Research Article

The Multivariable H-function and M-series involving the generalized Mellin-Barnes contour integral swith general class of polynomials

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ABSTRACT

In this research paper, we will study some important integral formulas, with the help of which they play a major role in the study of the hypergeometric functions and multivariable H-function. In this study we will use these formulas to get the solutions of M-series and general class of polynomials in the form of multivariable H-function. The results proved in general forms and besides of this have been put in a compact from avoiding the occurrence of infinite series and thus making them useful in applications. In this research paper we can obtain new and some modified known results which will brings new ideas and fact in field of hypergeometric and polynomials studies.

Keywords and Phrases: Generalized hypergeometric function, fractional calculus, General class of polynomial, M-series, Special function, Multivariable H-function.

1. Introduction and Preliminaries

The multivariable H-function which was introduced and investigated by Srivastava & Panda [7, p. 271, Eqn. (4.1)] in term of a multiple Mellin-Bernes type contour integral as

$H\,[z_1,\ldots,z_r]=$

$$\boldsymbol{H}^{o,n:m_1,n_1;..;m_s,n_s}_{p,q:p_1q_1;....;p_s,q_s}\left[\begin{array}{c} z_1 \\ \vdots \\ z_s \end{array}\middle| \begin{array}{l} \left(a_j:\alpha_j^1;\ldots;\alpha_j^s\right)_{1,p}:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\ldots;\left(c_j^s,\gamma_j^s\right)_{1,P_s} \\ \left(b_j:\beta_j^1,\ldots;\beta_j^s\right)_{1,q}:\left(d_j^1,\delta^1{}_j\right)_{1,q_1};\ldots;\left(d_j^s,\delta_j^s\right)_{1,q_s} \end{array}\right]$$

$$=\frac{1}{(2\pi\omega)^s}\int_{\xi_1}..\int_{\xi_s}\psi(\lambda_1,\ldots,\lambda_s)\phi_1(\lambda_1)..\phi_s(\lambda_s)z_1^{\lambda_1},\ldots,z_s^{\lambda_s}\,d\lambda_1,\ldots,d\lambda_s,$$

(1.1)

Where $\omega=\sqrt{-1}$; and

$$\psi(\lambda_1,\ldots,\lambda_s)=\frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\Sigma_{i=1}^{r}\alpha_j^{(i)}\lambda_i\right)}{\prod_{j=n+1}^{p}\Gamma(a_j-\Sigma_{i=1}^{r}\alpha_j^{(i)}\lambda_i)\prod_{j=1}^{q}\Gamma(1-b_j+\Sigma_{i=1}^{r}\beta_j^{(i)}\lambda_i)},\qquad(1.2)$$

$$\phi_i(\lambda_i)=\frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}\lambda_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}\lambda_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\lambda_i\right)\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\lambda_i\right)}\quad (i=1,...,s); \ (1.3)$$

The contour L_j lies in the complex plane λ_j is of Mellin-Barnes type which start at the point $\tau_j-\omega_\infty$ and terminate at the point $\tau_j+\omega_\infty$ with $\tau_j\in\Re=(-\infty,\infty)(j=1,...,s)$.

In case r = 2, (1) reduce to the H-function of two variables.

$$\nabla_i=\sum_{j=1}^{p}\alpha_j+\sum_{j=1}^{p_i}\gamma_j-\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{q_i}\delta_j\le 0$$

$$\Re_i=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p}\alpha_j+\sum_{j=1}^{q}\beta_j+\sum_{j=1}^{n_i}\gamma_j-\sum_{j=n_i+1}^{p_i}\gamma_j+\sum_{j=1}^{m_i}\delta_j-\sum_{j=m_i+1}^{q_i}\delta_j>0 \quad (1.4)$$

For a detailed definition and convergence conditions of the multivariable H-function, the reader is referred to the original paper by Srivastava and Panda [8, p. 131], we have

$$H[z_1,\ ...,z_s]=O\ \left(|z_1|^{e_1}\ ...\ |z_s|^{e_s}\right)\ \left(\max_{1\le j\le s}\|z_j\|\to 0\right), \qquad (1.5)$$

Where

$$e_i=\min_{1\le j\le m_i}\left[\frac{\mathrm{Re}(d_j^{(i)})}{\delta_j^{(i)}}\right]\ \ (i=1,...,\ s). \qquad (1.6)$$

For n = p = q = 0 the multivariable H-function reduced to the product of 'r' H-functions and consequently there holds the following result:

$$\mathrm{H}_{0,0:\mathrm{p}_1,\mathrm{q}_1;...;\mathrm{p}_\mathrm{s},\mathrm{q}_\mathrm{s}}^{0,0:\mathrm{m}_1,\mathrm{n}_1;...;\mathrm{m}_\mathrm{s},\mathrm{n}_\mathrm{s}}\left[\begin{matrix}z_1\\ \vdots\\ z_s\end{matrix}\left|\begin{matrix}\left(c_j^1,\gamma_j^1\right)_{1,p_1};...;\left(c_j^{(s)},\gamma_j^{(s)}\right)_{1,p_s}\\ \left(d_j^1,\delta_1^1\right)_{1,q_1};...;\left(d_j^{(s)},\delta_j^{(s)}\right)_{1,q_s}\end{matrix}\right.\right]=\prod_{i=1}^{s}\mathrm{H}_{\mathrm{p_i},\mathrm{q_i}}^{\mathrm{m_i},\mathrm{n_i}}\left[z\left|\begin{matrix}\left(c_j^{(i)},\gamma_j^{(i)}\right)_{1,p_i}\\ \left(d_j^{(i)},\delta_j^{(i)}\right)_{1,q_i}\end{matrix}\right.\right](1.7)$$

Where $\mathrm{H}_{\mathrm{p,q}}^{\mathrm{m,n}}(\cdot)$ is the familiar H-function .

The well known M-series, which is a particular case of $\overline{H}$ -function introduced by Inayat-Hussain [4] and is defined by means of the following series expansion

$$\mathrm{pM}_\mathrm{Q}^{\beta}(\mathrm{a}_1,...,\mathrm{a}_\mathrm{p};\mathrm{b}_1,...,\mathrm{b}_\mathrm{Q};Z)=\sum_{r=0}^{\infty}\frac{(\mathrm{a}_1)_s\ ...(\mathrm{a}_\mathrm{p})_s}{(\mathrm{b}_1)_1\ ...\ (\mathrm{b}_\mathrm{Q})_s}\ \frac{Z^s}{\Gamma(\beta s+1)} \qquad (1.8)$$

Provided that $\beta\in\mathrm{C}$, $\mathrm{R}(\beta)>0$, $(\mathrm{a}_\mathrm{j})_\mathrm{s}$ $(\mathrm{b}_\mathrm{j})_\mathrm{s}$ are Pochammer symbols.

The second class of multivariable polynomials introduced by Srivastava[5] is defined as follows.

$$S_{n_1,\dots,n_k}^{m_1,\dots,m_k}[y_1,\dots,y_k]=\sum_{r_1=0}^{[n_1/m_1]},\dots,\sum_{r_k=0}^{[n_k/m_k]}(-n_1)_{m_1r_1},\dots,(-n_k)_{m_kr_k}$$

$$A(n_1,r_1;\dots;n_k,r_k)\ \frac{y_1^{r_1}}{r_1!},\dots,\frac{y_k^{r_k}}{r_k!}$$

n = 0, 1, 2, … (1.9)

2. **Required consequences**

For $a>0; b\geq 0; c+4ab>0; \Re(\rho)+1/2>0$ the following formulas is introduced by Qureshi et al.[10, p.77, Eq^n, (3.1) – (3.3)]

$$\int_0^\infty \left[\left(az+\frac{b}{z}\right)^2+c\right]^{-\rho-1} dz=\frac{\sqrt{\pi}}{2a(4ab+c)^{\rho+\frac{1}{2}}}\ \frac{\Gamma(\rho+\frac{1}{2})}{\Gamma(\rho+1)}. \tag{2.1}$$

For $a\geq 0; b>0; 4ab+c>0; \Re(\rho)+1/2>0$,

$$\int_0^\infty \frac{1}{z^2}\left[\left(az+\frac{b}{z}\right)^2+c\right]^{-\rho-1} dz=\frac{\sqrt{\pi}}{2b(4ab+c)^{\rho+\frac{1}{2}}}\ \frac{\Gamma(\rho+\frac{1}{2})}{\Gamma(\rho+1)}. \tag{2.2}$$

For $a>0; b>0; 4ab+c>0; \Re(\rho)+1/2>0$,

$$\int_0^\infty \left(a+\frac{b}{z^2}\right)^2\left[\left(a+\frac{b}{z^2}\right)^2+c\right]^{-\rho-1} dz=\frac{\sqrt{\pi}}{(4ab+c)^{\rho+\frac{1}{2}}}\ \frac{\Gamma(\rho+\frac{1}{2})}{\Gamma(\rho+1)}. \tag{2.3}$$

The following formulas [14, p. 75] will also be necessary in our investigation.

$$(1-y)^{\ a+b-c}\ {}_2F_1\ (2a, 2b, 2c\,; y)=\sum_{r=0}^{\infty} a_r y^r\ , \tag{2.4}$$

and

$${}_2F_1(a, b, c+\frac{1}{2}; X)\ \ {}_2F_1\ (c-a, c-b,\ \ c+\frac{1}{2}; X)=\sum_{r=0}^{\infty}\frac{(c, r)}{\left(c+\frac{1}{2}, r\right)}a_r X^r. \tag{2.5}$$

3. **Main Results**

Theorem 3.1 Let $a>0, b\geq 0, 4ab+c>0, \sigma>0, \rho_i\geq 0, \Re(\Omega)+\frac{1}{2}>0,\ \Re(\rho+\sigma\, e_i)>0\ \ (i=1,\dots,s)$,

$-\frac{1}{2}<\ (a-b-c)\ <\ \frac{1}{2}$ the following formula holds

$$\int_0^{\infty}\left(az+\frac{b}{z}\right)^{-\Omega-1} {}_2F_1\left(a,b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)\ {}_2F_1\left(c-a,c-b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)$$

$$S_{n_1,\dots,n_k}^{m_1,\dots,m_k}\left[y_1\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_1},\dots,y_k\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_k}\right]$$

$$\times {}_P M_Q^{\beta}\left[(a_P);(b_Q);x\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\sigma}\right] H\left[\left\{x_1\left(az+\frac{b}{z}\right)^2+c\right\}^{-\rho_1},\dots,\left\{x_s\left(az+\frac{b}{z}\right)^2+c\right\}^{-\rho_s}\right]$$

$$=\frac{\sqrt{\pi}}{2a\,(4ab+c)^{\Omega+1/2}}\sum_{r=0}^{\infty}\sum_{r_1=0}^{[n_1/m_1]},\dots,\sum_{r_k=0}^{[n_k/m_k]}\frac{1}{(4ab+c)^{-r}}\frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\ a_r\left\{\sum_{s=0}^{\infty}G(s)\frac{1}{(4ab+c)^{\sigma s}}\right\}$$

$$(-n_1)_{m_1r_1},\dots,(-n_k)_{m_kr_k}\ A(n_1,r_1;\dots;n_k,r_k)\ \frac{y_1^{r_1}}{r_1!},\dots,\frac{y_k^{r_k}}{r_k!}\ \times H^{0,n+1:\ m_1,n_1;\dots;m_s,n_s}_{p+1,q+1:\ p_1,q_1;\dots;p_s,q_s}$$

$$\left[\begin{matrix} x_1(4ab+c)^{-\rho_1} \\ \vdots \\ x_s(4ab+c)^{-\rho_s}\end{matrix}\left|\begin{matrix}\left(\frac{1}{2}-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,\dots,\rho_s\right),\left(a_j;\alpha_j^1,\dots,\alpha_j^{(s)}\right)_{1,p}:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\dots;\left(c_j^{(s)},\gamma_j^{(s)}\right)_{1,p_s} \\ \left(b_j;\beta_j^1,\dots,\beta_j^{(s)}\right)_{1,q},\left(-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,\dots,\rho_s\right):\left(d_j^1,\delta_j^1\right)_{1,q_1};\dots;\left(d_j^{(s)},\delta_j^{(s)}\right)_{1,q_s}\end{matrix}\right.\right] \quad (3.1)$$

where e_i is defined in (1.6).

Proof: By virtue of equation (1.1), (1.8), (1.9), (2.1), and (2.5), we have the following result

$$\int_0^{\infty}\left(az+\frac{b}{z}\right)^{-\Omega-1} {}_2F_1\left(a,b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)\ {}_2F_1\left(c-a,c-b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)$$

$$S_{n_1,\dots,n_k}^{m_1,\dots,m_k}\left[y_1\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_1},\dots,y_k\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_k}\right]$$

$$\times {}_P M_Q^{\beta}\left[(a_P);(b_Q);x\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\sigma}\right] H\left[\left\{x_1\left(az+\frac{b}{z}\right)^2+c\right\}^{-\rho_1},\dots,\left\{x_s\left(az+\frac{b}{z}\right)^2+c\right\}^{-\rho_s}\right]$$

$$=\int_0^{\infty}\left(az+\frac{b}{z}\right)^{-\Omega-1}\sum_{r=0}^{\infty}\sum_{r_1=0}^{[n_1/m_1]},\dots,\sum_{r_k=0}^{[n_k/m_k]}\frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\ a_r\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{r}\sum_{s=0}^{\infty}\frac{\Pi_{j=1}^{p}(a_j)_s\ x^s}{\Pi_{j=1}^{Q}(b_j)_s\ \Gamma(\beta s+1)}\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\sigma s}$$

$$\times\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_1 r_1},\dots,\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_k r_k} A(n_1,r_1;\dots;n_k,r_k)\ \frac{y_1^{\mu_1 r_1}}{r_1!},\dots,\frac{y_k^{\mu_k r_k}}{r_k!}$$

$$\frac{1}{(2\pi\omega)^s}\int_{\xi_1}\cdots\int_{\xi_s}\psi(\lambda_1,\ldots,\lambda_s)\left\{\prod_{i=1}^{s}\phi_i(\lambda_i)\left[x_i\left(az+\frac{b}{z^2}\right)^2+c\right]^{\rho_i\lambda_i}\right\}d\lambda_1,\ldots,d\lambda_s \text{ dz}$$

$$=\sum_{r=0}^{\infty}\sum_{r_1=0}^{[n_1/m_1]},\ldots,\sum_{r_k=0}^{[n_k/m_k]}\frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\; a_r\sum_{s=0}^{\infty}G(s)\; A(n_1,r_1;\ldots;n_k,r_k)\;\frac{y_1^{\mu_1 r_1}}{r_1!},\ldots,\frac{y_k^{\mu_k r_k}}{r_k!}$$

$$\frac{1}{(2\pi\omega)^s}\int_{\xi_1}\cdots\int_{\xi_s}\psi(\lambda_1,\ldots,\lambda_s)\left\{\prod_{i=1}^{s}\phi_i(\lambda_i)\left[x_i^{\lambda_i}\right]\right\}d\lambda_1,\ldots,d\lambda_s \text{ dz}$$

where,

$$G(s)=\left\{\frac{\Pi_{j=1}^{P}(A_j)_s\; y^s}{\Pi_{j=1}^{Q}\,(B_j)_s\;\Gamma(\beta s+1)}\right.$$

$$=\frac{\sqrt{\pi}}{2a\,(4ab+c)^{\Omega+1/2}}\sum_{r=0}^{\infty}\sum_{r_1=0}^{[n_1/m_1]},\ldots,\sum_{r_k=0}^{[n_k/m_k]}\frac{1}{(4ab+c)^{-r}}\frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\; a_r\left\{\sum_{s=0}^{\infty}G(s)\frac{1}{(4ab+c)^{\sigma s}}\right\}$$

$$(-n_1)_{m_1r_1},\ldots,(-n_k)_{m_kr_k}\; A(n_1,r_1;\ldots;n_k,r_k)\;\frac{y_1^{r_1}}{r_1!},\ldots,\frac{y_k^{r_k}}{r_k!}$$

$$\frac{1}{(2\pi\omega)^s}\int_{\xi_1}\cdots\int_{\zeta_s}\psi(\lambda_1,\ldots,\lambda_s)\left\{\prod_{i=1}^{s}\phi_i(\lambda_i)\left\{x_i\,(4ab+c)^{-\rho_i}\right\}^{\lambda_i}\right\}\frac{\Gamma(\Omega-r+\sum_{1=1}^{k}\mu_i r+\sigma s+\sum_{i=1}^{s}\rho_i\,\lambda_i)+1/2)}{\Gamma(1+\Omega-r+\sum_{1=1}^{k}\mu_i r+\sigma s+\sum_{i=1}^{s}\rho_i\lambda_i)}\,d\lambda_1,\ldots,d\lambda_s \tag{3.1a}$$

Thus the definition of the multivariable H-function will give Theorem (3.1).

If we situate n = p = q = 0 then by virtue of the character (1.7), we get.

Corollary 3.2 If $a\ge 0,\ b>0;\ c+4ab>0,\ \sigma>0,\ \rho_i\ge 0,\ \Re(\Omega)+\frac{1}{2}>0,\ \Re(\rho+\sigma\ e_i)>0\ \ (i=1,\ldots,s),$

$-\frac{1}{2}<\ (a-b-c)\ <\frac{1}{2}$ then the following formula holds

$$\int_0^{\infty}\left(az+\frac{b}{z}\right)^{-\Omega-1}{}_2F_1\left(a,b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)\;{}_2F_1\left(c-a,c-b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)$$

$$S_{n_1,\ldots,n_k}^{m_1,\ldots,m_k}\left[y_1\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_1},\ldots,y_k\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_k}\right]$$

$$\times{}_PM_Q^{\beta}\left[(a_P);(b_Q);x\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\sigma}\right]\prod_{i=1}^{s}H_{p_i,q_i}^{m_i,n_i}\left[x_i\left\{\left(az+\frac{b}{z^2}\right)^2+c\right\}^{\rho_i}\middle|\begin{matrix}(c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\(d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$= \frac{\sqrt{\pi}}{2a\,(4ab+c)^{\Omega+1/2}} \sum_{r=0}^{\infty} \sum_{r_1=0}^{[n_1/m_1]},...,\sum_{r_k=0}^{[n_k/m_k]} \frac{1}{(4ab+c)^{-r}} \frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\; a_r \left\{ \sum_{s=0}^{\infty} G(s) \frac{1}{(4ab+c)^{\sigma s}} \right\}$$

$$(-n_1)_{m_1r_1},...,(-n_k)_{m_kr_k}\; A(n_1,r_1;...;n_k,r_k)\; \frac{y_1^{r_1}}{r_1!},....,\frac{y_k^{r_k}}{r_k!}$$

$$\times \mathrm{H}^{0,1:\; m_1,n_1;...;m_s,n_s}_{1,1:\; p_1,q_1;...;p_s,q_s} \left[\begin{matrix} x_1(4ab+c)^{-\rho_1} \\ \vdots \\ x_s(4ab+c)^{-\rho_s} \end{matrix} \middle| \begin{matrix} \left(1\!/\!2-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,...,\rho_s \right),:\left(c_j^1,\gamma_j^1\right)_{1,p_1};...;\left(c_j^{(s)},\gamma_j^{(s)}\right)_{1,p_s} \\ ,\left(-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,...,\rho_s \right):\left(d_j^1,\delta_j^1\right)_{1,q_1};...;\left(d_j^{(s)},\delta_j^{(s)}\right)_{1,q_s} \end{matrix} \right] \quad (3.2)$$

where e_i is defined in (1.6).

Following a similar process the next Theorems (3.3) and (3.5) can be proved.

Theorem 3.3 Let $a \geq 0$, $b > 0$; $4ab + c > 0$, $\sigma > 0$, $\rho_i \geq 0$, $\Re(\Omega) + \frac{1}{2} > 0$, $\Re(\rho + \sigma\, e_i) > 0$ $(i = 1,...,s)$,

$-\frac{1}{2} < (a-b-c) < \frac{1}{2}$ then the next formula holds:

$$\int_0^{\infty} \frac{1}{z^2} \left\{ \left(az + \frac{b}{z} \right) + c \right\}^{-\Omega-1} {}_2F_1\left(a,b,c+\frac{1}{2}; \left(az+\frac{b}{z}\right)^2 + c \right) {}_2F_1\left(c-a,c-b,c+\frac{1}{2}; \left(az+\frac{b}{z}\right)^2 + c \right)$$

$$S^{m_1,....,m_k}_{n_1,....,n_k}\left[y_1 \left\{ \left(az+\frac{b}{z}\right)^2 + c \right\}^{-\mu_1},...,y_k \left\{ \left(az+\frac{b}{z}\right)^2 + c \right\}^{-\mu_k} \right]$$

$$\times {}_P\mathrm{M}_Q^{\beta}\left[(a_P);(b_Q); x\left\{ \left(az+\frac{b}{z}\right)^2 + c \right\}^{-\sigma} \right] H\left[x_1\left\{ \left(az+\frac{b}{z}\right)^2 + c \right\}^{-\rho_1},...,x_s\left\{ \left(az+\frac{b}{z}\right)^2 + c \right\}^{-\rho_s} \right]$$

$$= \frac{\sqrt{\pi}}{2b\,(4ab+c)^{\Omega+1/2}} \sum_{r=0}^{\infty} \sum_{r_1=0}^{[n_1/m_1]},...,\sum_{r_k=0}^{[n_k/m_k]} \frac{1}{(4ab+c)^{-r}} \frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\; a_r \left\{ \sum_{s=0}^{\infty} G(s) \frac{1}{(4ab+c)^{\sigma s}} \right\}$$

$$(-n_1)_{m_1r_1},...,(-n_k)_{m_kr_k}\; A(n_1,r_1;...;n_k,r_k)\; \frac{y_1^{r_1}}{r_1!},....,\frac{y_k^{r_k}}{r_k!} \times \mathrm{H}^{0,n+1:\; m_1,n_1;...;m_s,n_s}_{p+1,q+1:\; p_1,q_1;...;p_s,q_s}$$

$$\left[\begin{matrix} x_1(4ab+c)^{-\rho_1} \\ \vdots \\ x_s(4ab+c)^{-\rho_s} \end{matrix} \middle| \begin{matrix} \left(1\!/\!2-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,...,\rho_s \right),\left(a_j;\alpha_j^1,...,\alpha_j^{(s)}\right)_{1,p}:\left(c_j^1,\gamma_j^1\right)_{1,p_1};...;\left(c_j^{(s)},\gamma_j^{(s)}\right)_{1,p_s} \\ \left(b_j;\beta_j^1,...,\beta_j^{(s)}\right)_{1,q},\left(-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,...,\rho_s \right):\left(d_j^1,\delta_j^1\right)_{1,q_1};...;\left(d_j^{(s)},\delta_j^{(s)}\right)_{1,q_s} \end{matrix} \right] \quad (3.3)$$

If we locate n=p=q=0 then by virtue of the observe (1.7), we get the following :

Corollary 3.4 If $a \geq 0$; $b > 0$; $4ab + c > 0$, $\sigma > 0$, $\rho_i \geq 0$, $\Re(\Omega) + \frac{1}{2} > 0$, $\Re(\rho + \sigma\, e_i) > 0$ $(i = 1,...,s)$,

$-\frac{1}{2} < (a-b-c) < \frac{1}{2}$ then there holds the next result

$$\int_0^\infty \frac{1}{z^2}\left\{\left(az+\frac{b}{z}\right)+c\right\}^{-\Omega-1} {}_2F_1\left(a,b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right) {}_2F_1\left(c-a,c-b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)$$

$$S_{n_1,\ldots,n_k}^{m_1,\ldots,m_k}\left[y_1\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_1},\ldots,y_k\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_k}\right]$$

$$\times {}_P M_Q^{\beta}\left[(a_P);(b_Q);x\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\sigma}\right]\prod_{i=1}^{s} H_{p_i,q_i}^{m_i,n_i}\left[x_i\left\{\left(az+\frac{b}{z^2}\right)^2+c\right\}^{\rho_i}\left|\begin{matrix}(c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\(d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right.\right]$$

$$=\frac{\sqrt{\pi}}{2b\,(4ab+c)^{\Omega+1/2}}\sum_{r=0}^{\infty}\sum_{r_1=0}^{[n_1/m_1]},\ldots,\sum_{r_k=0}^{[n_k/m_k]}\frac{1}{(4ab+c)^{-r}}\frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\;a_r\left\{\sum_{s=0}^{\infty}G(s)\frac{1}{(4ab+c)^{\sigma s}}\right\}$$

$$(-n_1)_{m_1r_1},\ldots,(-n_k)_{m_kr_k}\;A(n_1,r_1;\ldots;n_k,r_k)\;\frac{y_1^{r_1}}{r_1!},\ldots,\frac{y_k^{r_k}}{r_k!}\;\times H_{1,1:\;p_1,q_1;\ldots;p_s,q_s}^{0,1:\;m_1,n_1;\ldots;m_s,n_s}$$

$$\left[\begin{matrix}x_1\,(4ab+c)^{-\rho_1}\\ \vdots\\ x_s\,(4ab+c)^{-\rho_s}\end{matrix}\left|\begin{matrix}\left(\tfrac{1}{2}-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,\ldots,\rho_s\right),:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\ldots;\left(c_j^{(s)},\gamma_j^{(s)}\right)_{1,p_s}\\ ,\left(-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,\ldots,\rho_s\right):\left(d_j^1,\delta_j^1\right)_{1,q_1};\ldots;\left(d_j^{(s)},\delta_j^{(s)}\right)_{1,q_s}\end{matrix}\right.\right] \tag{3.4}$$

Theorem 3.5 If $a>0,\ b>0;\ 4ab+c>0,\sigma>0,\ \rho_i\ge 0,\Re(\Omega)+\frac{1}{2}>0,\ \Re(\rho+\sigma\ e_i)>0\ \ (i=1,\ldots,s),$

$-\frac{1}{2}<\ (a-b-c)\ <\frac{1}{2}$ then the subsequently formula holds:

$$\int_0^\infty\left(a+\frac{b}{z^2}\right)\left\{\left(az+\frac{b}{z}\right)+c\right\}^{-\Omega-1} {}_2F_1\left(a,b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right) {}_2F_1\left(c-a,c-b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)$$

$$S_{n_1,\ldots,n_k}^{m_1,\ldots,m_k}\left[y_1\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_1},\ldots,y_k\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_k}\right]$$

$$\times {}_P M_Q^{\beta}\left[(a_P);(b_Q);x\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\sigma}\right]H\left[x_1\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\rho_1},\ldots,x_s\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\rho_s}\right]$$

$$=\frac{\sqrt{\pi}}{(4ab+c)^{\Omega+1/2}}\sum_{r=0}^{\infty}\sum_{r_1=0}^{[n_1/m_1]},\ldots,\sum_{r_k=0}^{[n_k/m_k]}\frac{1}{(4ab+c)^{-r}}\frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\;a_r\left\{\sum_{s=0}^{\infty}G(s)\frac{1}{(4ab+c)^{\sigma s}}\right\}$$

$$(-n_1)_{m_1r_1},\ldots,(-n_k)_{m_kr_k}\;A(n_1,r_1;\ldots;n_k,r_k)\;\frac{y_1^{r_1}}{r_1!},\ldots,\frac{y_k^{r_k}}{r_k!}\;\times H_{p+1,q+1:\;p_1,q_1;\ldots;p_s,q_s}^{0,n+1:\;m_1,n_1;\ldots;m_s,n_s}$$

$$\left[\begin{array}{c} x_1(4ab+c)^{-\rho_1} \\ \vdots \\ x_s(4ab+c)^{-\rho_s} \end{array}\left| \begin{array}{l} \left(\tfrac{1}{2}-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,\dots,\rho_s\right),\left(a_j;\alpha_j^1,\dots,\alpha_j^{(s)}\right)_{1,p}:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\dots;\left(c_j^{(s)},\gamma_j^{(s)}\right)_{1,p_s} \\ \left(b_j;\beta_j^1,\dots,\beta_j^{(s)}\right)_{1,q},\left(-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,\dots,\rho_s\right):\left(d_j^1,\delta_j^1\right)_{1,q_1};\dots;\left(d_j^{(s)},\delta_j^{(s)}\right)_{1,q_s} \end{array}\right.\right] \tag{3.5}$$

where e_i is defined in (1.6).

If we locate n=p=q=0 then by virtue of the identify (1.7) , we get then following

Corollary 3.6 If $a>0, b>0; 4ab+c>0, \sigma>0, \rho_i\geq 0, \Re(\Omega)+\frac{1}{2}>0, \Re(\rho+\sigma e_i)>0\ (i=1,\dots,s),$

$-\frac{1}{2}<\ (a-b-c)\ <\frac{1}{2}$ then there holds the next result

$$\int_0^\infty\left(a+\frac{b}{z^2}\right)\left\{\left(az+\frac{b}{z}\right)+c\right\}^{-\Omega-1}{}_2F_1\left(a,b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)\ {}_2F_1\left(c-a,c-b,c+\frac{1}{2};\left(az+\frac{b}{z}\right)^2+c\right)$$

$$S_{n_1,\dots,n_k}^{m_1,\dots,m_k}\left[y_1\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_1},\dots,y_k\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\mu_k}\right]\times{}_PM_Q^\beta\left[(a_P);(b_Q);x\left\{\left(az+\frac{b}{z}\right)^2+c\right\}^{-\sigma}\right]$$

$$\prod_{i=1}^{s}H_{p_i,q_i}^{m_i,n_i}\left[x_i\left\{\left(az+\frac{b}{z^2}\right)^2+c\right\}^{-\rho_i}\left|\begin{array}{l}\left(c_j^{(i)},\gamma_j^{(i)}\right)_{1,p_i}\\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{array}\right.\right]$$

$$=\frac{\sqrt{\pi}}{(4ab+c)^{\Omega+1/2}}\sum_{r=0}^{\infty}\sum_{r_1=0}^{[n_1/m_1]},\dots,\sum_{r_k=0}^{[n_k/m_k]}\frac{1}{(4ab+c)^{-r}}\frac{(c,r)}{\left(c+\frac{1}{2},r\right)}\ a_r\left\{\sum_{s=0}^{\infty}G(s)\frac{1}{(4ab+c)^{\sigma s}}\right\}$$

$$(-n_1)_{m_1r_1},\dots,(-n_k)_{m_kr_k}\ A(n_1,r_1;\dots;n_k,r_k)\ \frac{y_1^{r_1}}{r_1!},\dots,\frac{y_k^{r_k}}{r_k!}$$

$$\times H_{1,1:\ p_1,q_1;\dots;p_s,q_s}^{0,1:\ m_1,n_1;\dots;m_s,n_s}\left[\begin{array}{c} x_1(4ab+c)^{-\rho_1} \\ \vdots \\ x_s(4ab+c)^{-\rho_s} \end{array}\left| \begin{array}{l} \left(\tfrac{1}{2}-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,\dots,\rho_s\right),:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\dots;\left(c_j^{(s)},\gamma_j^{(s)}\right)_{1,p_s} \\ ,\left(-\Omega+r-\sigma s-\sum_{i=1}^{k}\mu_i k_i;\rho_1,\dots,\rho_s\right):\left(d_j^1,\delta_j^1\right)_{1,q_1};\dots;\left(d_j^{(s)},\delta_j^{(s)}\right)_{1,q_s} \end{array}\right.\right] \tag{3.6}$$

where e_i is defined in (1.6).

Remark.*If we further take r = 1 in Corollaries (3.2), (3.4) and (3.6), then we can simply get the results in term of single H-function.*

4. Conclusion

Conclusion: In the present Research paper we derived some conclusions and results on the generalized fractional calculus, involving definite integrals of Gradshteyn- Ryzhik of the Multivariable H -function. We have also given the number of theorems and corollaries for special functions as of our main results, which are related to the multivariable H-function with M-series and polynomials. The results derived in present investigation are general in nature and can present certain very interesting results in the form of several theorems of various fields which will help in various research field as well as applications.

Competing interests.
The authors declare that they have no competing interests.

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