac{t}{b}\Big) =$$
$$*_{i=1}^m \lim_{r\to\infty} V\Big(gF\big(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},....., x_r^{\sigma_i(m)}\big), gF\big(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},....., x_r^{\sigma_i(m)}\big),...,$$
$$gF\left(x_r^{\sigma_i(1)}, x_r^{\sigma_i(2)},....., x_r^{\sigma_i(m)}\right), F\left(gx_r^{\sigma_i(1)}, gx_r^{\sigma_i(2)},....., gx_r^{\sigma_i(m)}\right), \tfrac{t}{b}\Big)$$
$$= 1$$

all t > 0. Hence T and G are compatible mapping in $\big(X^m,V^m,*\big)$.

Theorem 3.10: Let $(X,V,*)$ be a complete V-Fuzzy b-Metric Space and $(X,\preceq)$ a partially ordered set. Let $\{A,B\}$ be any partition of $\Lambda_m=\{1,2,3,...,m\}$ and $\Phi=(\sigma_1,\sigma_2,.....,\sigma_m)$ an n-tuple of mappings from Λ_m into itself verifying that $\sigma_i\in\Omega_{A,B}$ if $i\in A$ and $\sigma_i\in\Omega'_{A,B}$ if $i\in B$. Let $F:X^m\to X$ and $g:X\to X$ be two mappings such that

(i) $F(X^m)\subseteq g(X)$.

(ii) F has the mixed g-monotone property on X.

(iii) Assume that there exists a function $\phi\in\Phi_w$, such that

$$V\big(F(x_1,x_2....,x_m),F(x_1,x_2....,x_m),......,F(x_1,x_2....,x_m),\; F(y_1,y_2....,y_m),\phi(t)\big)\geq\gamma\left(*_{i=1}^{m}V(gx_i,gx_i,.....,gx_i,gy_i,\tfrac{t}{b})\right) \quad .....(13)$$

for all t > 0, $b\geq 1$ and $x_1,x_2,....,x_m,y_1,y_2,.....,y_m\in X$ and $gx_i\preceq_i gy_i$ for all $i\in(1,2,....,m)$, where $\gamma:[0,1]\to[0,1]$ such that $*_{i=1}^{m}\gamma(a)\geq a$ for all $a\in[0,1]$. Suppose that

$$\gamma\left(*_{i=1}^{m}V\left(gx_{\sigma_j(i)},gx_{\sigma_j(i)},....,gx_{\sigma_j(i)},gx_{\sigma_j(i)},\tfrac{t}{b}\right)\right)\geq\gamma\left(*_{i=1}^{m}V(gx_i,gx_i,....,gx_i,gy_i,\tfrac{t}{b})\right) \quad .....(14)$$

for $i,j\in(1,2,...,m)$ and for all $x_1,x_2,...,x_m,y_1,y_2,....,y_m\in X$ with $gx_i\preceq_i gy_i$.

Also, suppose that either

(b_1) F and g are continuous and Φ-compatible or

(b_2) X has the following properties

(a) If $\{x_n\}$ is a non-decreasing sequence such that $x_n\to x$ then $x_n\leq x\,\forall n\in\mathbf{N}$.

(b) If $\{x_n\}$ is a non-increasing sequence such that $x_n\to x$ then $x_n\geq x\,\forall n\in\mathbf{N}$,

and also suppose that either

$(b_{2.1})$ g is continuous and F, g are Φ-compatible maps

$(b_{2.2})$ g(X) is closed.

If there exist $x_0^1,x_0^2,.....,x_0^m\in X$ satisfying $gx_0^i\preceq_i F\left(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},....,x_0^{\sigma_i(m)}\right)$ for $i\in(1,2,...,m)$ then F and g have Φ-coincidence point.

Proof: Let $(X,V,*)$ be a V-Fuzzy b-Metric Space such that * is a continuous t-norm and $(X,\preceq)$ a partially ordered set. According to Lemma 3.7 $(X^m,V^m,*)$ is also a V-Fuzzy b-Metric Space. Define mappings $T:X^m\to X$ and $G:X^m\to X$ as

$$T(Y)=\Big(F\big(x_{\sigma_1(1)},x_{\sigma_1(2)},....,x_{\sigma_1(m)}\big),....,F\big(x_{\sigma_i(1)},x_{\sigma_i(2)},....,x_{\sigma_i(m)}\big),...,\; F\big(x_{\sigma_m(1)},x_{\sigma_m(2)},....,x_{\sigma_m(m)}\big)\Big) \quad ....(15)$$

and $G(Y)=(gx_1,gx_2,.....,gx_m)$ ....(16)

For all $Y\in X^m$ and $x_i\in X$ for $i\in(1,2,.....,m)$

Since $F(X^m)\subseteq g(X)$ which implies that $T(X^m)\subseteq G(X^m)$.

Suppose $gx_0^i\preceq_i F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},....,x_0^{\sigma_i(m)})$ then there exists $X_0\in X^m$ such that $G(X_0)\preceq_m T(X_o)$.

Now, to prove that T is a G-isotone, let there exists $U,Y\in X^m$ such that $G(Y)\preceq G(U)$ which implies that $gx_j\preceq gu_j$ when $j\in A$ and $gx_j\succeq gu_j$ when $j\in B$, we have $\sigma_i\in\Omega_{A,B}=$ { $\sigma:\Lambda_m\to\Lambda_m$ such that $\sigma(A)\subseteq A$ and $\sigma(B)\subseteq B$ } when $i\in A$ and $\sigma_i\in\Omega_{A,B}=$ {

$\sigma:\Lambda_m \to \Lambda_m$ such that $\sigma(A)\subseteq B$ and $\sigma(B)\subseteq A$ } when $i\in B$. Thus, we have $gx_{i(j)} \preceq gu_{i(j)}$ when $j\in A$ and $gx_{i(j)} \succeq gu_{i(j)}$ when $j\in A$, for fixed $i\in A$. As, F is mixed g-monotone, when $j\in A$ and fixed $i\in A$, we have

$$F\left(x_{\sigma_i(1)},....,x_{\sigma_i(j-1)},x_{\sigma_i(j)},x_{\sigma_i(j+1)},....,x_{\sigma_i(m)}\right)\preceq$$
$$F\left(x_{\sigma_i(1)},....,x_{\sigma_i(j-1)},x_{\sigma_i(j)},x_{\sigma_i(j+1)},....,x_{\sigma_i(m)}\right)$$

Similarly, when $j\in B$ and fixed $i\in A$ the inequality (17) hold. Hence, for fixed $i\in A$ and for all j the inequality (17) also holds. Thus, we conclude that

$$F\left(x_{\sigma_i(1)},x_{\sigma_i(2)},......,x_{\sigma_i(m)}\right)\preceq\left(u_{\sigma_i(1)},u_{\sigma_i(2)},......,u_{\sigma_i(m)}\right) \quad .....(18)$$

for $i\in A$. Same for $i\in B$, we have

$$F\left(x_{\sigma_i(1)},x_{\sigma_i(2)},......,x_{\sigma_i(m)}\right)\succeq\left(u_{\sigma_i(1)},u_{\sigma_i(2)},......,u_{\sigma_i(m)}\right) \quad .....(19)$$

From (18) and (19), we get $T(\mathrm{Y})\preceq_{\mathrm{m}} T(\mathrm{U})$ for $Y,U\in X^m$.
Hence, T is a G-isotone mapping.
Now, for $Y,U\in X^m$ we have $G(Y)\preceq_m G(U)$. From Proposition 2.15 $\left(x_{\sigma(1)},x_{\sigma(2)},....,x_{\sigma(m)}\right)$ and $\left(u_{\sigma(1)},u_{\sigma(2)},....,u_{\sigma(m)}\right)$ are comparable. By using (15) and (16) in the view of these points and it follows that for all t > 0,

$$\begin{aligned}
&V^m\left(T(Y),T(Y),.....,T(Y),T(U),\phi(t)\right)\geq *_{i=1}^m V\Big(F\left(x_{\sigma_i(1)},x_{\sigma_i(2)},.....,x_{\sigma_i(m)}\right),\\
&F\left(x_{\sigma_i(1)},x_{\sigma_i(2)},.....,x_{\sigma_i(m)}\right),....,F\left(x_{\sigma_i(1)},x_{\sigma_i(2)},.....,x_{\sigma_i(m)}\right),\\
&F\left(x_{\sigma_i(1)},x_{\sigma_i(2)},.....,x_{\sigma_i(m)}\right),\varphi(\tfrac{t}{b})\Big)\\
&\geq *_{i=1}^m\gamma\left(*_{j=1}^m V\left(gx_{\sigma_i(j)},gx_{\sigma_i(j)},....,gx_{\sigma_i(j)},gu_{\sigma_i(j)},\tfrac{t}{b}\right)\right.\\
&\geq *_{i=1}^m\gamma\left(*_{j=1}^m V\left(gx_j,gx_j,....,gx_j,gu_j,\tfrac{t}{b}\right)\right.\\
&= *_{i=1}^m\gamma\left(V^m(G(Y),G(Y),....,G(Y),G(U),\tfrac{t}{b})\right)\\
&= *^m\gamma\left(V^m(G(Y),G(Y),....,G(Y),G(U),\tfrac{t}{b})\right)\\
&\geq V^m(G(Y),G(Y),....,G(Y),G(U),\tfrac{t}{b}) \quad .....(20)
\end{aligned}$$

Hence, condition (iii) of Theorem 3.8 implies that condition (iii) of Theorem 3.3 w.r.t. $(X^m,V^m,*)$. Now, we have to deduce other conditions of Theorem 3.3. According to condition (b_1) of Theorem 3.10, we have F, g are continuous and Φ-compatible, by (15) and (16) T and G are continuous and using Lemma 3.10, T and G are compatible. Hence condition (a_1) of Theorem 3.3 holds w.r.t. $(X^m,V^m,*)$.
Now, we assume a non-decreasing sequence $\{Y_r\}\in X^m$ such that $Y_r\to Y$ as $r\to\infty$ for some $Y\in X^m$. By Lemma 3.6, $x_r^i\to x^i$ as $r\to\infty$ and for all $i\in(1,2,....,m)$. Since, $Y_r\preceq_{\mathrm{m}} Y_{r+1}$ for all $r\in N\cup\{0\}$ then $\{x_r\}$ is non-decreasing sequence when $i\in A$ and $\{x_r\}$ is non-increasing sequence when $i\in B$ for all $r\in N\cup\{0\}$. By condition (b_2) of Theorem 3.10, we have $x_r^i\preceq x^i$ when $i\in A$ and $x_r^i\succeq x^i$ when $i\in B$ for all $r\in N\cup\{0\}$. Which implies that, $Y_r\preceq_m Y$ for all $r\in N\cup\{0\}$.
Similarly, by assuming Y_r as a non-increasing sequence we get $Y_r\succeq_m Y$ for all $r\in N\cup\{0\}$.

$g(X)$ is closed i.e., $G(X^m)$ is closed.

Hence, all conditions of Theorem 3.3 hold w.r.t $(X^m, V^m, *)$. Therefore, according to the Theorem 3.3, T and G have a coincidence point, and from (15) and (16) that coincidence point will be the Φ-coincidence point of F and g.

Corollary 3.11: In addition to hypothesis of Theorem 3.11, suppose that X is a totally ordered set then F and g have a unique Φ-coincidence point. Moreover, if g is Φ-weakly compatible with F then F and G have a unique Φ-common fixed point.

Proof: From (15), (16) and Theorem 3.4, we can easily deduce that T and G have a unique coincidence point w.r.t. $(X^m, V^m, *)$, which is unique Φ-coincidence point of F and G.
Let $v_i = gx_i = F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ...x_{\sigma_i(m)})$ and F and g are Φ-weakly compatible. We get,

$$gv_i = gF(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ...x_{\sigma_i(m)}) = F(gx_{\sigma_i(1)}, gx_{\sigma_i(2)}, ...gx_{\sigma_i(m)})$$
$$= F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, ...v_{\sigma_i(m)})$$

Hence, $v_i \in X^m$ is a Φ-coincidence point of F and g which implies that $v_i = gx_i = gv_i$. Hence, v_i is a Φ-common fixed point and Φ-coincidence point of F and g. In order to prove uniqueness suppose that is another Φ-common fixed point of F and g then, we have $u_i = gu_i = gv_i = v_i$, which proves uniqueness.
Taking m = 2, A = {1}, B = {2} in Theorem 3.11 and corollary 3.12, we get coupled coincidence and coupled fixed point result as:

Corollary 3.12: Let $(X, V, *)$ be a complete V-Fuzzy b-Metric Space with * is a continuous t-norm. Let $(X, \preceq)$ be, partially ordered set. Let $F : X^2 \to X$ and $g : X \to X$ be two mappings such that

(b) F has the mixed g-monotone properties on X.

(ii) $F(X^2) \subseteq g(X)$.

(iii) If there exist $\phi \in \Phi_w$, such that

$$V(F(x_1,x_2), F(x_1,x_2), ..., F(x_1,x_2), F(v_1,v_2), \varphi(\tfrac{t}{b}))$$
$$\geq \gamma(V(g(x_1), g(x_1), ..., g(x_1), g(v_1), \tfrac{t}{b})$$
$$*V(g(x_2), g(x_2), ..., g(x_2), g(v_2), \tfrac{t}{b})$$

For all t > 0 and $x_1, x_2, v_1, v_2 \in X$ and $g(x_1) \preceq g(v_1)$ and $g(x_2) \succeq g(v_2)$, where $\gamma : [0,1] \to [0,1]$ such that $\gamma(a) * \gamma(a) \geq a$ for all $a \in [0,1]$. Also, suppose that either

(c_1) F and g are continuous and compatible or

(c_2) X has the following property

(a) If $\{x_n\}$ is non-decreasing sequence such that $x_n \to x$ then $x_n \leq x \forall n \in \mathbb{N}$.

(b) If $\{x_n\}$ is non-increasing sequence such that $x_n \to x$ then $x_n \geq x \forall n \in \mathbb{N}$,

and also suppose that either

$(b_{2.1})$ g is continuous and F, g are Φ-compatible maps or

$(b_{2.2})$ $g(X)$ is closed.

If there exists $x_0^1, x_0^2 \in X$ satisfying $g(x_0^1) \preceq F(x_0^1, x_0^2)$ and $g(x_0^2) \preceq F(x_0^2, x_0^1)$ then F and g have coupled coincidence point. Assume that X is a totally ordered set then F and g have a unique coincidence point. Moreover, F and g are weakly compatible then F and g have a unique common coupled fixed point.

Remark 3.13: We can omit continuity of Q and compatibility of P, Q in the hypothesis of the Theorem 3.1 and Theorem 3.2 in [13], instead of these points add condition ($C_{2.2}$) (i.e., Q(X) is closed) of the Corollary 3.12, we will get the coupled fixed point. For example, take a discontinuous function

$$Q(x) = \begin{cases} x & \text{if} \quad x \le \frac{2}{3} \\ 1 & \text{if} \quad x > \frac{2}{3} \end{cases}$$

We use the family of the function Φ_w which is more general class of the functions than $\phi(t) = kt$, $k \in (0,1)$. In this manner, we generalize and improve the coupled fixed pointresults in [13].

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