Developable Hermite Surfaces Constructed between Two Parallel and Oblique Planes

Kusno

Department of Mathematics, University of Jember, Indonesia
kusno.fmipa@unej.ac.id (https://orcid.org/0000-0001-6704-0996)

Abstract: The developable surfaces are useful in modeling plywood sheet installation and plat-metal-based industries. For this reason, this paper introduces the construction of cubic, quartic, and quintic developable Hermite surfaces with their boundary curves placed in two parallel and oblique planes. The steps are as follows. First, it defines the developable piece in the form of algebraic equations. Second, the criteria equation of the developable surfaces is determined for modeling the patches. Finally, we introduce a method for constructing the developable Hermite pieces between both planes. As a result, the presented technique is handy and straightforward for designing these developable surface types.

Keywords: Developable criteria, cubic, quartic, quintic curves, developable Hermite surfaces

1. Introduction

Mathematical and numerical studies for designing developable surfaces have been introduced. From a given curve as a geodesic of the developable piece, Al-Ghefari and Abdel-Baky [5] presented a method for constructing a cone, cylinder, or tangent lines surface. Xu et al. [16] studied a novel calculation to design a minimal surface from a boundary curve of the surface by implementing quasi-harmonic Bézier approximation and quasi-harmonic mask methods. Also, Xu et al. [15] reported general framework for formulating IGA-suitable planar B-spline parameterizations from complex CAD boundaries. However, it requires many complex operations to obtain this surface type. Hu et al. [6] presented the generalized developable H-Bézier surfaces by applying control planes with generalized H-Bézier basis functions.

In connection with the industrial application aspects of the surface, Aumann [1] introduced a developable Bézier patch using two boundary curves. In this case, the patch was placed in parallel planes, and the curves' tangent vectors must be parallel. Frey and Bindschadler [4] continued Aumann's work by generalizing the degree of both boundary curves. Implementing regularity conditions of Bézier developable surface, they could generate developable Bézier strip patches. Then, Elber [2] presented an approximation method called trimming surface for finding the developable surfaces. After that, Kusno [7] studied the construction of regular developable Bézier patches with the boundary curves from four, five, and six degrees. Using polynomial curve and tangent vectors criteria of surface's boundary curves, he introduced the method to design the developable Hermite patches [8–10]. Recently, Fernández-Jambrina and Pérez-Arribas[3] discussed the developable patches construction bounded by two rational or NURBS curves.

The developable surface is essential for designing the industrial objects constructed by plywood sheets and plat-metal, for example, aircraft, ship hull, and train industries [2, 4, 6, 13]. The modeling of each surface part of the objects, generally, must be bounded by two curves $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ laid in the different planes. For this reason, this paper that will be discussed here is to formulate the developable Hermite surfaces using theseboundary curves in two planes $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, respectively. In this case, we propose a new approach to design these surfaces employing somecontrol points, the tangent vectors, and the generatrix of bothboundary curves.

2. Cubic, Quartic, and Quintic Hermite Curves

Consider a cubic curve $\mathbf{p}_3(u) = \mathbf{a}_1u^3 + \mathbf{b}_1u^2 + \mathbf{c}_1u + \mathbf{d}_1$ with the conditions at $\mathbf{p}_3(0) = \mathbf{p}_0$, $\mathbf{p}_3(1) = \mathbf{p}_1$, $\mathbf{p}_3^u(0) = \mathbf{p}_0^u$, and at , $\mathbf{p}_3^u(1) = \mathbf{p}_1^u$with the parameter u in interval $0 < u < 1$. It obtains four equations for computing the coefficient vectors $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$, and $\mathbf{d}_1$, i.e., $\mathbf{p}_0 = \mathbf{d}_1$, $\mathbf{p}_1 = \mathbf{a}_1 + \mathbf{b}_1 + \mathbf{c}_1 + \mathbf{d}_1$, $\mathbf{p}_3^u(0) = \mathbf{c}_1$, and $\mathbf{p}_3^u(1) = 3\mathbf{a}_1 + 2\mathbf{b}_1 + \mathbf{c}_1$. The solution of the equation system is $\mathbf{a}_1 = 2\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_0^u + \mathbf{p}_1^u$; $\mathbf{b}_1 = -3\mathbf{p}_0 + 3\mathbf{p}_1 - 2\mathbf{p}_0^u - \mathbf{p}_1^u$; $\mathbf{c}_1 = \mathbf{p}_0^u$; $\mathbf{d}_1 = \mathbf{p}_0$. Then the cubic Hermite curve can, respectively, formulate in the algebraic and geometric presentation [12, 14]

$$\mathbf{p}_3(u) = (2\mathbf{p}_0-2\mathbf{p}_1+\mathbf{p}_0^u+\mathbf{p}_1^u)u^3+(-3\mathbf{p}_o+3\mathbf{p}_1-2\mathbf{p}_0^u-\mathbf{p}_1^u)u^2+\mathbf{p}_0^u u+\mathbf{p}_0 \quad (2.1a)$$

or

$$\mathbf{p}_3(u) = H_1(u)\mathbf{p}_o + H_2(u)\mathbf{p}_1 + H_3(u)\mathbf{p}_0^u + H_4(u)\mathbf{p}_1^u \quad (2.1b)$$

with

$$H_1(u) = 2u^3 - 3u^2 + 1; \quad H_2(u) = -2u^3 + 3u^2; H_3(u) = u^3 - 2u^2 + u; H_4(u) = u^3 - u^2.$$

Using these calculation steps, we can formulate and evaluate the equation forms of the quartic and quintic Hermite curves from the following determinations.

Let a quartic curve $\mathbf{p}_4(u) = \mathbf{a}_1u^4 + \mathbf{b}_1u^3 + \mathbf{c}_1u^2 + \mathbf{d}_1u + \mathbf{e}_1$. We pose to this curve at $\mathbf{p}_4(0) = \mathbf{p}_0$, $\mathbf{p}_4(0.5) = \mathbf{p}$, $\mathbf{p}_4(1) = \mathbf{p}_1$, $\mathbf{p}_4^u(0) = \mathbf{p}_0^u$, and at $\mathbf{p}_4^u(1) = \mathbf{p}_1^u$with $0 < u < 1$. It gives five equations with the solutions for the coefficient vectors

$\mathbf{a}_1 = -8\mathbf{p}_1 - 8\mathbf{p}_0 + 16\mathbf{p} + 2\mathbf{p}_1^u - 2\mathbf{p}_0^u$; $\quad$ $\mathbf{b}_1 = 14\mathbf{p}_1 + 18\mathbf{p}_0 - 32\mathbf{p} - 3\mathbf{p}_1^u + 5\mathbf{p}_0^u$;

$\mathbf{c}_1 = -5\mathbf{p}_1 - 11\mathbf{p}_0 + 16\mathbf{p} + \mathbf{p}_1^u - 4\mathbf{p}_0^u$; $\quad$ $\mathbf{d}_1 = \mathbf{p}_0^u$; $\quad$ $\mathbf{e}_1 = \mathbf{p}_0$.

Therefore, the quartic Hermite curve in the algebraic formula of the canonical basis $[u^4, u^3, u^2, u, 1]$ is

$$\mathbf{p}_4(u) = (-8\mathbf{p}_1 - 8\mathbf{p}_0 + 16\mathbf{p} + 2\mathbf{p}_1^u - 2\mathbf{p}_0^u)u^4 + (14\mathbf{p}_1 + 18\mathbf{p}_0 - 32\mathbf{p} - 3\mathbf{p}_1^u + 5\mathbf{p}_0^u)u^3 + (-5\mathbf{p}_1 - 11\mathbf{p}_0 + 16\mathbf{p} + \mathbf{p}_1^u - 4\mathbf{p}_0^u)u^2 + \mathbf{p}_0^u u + \mathbf{p}_0. \quad (2.2)$$

The geometric form of the curve in the Hermite basis functions $[H_1, H_2, H_3, H_4, H_5]$ is

$$\mathbf{p}_4(u) = H_1(u)\mathbf{p}_0 + H_2(u)\mathbf{p} + H_3(u)\mathbf{p}_1 + H_4(u)\mathbf{p}_0^u + H_5(u)\mathbf{p}_1^u \quad (2.3)$$

with

$H_1(u) = -8u^4 + 18u^3 - 11u^2 + 1$; $\quad$ $H_2(u) = 16u^4 - 32u^3 + 16u^2$; $\quad$ $H_3(u) = -8u^4 + 14u^3 - 5u^2$;

$H_4(u) = -2u^4 + 5u^3 - 4u^2 + u$; $\quad$ $H_5(u) = 2u^4 - 3u^3 + u^2$.

Consider a quintic curve in the form $\mathbf{p}_5(u) = \mathbf{a}_1u^5 + \mathbf{b}_1u^4 + \mathbf{c}_1u^3 + \mathbf{d}_1u^2 + \mathbf{e}_1u + \mathbf{f}_1$. It is determined at $\mathbf{p}_5(0) = \mathbf{P}_0$, $\mathbf{p}_5(0.5) = \mathbf{p}$, $\mathbf{p}_5(1) = \mathbf{p}_1$, $\mathbf{p}_5^u(0) = \mathbf{p}_0^u$, $\mathbf{p}_5^u(0.5) = \mathbf{p}^u$, and at $\mathbf{p}_5^u(1) = \mathbf{p}_1^u$in interval $0<u<1$. It can find the coefficient vectors

$\mathbf{a}_1 = -24\mathbf{p}_1 + 24\mathbf{p}_0 + 16\mathbf{p}^u + 4\mathbf{p}_0^u + 4\mathbf{p}_1^u$;

$\mathbf{b}_1 = 52\mathbf{p}_1 - 68\mathbf{p}_0 + 16\mathbf{p} - 40\mathbf{p}^u - 12\mathbf{p}_0^u - 8\mathbf{p}_1^u$;

$\mathbf{c}_1 = -34\mathbf{p}_1 + 66\mathbf{p}_0 - 32\mathbf{p} + 32\mathbf{p}^u + 13\mathbf{p}_0^u + 5\mathbf{p}_1^u$;

$\mathbf{d}_1 = 7\mathbf{p}_1 - 23\mathbf{p}_0 + 16\mathbf{p} - 8\mathbf{p}^u - 6\mathbf{p}_0^u - 1\mathbf{p}_1^u$;

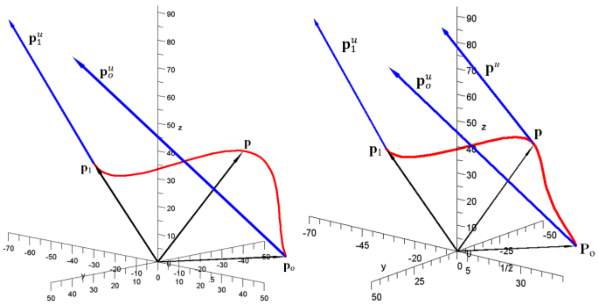
$\mathbf{e}_1 = \mathbf{p}_0^u$; $\quad$ $\mathbf{f}_1 = \mathbf{P}_0$.

Thus, the quintic Hermite curve in the algebraic formula is

$$\mathbf{p}_5(u) = (-24\mathbf{p}_1 + 24\mathbf{p}_0 + 16\mathbf{p}^u + 4\mathbf{p}_0^u + 4\mathbf{p}_1^u)u^5 + (52\mathbf{p}_1 - 68\mathbf{p}_0 + 16\mathbf{p} - 40\mathbf{p}^u - 12\mathbf{p}_0^u - 8\mathbf{p}_1^u)u^4 + (-34\mathbf{p}_1 + 66\mathbf{p}_0 - 32\mathbf{p} + 32\mathbf{p}^u + 13\mathbf{p}_0^u + 5\mathbf{p}_1^u)u^3 + (7\mathbf{p}_1 - 23\mathbf{p}_0 + 16\mathbf{p} - 8\mathbf{p}^u - 6\mathbf{p}_0^u - \mathbf{p}_1^u)u^2 + (\mathbf{p}_0^u)\,u + \mathbf{p}_0. \quad (2.4)$$

Equations (2.1) up to (2.4) facilitate to design the curves. Arranging the position of these curves in space can use the control points $\mathbf{p}_0$, $\mathbf{p}$, and $\mathbf{p}_1$ meanwhile, the shapes of the curves can be designed by the tangent vectors $\mathbf{p}_0^u$, $\mathbf{p}^u$, and $\mathbf{p}_1^u$ at these control points, respectively.

Figure (1a) presents the quartic curve with the control points $\mathbf{p}_0 = < -20, 40, 5 >$, $\mathbf{p} = < -20, 20, 40 >$, $\mathbf{p}_1 = < -20, -50, 25 >$, and the tangent vectors $\mathbf{p}_0^u = < 40, 0, 80 >$, and $\mathbf{p}_1^u = < 10, -60, 80 >$. On another side, Figure (1b) illustrates the quintic curve by using the control points and the tangent vectors data of the quartic curve, and the tangent vector $\mathbf{p}^u = < 60, 5, 60 >$. It shows that this tangent vector $\mathbf{p}^u$changes the shape of the curve in the middle part. The next section, we will introduce the construction of developable Hermite surfaces bounded with these formulated curves laid in the planes.



(a) (b)

Figure 1: Quartic (a) and quintic (b) Hermite curves

3. Construction of Developable Surfaces between Two Parallel Planes

This section presents the following topics. It starts with the definition of a developable surface in algebraic equations. Then, it determines equation criteria to design the developable surfaces numerically. After that, we introduce the methods for constructing cubic, quartic, and quintic developable Hermite surfaces bounded by two parallel planes.

The developable surface definition in mathematical equations is expressed as follows [7, 11].

Definition 3.1. A regular ruled surface $\mathbf{R}(u,v)$ is a surface constructed by a one parameter of lines $\mathbf{R}(u,v) = \mathbf{p}(u) + v.\,\mathbf{g}(u)$ with $\mathbf{p}(u)$ and $\mathbf{g}(u)$ of class C^n and $[(\mathbf{p}'(u) + v.\mathbf{g}'(u)) \wedge \mathbf{g}(u)] \neq \mathbf{0}$.

Definition 3.2. The ruled surface $\mathbf{R}(u,v) = \mathbf{p}(u) + v.\mathbf{g}$ is developable, if the tangent plane along each generatrix is constant that is the vectors $[\mathbf{g}'(u), \mathbf{p}'(u), \mathbf{g}]$ are coplanar.

Definition 3.2 means that the ruled surface $\mathbf{R}(u,v)$ is developable if the vector $\mathbf{g}'(u)$ can be presented in linear combination of the vectors $\mathbf{p}'(u)$ and $\mathbf{g}(u)$ such that $\mathbf{g}'(u) = \alpha(u)\,\mathbf{p}'(u) + \beta(u)\,\mathbf{g}(u)$ with $\alpha(u)$ and $\beta(u)$ of real scalars. In this case, the curves $\mathbf{p}(u)$ is called the directrix curve and $\mathbf{g}(u)$ is the generatrix (ruling) lines.

Consider the generatrix lines $\mathbf{g}(u)$ determined from two boundary curves $\mathbf{p}(u)$ and $\mathbf{q}(u)$ of the ruled surface $\mathbf{R}(u,v)$, that is $\mathbf{g}(u) = \mathbf{q}(u) - \mathbf{p}(u)$. Consequently, this developable criteria of the surface $\mathbf{R}(u,v)$ must be$\mathbf{g}'(u) = [\alpha(u) + 1]\,\mathbf{p}'(u) + \beta(u)\,\mathbf{g}(u)$ or

$$\mathbf{q}'(u) = \rho(u)\,\mathbf{p}'(u) + \beta(u)\,\mathbf{g}(u) \tag{3.1}$$

with $\rho(u)$ and $\beta(u)$ of real scalars.

In general, Equation 3.1 defines the developable surfaces of the cylinder, cone, or tangent lines surface types. Employing the tangent vector criteria $\mathbf{q}'(u) = \rho(u)\,\mathbf{p}'(u)$ of Equation (3.1), Frey and Bindschadler [4] designed the developable Bézier surfaces with the boundary curves in parallel planes. Then, Kusno [8] use these criteria with $\rho(u)$ real constant, linear, and quadratic for modeling the developable Bézier, and on the other hand, $\rho(u)$ real constant for modeling a cylinder and cone Hermite surfaces [8, 10]. Unfortunately, this tangent vector criteria of these curves can make some difficulties in positioning endpoints and center control points of both boundary curves.

To control the position and shapes of a developable surface bounded by two parallel and oblique planes, we introduce the new developable criteria from Equation (3.1). These criteria can facilitate to arrange the boundary curves' tangent vectors and the surface's generatrix with both boundary curves placed in the different planes, that is

$$\mathbf{q}'(u) = k\mathbf{p}'(u) - \mathbf{g}(u) \tag{3.2}$$

with k constant. If the generatrix lines $\mathbf{g}(u)$ of developable surface are defined from the summit point $O$of the cone developable surface with the position vector$\mathbf{o}$ relative to the boundary curves $\mathbf{p}(u)$ and $\mathbf{q}(u)$, then we can determine $\mathbf{g}(u) = [\mathbf{q}(u) - \mathbf{o}] - k\,[\mathbf{p}(u) - \mathbf{o}]$. Therefore, it can state this developable criteria (3.2) in the form

$$[(\mathbf{q}(u) - \mathbf{o}) + \mathbf{q}'(u)] = k\,[(\mathbf{p}(u) - \mathbf{o}) + \mathbf{p}'(u)] \tag{3.3a}$$

or

$$[(\mathbf{q}(u) + \mathbf{q}'(u)] = k\,[\mathbf{p}(u) + \mathbf{p}'(u)] + (1 - k)\,\mathbf{o}. \tag{3.3b}$$

Equation (3.3a) shows that the vectors$\overrightarrow{\mathbf{oq}}$and $\mathbf{q}'(u)$ are orderly the multiplication of the vectors $\overrightarrow{\mathbf{op}}$and $\mathbf{p}'(u)$. If the value k is in interval $0 < k < 1$ the curve $\mathbf{q}(u)$ will lay between the summit point O and the curve $\mathbf{p}(u)$. When $k = 1$, both curves $\mathbf{p}(u)$ and $\mathbf{q}(u)$ will be identical. If $k > 1$, then the curve $\mathbf{p}(u)$ will be placed betweenthe summit point$O$ and the curve $\mathbf{q}(u)$.

Let the curves $\mathbf{p}(u)$ and $\mathbf{q}(u)$ respectively placed in the parallel planes $\mathbf{\Gamma}_1 // \mathbf{\Gamma}_2$ with the summit point O of the cone developable surface positioned outer the planes. Using these data, Equation (3.3), and Maple software, we can introduce a new approach of the construction method for designing cubic, quartic, and quintic Hermite developable surfaces

$$\mathbf{D}(u,v) = \mathbf{p}(u) + v.\mathbf{g} = (1- v)\, \mathbf{p}(u) + v\mathbf{q}(u) \quad (3.4)$$

as explainedin the following sections.

3.1 Cubic Developable Hermite Surfaces

Two cubic Hermite curves $\mathbf{p}_3(u)$ and $\mathbf{q}_3(u)$ in the parallel plane $\mathbf{\Gamma}_1 // \mathbf{\Gamma}_2$ are presented in the polynomial formula and their first derivations, respectively, as follows

$$\mathbf{p}_3(u) = \mathbf{a}_1 u^3 + \mathbf{b}_1 u^2 + \mathbf{c}_1 u + \mathbf{d}_1; \qquad \mathbf{p}_3'(u) = 3\mathbf{a}_1 u^2 + 2\mathbf{b}_1 u + \mathbf{c}_1,$$
$$\mathbf{q}_3(u) = \mathbf{a}_2 u^3 + \mathbf{b}_2 u^2 + \mathbf{c}_2 u + \mathbf{d}_2; \qquad \mathbf{q}_3'(u) = 3\mathbf{a}_2 u^2 + 2\mathbf{b}_2 u + \mathbf{c}_2.$$

Substituting these equations to Equation (3.3b) of developable surface criteria find

$$\mathbf{a}_2 u^3 + (\mathbf{b}_2+3\mathbf{a}_2)u^2 + (\mathbf{c}_2+2\mathbf{b}_2)u + (\mathbf{d}_2+\mathbf{c}_2) = k[\mathbf{a}_1 u^3 + (\mathbf{b}_1+3\mathbf{a}_1)u^2 + (\mathbf{c}_1+2\mathbf{b}_1)u + (\mathbf{d}_1+\mathbf{c}_1)] + (1-k)\,\mathbf{o}.$$

It obtains four equations relating to this developable surface criteria, i.e., $\mathbf{a}_2 = k.\mathbf{a}_1$; $\mathbf{b}_2+3\mathbf{a}_2 = k.(\mathbf{b}_1+3\mathbf{a}_1)$; $\mathbf{c}_2+2\mathbf{b}_2 = k.(\mathbf{c}_1+2\mathbf{b}_1)$; $\mathbf{d}_2+\mathbf{c}_2 = k.(\mathbf{d}_1+\mathbf{c}_1)+ (1-k)\,\mathbf{o}$. When we replace these coefficient vectors $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$, $\mathbf{d}_1$, and $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{c}_2$, $\mathbf{d}_2$ in the form of control points and tangent vectors values shown in Equation (2.1a), it gives the equations

$$2\mathbf{q}_0-2\mathbf{q}_1 +\mathbf{q}_0^u + \mathbf{q}_1^u = k.(2\mathbf{p}_0-2\mathbf{p}_1 + \mathbf{p}_0^u + \mathbf{p}_1^u)$$
$$3\mathbf{q}_0-3\mathbf{q}_1 + \mathbf{q}_0^u + 2\mathbf{q}_1^u = k.(3\mathbf{p}_0-3\mathbf{p}_1 + \mathbf{p}_0^u + 2\mathbf{p}_1^u)$$
$$-6\mathbf{q}_0+6\mathbf{q}_1-3\mathbf{q}_0^u-2\mathbf{q}_1^u = k.(-6\mathbf{p}_0+6\mathbf{p}_1-3\mathbf{p}_0^u-2\mathbf{p}_1^u)$$
$$\mathbf{q}_0 + \mathbf{q}_0^u = k.(\mathbf{p}_0 + \mathbf{p}_0^u) + (1-k)\mathbf{o}.$$

If in these equations, we determine some values for the summit point $\mathbf{o}$, the control points $\mathbf{q}_0$, $\mathbf{q}_1$, and the tangent vectors $\mathbf{q}_0^u$ and $\mathbf{q}_1^u$, then it gets the equation system in matrix form equations as follows

$$\begin{pmatrix} 2k & -k & -k & -2k \\ 3k & -k & -2k & -3k \\ -6k & 3k & 2k & 6k \\ 0 & -k & 0 & -k \end{pmatrix} \times \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_0^u \\ \mathbf{p}_1^u \\ \mathbf{p}_0 \end{pmatrix} = \begin{pmatrix} 2\mathbf{q}_1 - 2\mathbf{q}_0 - \mathbf{q}_0^u - \mathbf{q}_1^u \\ 3\mathbf{q}_1 - 3\mathbf{q}_0 - \mathbf{q}_0^u - 2\mathbf{q}_1^u \\ -6\mathbf{q}_1 + 6\mathbf{q}_0 + 3\mathbf{q}_0^u + 2\mathbf{q}_1^u \\ (1-k)\mathbf{o} - \mathbf{q}_0 - \mathbf{q}_0^u \end{pmatrix} \quad (3.5)$$

$$\mathbf{A}_{4\times4} \times \mathbf{B}_{4\times1} = \mathbf{C}_{4\times1}.$$

This equation system (3.5) has a unique solution if the determinant value of the coefficient matrix $\mathbf{A}$ is different from zero. Due to $det(\mathbf{A}) = k^4$ and the defined developable surfaces are positioned in the same side to the summit point O, we elect the value $k > 0$ and $k \neq 1$. It obtains the solutions

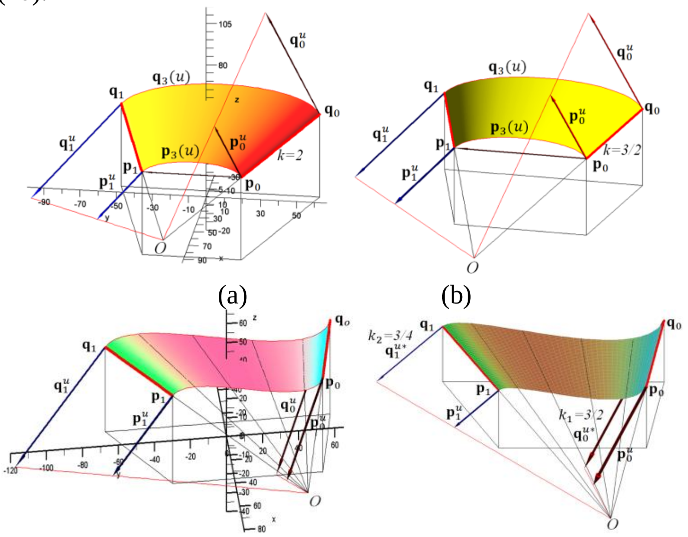
$$\mathbf{p}_1 = [\mathbf{q}_1 + (k-1)\mathbf{o}]/k; \qquad \mathbf{p}_0^u = \mathbf{q}_0^u/k; \qquad \mathbf{p}_1^u = \mathbf{q}_1^u/k; \qquad \mathbf{p}_0 = [\mathbf{q}_0 + (k-1)\mathbf{o}]/k.$$

It can conclude that using two cubic Hermite curves $\mathbf{p}_3(u)$ and $\mathbf{q}_3(u)$ in the parallel planes $\mathbf{\Gamma}_1 // \mathbf{\Gamma}_2$, correspondingly, can be designed a cubic developable Hermite surface $\mathbf{D}_3(u, v) = (1 - v)\,\mathbf{p}_3(u) + v\mathbf{q}_3(u)$ through the steps:

1. Substitute $\mathbf{p}_3(u)$, $\mathbf{p}'(u)$, $\mathbf{q}_3(u)$, $\mathbf{q}'(u)$ in the developable surface criteria of Equation (3.3);
2. Determine the values for the summit point O, the control points and the tangent vectors $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_0^u$, $\mathbf{q}_1^u$ in the polynomial coefficient equations of the step result (1);
3. Determine the value $k>0$ and $k\neq1$, after that, calculate Equation (3.5) to find the control points $\mathbf{p}_0$, $\mathbf{p}_1$and the tangent vectors $\mathbf{p}_0^u$, $\mathbf{p}_1^u$.

If in step (2), we respectively give some tension values k_1 and k_2 to the tangent vectors $\mathbf{q}_0^u$ and $\mathbf{q}_1^u$, i.e., $\mathbf{q}_0^{u*} = k_1\mathbf{q}_0^u$ and $\mathbf{q}_1^{u*} = k_2\mathbf{q}_1^u$, then in step (3), it finds the values $\mathbf{p}_0^u = [k_1\mathbf{q}_0^u]/k$ and $\mathbf{p}_1^u = [k_2\mathbf{q}_1^u]/k$.

When the vector position $\mathbf{o}$ for the summit point O, the control points $\mathbf{q}_0$, $\mathbf{q}_1$, and the tangent vectors $\mathbf{q}_0^u$, $\mathbf{q}_1^u$ are $\mathbf{o}$ =< 170, 0, 45 >, $\mathbf{q}_0$ =< −20, 60, 45 >, $\mathbf{q}_1$ =< −20, −50, 45 >, $\mathbf{q}_0^u$ =< 0, −30, 60 >, and $\mathbf{q}_1^u$ =< 0, −50, −60) >, meanwhile, the value k is elected $k = 2$, it can find $\mathbf{p}_1$ =< 75, −25, 45 >, $\mathbf{p}_0^u$ =< 0, −15, 30 >, $\mathbf{p}_1^u$ =< 0, −25, −30 >, and $\mathbf{p}_0$ =< 75, 30, 45 >. The cubic developable Hermite surface presents in Figure (2a). If we determine $k = 3/2$, then the surface's width (generatrix measure) changes as shown in Figure (2b). On the other hand, when we replace $\mathbf{o}$ =< 125, 30, 45 >, $\mathbf{q}_1$ =< −20, −65, 40 >, $\mathbf{q}_0^u$ =< 0, −30, −80 >, and $k = 3/2$, it finds the developable surface presented in Figure (2c). From these data, if we respectively gives the tension values $k_1 = 3/2$ and $k_2 = 3/4$ to the tangent vectors $\mathbf{q}_0^u$ and $\mathbf{q}_1^u$, i.e., $\mathbf{q}_0^{u*} = 3/2\mathbf{q}_0^u$ and $\mathbf{q}_1^{u*} = 3/4\mathbf{q}_1^u$, then the shape of the developable surface modify as shown in Figure (2d). The surface modeling results in Figure (2a) and Figure (2b) show that the determination of the point O and the parameter k can arrange the position ofcontrol points $\mathbf{p}_0$, $\mathbf{p}_1$, and the measure of generatrix $\mathbf{g}$ between the parallel planes $\mathbf{\Gamma}_1//\mathbf{\Gamma}_2$. Then, the choices of different tangent vectors $\mathbf{q}_0^u$ and $\mathbf{q}_1^u$ will produce various shapes of the surfaces as presented in Figure (2a) and Figure (2c). In addition, giving tensions to the tangent vectors $\mathbf{q}_0^u$ and $\mathbf{q}_1^u$will effect the surface tangent form, sharply or weakly, in area near the control points $\mathbf{q}_0$ and $\mathbf{q}_1$ (Figure (2d).



(c) (d)

Figure 2: Cubic developable Hermite surfaces

3.2 Quartic Developable Hermite Surfaces

Consider the quartic Hermite curves $\mathbf{p}_4(u) = \mathbf{a}_1u^4 + \mathbf{b}_1u^3 + \mathbf{c}_1u^2 + \mathbf{d}_1u + \boldsymbol{e}_1$ and $\mathbf{q}_4(u) = \mathbf{a}_2u^4+\mathbf{b}_2u^3+\mathbf{c}_2u^2+\mathbf{d}_2u+\boldsymbol{e}_2$ in the parallel plane $\mathbf{\Gamma}_1//\mathbf{\Gamma}_2$, correspondingly, to construct the quartic developable Hermite surface $\mathbf{D}(u,v) = (1-v)\mathbf{p}_4(u) + v\mathbf{q}_4(u)$. Computing $\mathbf{p}_4'(u)$, and $\mathbf{q}_4'(u)$, substituting these results to the developable surface criteria in Equation (3.3), and determining control points and the tangent vectors $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_0^u$, $\mathbf{q}_1^u$, $\mathbf{p}_0$, and $\mathbf{p}$in these polynomial coefficient equations, it finds the equation system in matrix form equations

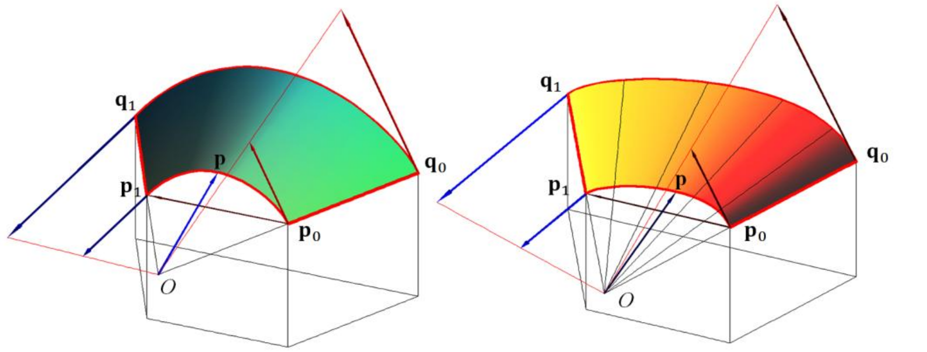
$$\begin{pmatrix} 16 & 8k & 2k & -2k & 0 \\ 32 & 18k & 3k & -5k & 0 \\ -80 & -37k & -11k & 8k & 0 \\ 32 & 10k & 7k & -2k & 0 \\ 0 & 0 & -k & 0 & k-1 \end{pmatrix} \times \begin{pmatrix} \mathbf{q} \\ \mathbf{p}_1 \\ \mathbf{p}_0^u \\ \mathbf{p}_1^u \\ \mathbf{o} \end{pmatrix} = \begin{pmatrix} -8k\mathbf{p}_0 + 16k\mathbf{p} + 8\mathbf{q}_1 + 8\mathbf{q}_0 - 2\mathbf{q}_1^u + 2\mathbf{q}_0^u \\ -14\mathbf{p}_0 + 32k\mathbf{p} + 18\mathbf{q}_1 + 14\mathbf{q}_0 - 5\mathbf{q}_1^u + 3\mathbf{q}_0^u \\ 43k\mathbf{p}_0 - 80k\mathbf{p} - 37\mathbf{q}_1 - 43\mathbf{q}_0 + 8\mathbf{q}_1^u - 11\mathbf{q}_0^u \\ -22k\mathbf{p}_0 + 32k\mathbf{p} + 10\mathbf{q}_1 + 22\mathbf{q}_0 - 2\mathbf{q}_1^u + 7\mathbf{q}_0^u \\ k\mathbf{p}_0 - \mathbf{q}_0 - \mathbf{q}_1^u \end{pmatrix} \quad (3.6)$$

$\mathbf{A}_{5\times5} \times \mathbf{B}_{5\times1} = \mathbf{C}_{5\times1}$.

The determinant value of the coefficient matrix $\mathbf{A}$ is $det(\mathbf{A}) = -2176k^5 + 4336k^4 - 2160k^3 = -16k^3(136k - 135)(k-1)$. If we choice $k > 0$, $k \neq 1$ and $k \neq (135/136)$, then it obtains the solutions

$$\mathbf{q} = -k\mathbf{p}_0 + \mathbf{q}_0 + k\mathbf{p}; \mathbf{p}_1 = (k\mathbf{p}_0 + \mathbf{q}_1 - \mathbf{q}_0)/k; \mathbf{p}_0^u = (\mathbf{q}_0^u)/k; \mathbf{p}_1^u = \mathbf{q}_1^u/k; \mathbf{o} = (k\mathbf{p}_0 - \mathbf{q}_0)/(k-1).$$

Let $k = 2$, $\mathbf{p}_0$ =< 40, 40, 45 >, $\mathbf{p}$ =< 40, 12.5, 60 >, $\mathbf{q}_0$=< −20, 60, 45 >, $\mathbf{q}_1$ =< −20, −50, 45 >, $\mathbf{q}_0^u$ =< 0, −30, 60 >, and $\mathbf{q}_1^u$ =< 0, −50, −60 >. From Equation (3.6), it obtains the solution $\mathbf{q}$ =<−20, 5, 75 >, $\mathbf{p}_1$ =< 40, −15, 45 >, $\mathbf{p}_0^u$ =< 0, −15, 30 >, $\mathbf{p}_1^u$ =< 0, −25, −30 >, and $\mathbf{o}$ =< 100, 20, 45 >. Implementing this data constructs the quartic developable Hermite surface in Figure (3a). When we replace $\mathbf{p}$ =< 40, 19, 55 >, the surface modify as shown in Figure (3b).



(a) (b)

Figure 3: Quartic developable Hermite surfaces

The advantage of this quartic developable Hermite surface over the cubic developable Hermite surface is that we can move the intermediate control point **p** or **q** for changing the shape of the middle part of the developable surface (Figure (3b)).

3.3 Quintic Developable Hermite Surfaces

Let two quintic Hermite curves $\mathbf{p}_5(u) = \mathbf{a}_1u^5 + \mathbf{b}_1u^4 + \mathbf{c}_1u^3 + \mathbf{d}_1u^2 + \boldsymbol{e}_1u + \boldsymbol{f}_1$ and $\mathbf{q}_5(u) = \mathbf{a}_2u^5 + \mathbf{b}_2u^4 + \mathbf{c}_2u^3 + \mathbf{d}_2u^2 + \boldsymbol{e}_2u + \boldsymbol{f}_2$ in the parallel plane $\boldsymbol{\Gamma}_1//\boldsymbol{\Gamma}_2$, consecutively, to design the quintic developable Hermite surface $\mathbf{D}(u,v) = (1-v)\,\mathbf{p}_5(u) + v\mathbf{q}_5(u)$. Calculating $\mathbf{p}_5'(u)$, and $\mathbf{q}_5'(u)$, substituting these results to the developable surface criteria in Equation (3.3), and posing control points and the tangent vectors $\mathbf{p}_0$, $\mathbf{p}$, $\mathbf{q}_0^u$, $\mathbf{q}_1$, $\mathbf{q}_0$, $\mathbf{q}_1^u$, and $\mathbf{p}^u$in these polynomial coefficient equations, it gets the equation system in matrix form equations

$$\mathbf{A}_{6\times6} \times \mathbf{B}_{6\times1} = \mathbf{C}_{6\times1}. \tag{3.7}$$

with

$$\mathbf{A} = \begin{pmatrix} -4k & 16 & -4k & 0 & 24k & 0 \\ -8k & 40 & -12k & 16 & 68k & 0 \\ 35k & -128 & 27k & 32 & -174k & 0 \\ -33k & 88 & -14k & -80 & 95k & 0 \\ 11k & -16 & 2k & 32 & -14k & 0 \\ -k & 0 & 0 & 0 & 0 & k-1 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} \mathbf{p}_0^u \\ \mathbf{q}^u \\ \mathbf{p}_1^u \\ \mathbf{q} \\ \mathbf{p}_1 \\ \mathbf{o} \end{pmatrix};$$

$$\mathbf{C} = \begin{pmatrix} 24k\mathbf{p}_0 + 24\mathbf{q}_1 - 24\mathbf{q}_0 - 4\mathbf{q}_0^u - 4\mathbf{q}_1^u + 16k\mathbf{p}^u \\ 52k\mathbf{p}_0 + 16k\mathbf{p} + 68\mathbf{q}_1 - 52\mathbf{q}_0 - 8\mathbf{q}_0^u - 12\mathbf{q}_1^u + 40k\mathbf{p}^u \\ -206k\mathbf{p}_0 + 32k\mathbf{p} - 174\mathbf{q}_1 + 206\mathbf{q}_0 + 35\mathbf{q}_0^u + 27\mathbf{q}_1^u - 128k\mathbf{p}^u \\ 175k\mathbf{p}_0 - 80k\mathbf{p} + 95\mathbf{q}_1 - 175\mathbf{q}_0 - 33\mathbf{q}_0^u - 14\mathbf{q}_1^u + 88k\,\mathbf{p}^u \\ -46k\,\mathbf{p}_0 + 32k\mathbf{p} - 14\mathbf{q}_1 + 46\mathbf{q}_0 + 11\mathbf{q}_0^u + 2\mathbf{q}_1^u - 16k\mathbf{p}^u \\ k\mathbf{p}_0 - \mathbf{q}_0 - \mathbf{q}_0^u \end{pmatrix}.$$

The matrix **A** has $det(\mathbf{A}) = 256k^3\,(k-1)$. If we determine $k > 0$ and $k \neq 1$, then it finds the solutions

$\mathbf{p}_0^u = (\mathbf{q}_0^u)/k$; $\quad \mathbf{q}^u = k\mathbf{p}^u$; $\quad \mathbf{p}_1^u = \mathbf{q}_1^u/k$;
$\mathbf{q} = -k\mathbf{p}_0 + \mathbf{q}_0 + k\mathbf{p}$; $\quad \mathbf{p}_1 = (k\mathbf{p}_0 + \mathbf{q}_1 - \mathbf{q}_0)/k$; $\quad \mathbf{o} = (k\mathbf{p}_0 - \mathbf{q}_0)/(k-1)$.

The validation of the method presents as follows. Let the data $k = 2$, $\mathbf{p}_0 =< 40, 40, 45 >$, $\mathbf{p} =< 40, 12.5, 40 >$, $\mathbf{q}_0^u =< 0, -60, 90 >$, $\mathbf{q}_1 =< -20, -50, 45 >$, $\mathbf{q}_0 =< -20, 60, 45 >$, $\mathbf{q}_1^u =< 0, -50, -30 >$, and $\mathbf{p}^u =< 0, -25, 35 >$. Equation (3.6) gives the solution $\mathbf{p}_0^u =< 0, -30, 45 >$, $\mathbf{q}^u =< 0, -50, 70 >$, $\mathbf{p}_1^u =< 0, -25, -15 >$, $\mathbf{q} =< -20, 5, 35 >$, $\mathbf{p}_1 =< 40, -15, 45 >$, and $\mathbf{o} =< 100, 20, 45 >$. The constructed developable surface is presented in Figure (4a). If we change $\mathbf{p}^u =< 0, -25, -35 >$, the surface shape modify as shown in Figure (4b). Moreover, when the tangent vector is replaced $\mathbf{q}_1^u =< 0, -30, -50 >$, it finds Figure (4c). Figure (4d) shows that the tangent planes a long the generatrix of the developable surfaces of Figure (4b) are constant. The advantage of these developable surfaces rather than cubic and quartic developable Hermite surface is that the choice of vector tangent $\mathbf{p}^u$ can modify straightforward the surface shape in the middle part of the surface (Figure (4b) and (4c)).

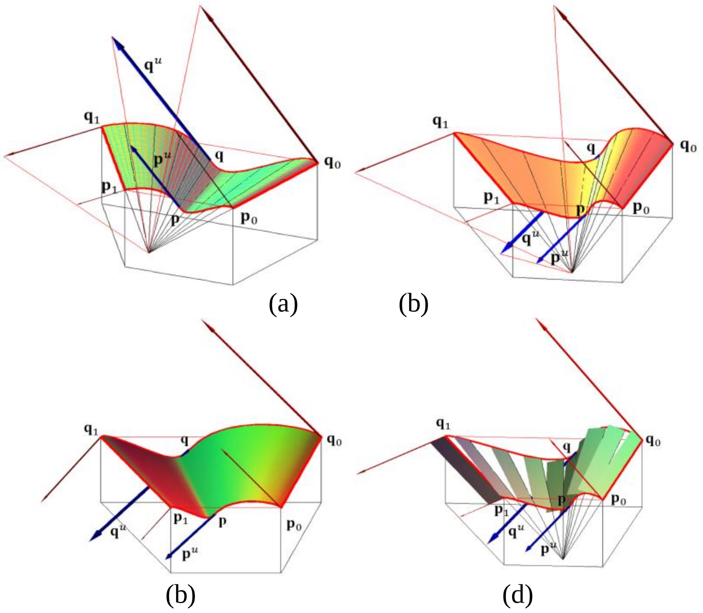


Figure 4: Quintic developable Hermite surfaces

4. Construction of Developable Surfaces between Two Oblique Planes

Consider a quintic developable Hermite surface of cone surface type $\mathbf{D}_5(u,v) = (1-v)\,\mathbf{p}_5(u) + v\mathbf{q}_5(u)$, as shown in Figure (5a), with $\mathbf{p}_5(u)$ and $\mathbf{q}_5(u)$ in the planes $\mathbf{\Gamma}_1//\mathbf{\Gamma}_2$, respectively, and the summit of this cone surface at the point O. A plane $\mathbf{\Gamma}_3$ contains the control point $\mathbf{q}_0$ of the curve $\mathbf{q}_5(u)$ in the form

$$\mathbf{\Gamma}_3(u,v) = \mathbf{q}_0 + u\mathbf{a} + v\mathbf{d} \tag{4.1}$$

and $\mathbf{\Gamma}_3$ is in oblique position to the plane $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, i.e., $\mathbf{\Gamma}_3 \nparallel \mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_3 \nparallel \mathbf{\Gamma}_2$. This section discusses the transformation of the curve $\mathbf{q}_5(u)$ in $\mathbf{\Gamma}_2$ to the curve image $\mathbf{r}_5(u)$ in $\mathbf{\Gamma}_3$ relative to the summit point O of the cone surface $\mathbf{D}_5(u,v)$ such that $\mathbf{p}_5(u)$, $\mathbf{q}_5(u)$, and $\mathbf{r}_5(u)$ construct the developable surface. In other words, we have to find the control points and the tangent vectors $\mathbf{r}_0$, $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_0^u$, $\mathbf{r}^u$, and $\mathbf{r}_1^u$in $\mathbf{\Gamma}_3$ as the projection images of the $\mathbf{q}_0$, $\mathbf{q}$, $\mathbf{q}_1$, $\mathbf{q}_0^u$, $\mathbf{q}^u$, and $\mathbf{q}_1^u$ in $\mathbf{\Gamma}_2$, correspondingly, for designing the curve $\mathbf{r}_5(u)$. The method is as follows.

We determine the control point $\mathbf{r}_0 = \mathbf{q}_0$. The projection of tangent vector $\mathbf{q}_0^u$ in $\mathbf{\Gamma}_2$ to $\mathbf{r}_0^u$ in $\mathbf{\Gamma}_3$ can be calculated using these ways. If the vector $\mathbf{o}+w[(\mathbf{q}_0\text{-}\ \mathbf{o}) + \mathbf{q}_0^u]$ and the plane $\mathbf{\Gamma}_3(u,v) = \mathbf{q}_0 + u\mathbf{a} + v\mathbf{d}$ intersect at a point with position vector$\mathbf{I}_1$, then it finds$\mathbf{I}_1 = \mathbf{o}+w[(\mathbf{q}_0\text{-}\mathbf{o})+\mathbf{q}_0^u]= \mathbf{q}_0+u\mathbf{a}+v\mathbf{d}$and $w = [(\mathbf{q}_0\text{-}\ \mathbf{o}).(\mathbf{a}\wedge\mathbf{d})]/[((\mathbf{q}_0\text{-}\mathbf{o})+\mathbf{q}_0^u).(\mathbf{a}\wedge\mathbf{d})]$. We get

$$\mathbf{r}_0^u = \left[\frac{(\mathbf{q}_0-\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})}{((\mathbf{q}_0-\mathbf{o})+\mathbf{q}_0^u).(\mathbf{a}\wedge\mathbf{d})}\right][(\mathbf{q}_0-\mathbf{o})+\mathbf{q}_0^u] - (\mathbf{q}_0-\mathbf{o}). \tag{4.2}$$

The projection $\mathbf{q}$ in $\mathbf{\Gamma}_2$ to $\mathbf{r}$ in $\Gamma3$ is $\mathbf{r} = \mathbf{o} + s(\mathbf{q}-\mathbf{o}) = \mathbf{q}_0 + u\mathbf{a} + v\mathbf{d}$and $s = [(\mathbf{q}_0\text{-}\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})]/[(\mathbf{q}\text{-}\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})]$. Therefore, the control point value $\mathbf{r}$ is

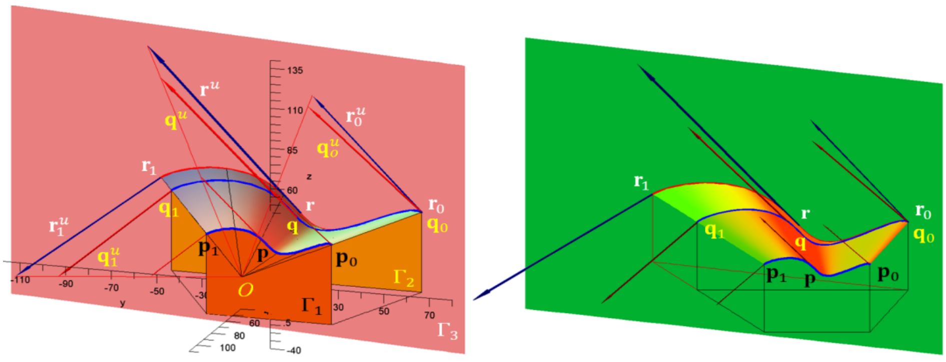
$$\mathbf{r} = \mathbf{o} + \left[\frac{(\mathbf{q}_0-\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})}{(\mathbf{q}-\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})}\right](\mathbf{q}\text{-}\ \mathbf{o}). \tag{4.3}$$

Applying this method of computation, it can be found $\mathbf{r}^u$, $\mathbf{r}_1$, and $\mathbf{r}_1^u$ in the form

$$\mathbf{r}^u = t[(\mathbf{q}-\mathbf{o})+\mathbf{q}^u] - (\mathbf{r}-\mathbf{o}); \qquad \mathbf{r}_1 = \mathbf{o} + n\,(\mathbf{q}_1-\mathbf{o}); \qquad \mathbf{r}_1^u = m[(\mathbf{q}_1-\mathbf{o}) + \mathbf{q}_1^u] - (\mathbf{r}_1-\mathbf{o}). \tag{4.4}$$

with $t = \left[\frac{(\mathbf{q}_0-\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})}{((\mathbf{q}-\mathbf{o})+\mathbf{q}^u).(\mathbf{a}\wedge\mathbf{d})}\right]$, $n = \left[\frac{(\mathbf{q}_0-\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})}{(\mathbf{q}_1-\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})}\right]$, $m = \left[\frac{(\mathbf{q}_0-\mathbf{o}).(\mathbf{a}\wedge\mathbf{d})}{((\mathbf{q}_1-\mathbf{o})+\mathbf{q}_1^u).(\mathbf{a}\wedge\mathbf{d})}\right]$.

Using quintic developable Hermite surface data of Figure (4a) in section 3.3, it has $\mathbf{o} =< 100, 20, 45 >$,$\mathbf{q}_0 =<-20, 60, 45>$, $\mathbf{q}_0^u =<0, -60, 90>$, $\mathbf{q} =<-20, 5, 35>$, $\mathbf{q}^u =<0, -50, 70>$, $\mathbf{q}_1 =<-20, -50, 45>$, $\mathbf{q}_1^u =<0, -50, -30>$. The data projection results in the plane$\mathbf{\Gamma}_3(u,v) =<-20, 60, 45> + u.\ <-30, -200, 0> + v.\ <0, 0, 200>$find $\mathbf{r}_0 =<-20, 60, 45>$,$\mathbf{r}_0^u =<-7.6, -50.6, 63.8>$,$\mathbf{r} =<-28.4, 3.9, 34.3>$,$\mathbf{r}^u =<-10.6, -70.8, 91, 8>$,$\mathbf{r}_1 =<-38, -60.5, 45>$,$\mathbf{r}_1^u =<-10.2, -67.7, -74.1>$. It constructs the quintic developable Hermite surface presented in Figure (5a). If $\mathbf{\Gamma}_3(u,v) =<-20, 60, 45> + u.\ <-60, -200, 0> + v.\ <0, 0, 200>$ gives the quintic developable Hermite surface in Figure (5b).



(a) (b)

Figure 5: Developable surface between two oblique planes

5.Conclusion

It has formulated the mathematics equations of the developable surface criteria that can be used to model the cubic, quartic, and quinticdevelopable Hermite surfaces. Using some control points, the tangent vectors and the generatrix of boundary curves of the surface, this formulacan design the desired forms of the surfaces. Therefore, it is expected that the construction of any developable Hermite surfaceswill be more effective and satisfied.

The construction methods of the developable surface laid between two parallel and oblique planeshave been presented. In the future works, we need to develop theconstruction of these developable surfaces with their boundary curves laid in the space.

References

[1] G. Aumann: Interpolation with developable Bézier patches, Computer Aided Geometric Design 8 (1991), pp. 409–420.

[2] G. Elber: Model Fabrication Using Surface layout Projection, Computer-Aided Design, 27 (1995), pp.283-291.

[3] L. Fernández-Jambrina, F.Pérez-Arribas: Developable Surface Patches Bounded by NURBS Curves, Journal of Computational Mathematics 38 (2020), pp. 715–731, doi: https://doi.org/10.4208/jcm.1904-m2018-0209.

[4] W. Frey, D. Bindschadler: Computer-Aided Design of a Class of Developable Bézier Surfaces, General Motors RD Publication: Detroit, MI, USA (1993), pp. 1–28.

[5] R. A. Al-Ghefari, R. A. Abdel-Baky: An approach for designing developable surface with a common geodesic curve, International Journal of Contemporary Mathematical Sciences 8 (2013), pp. 875–891, doi: http://dx.doi.org/10.12988/ijcms.2013.39101.

[6] G. Hu, J. Wu, X. Qin: A new approach in designing of local controlled developable HBézier surfaces, Advances Engineering Software 121 (2018), pp. 26–38. doi: http://dx.doi.org/10.1016/j.advengsoft.2018.03.003.

[7] Kusno: Construction of Regular Developable Bézier Patches, Mathematical and Computational Applications 24.4 (2019), pp. 1–13, doi: https://doi.org/10.3390/mca24010004.

[8] Kusno: Modeling of developable surfaces using Hermite spline interpolation curves, Advances in Mathematics: Scientific Journal 9.10 (2020), pp. 8431–8442, doi: https://doi.org/10.37418/amsj.9.10.72.

[9] Kusno: Fitting a curve, cutting surface, and adjusting the shapes of developable Hermite patches, Mathematics and Statistics 8.6 (2020), pp. 740–746, doi: https://doi.org/10.13189/ms.2020.080615.

[10] Kusno: On the modeling of developable Hermite patches, Journal of Mathematical and Computational Science 11.2 (2021), pp. 1145–1165, doi: https://doi.org/10.28919/jmcs/5293.

[11] M. Lipschutz: Theory and Problems of Differential Geometry, Schaum's Outline Series, New York: McGraw-Hill, 1969.

[12] M. E. Mortenson: Geometric Modeling, New York: Jonh Wiley and Sons Inc., 1996.

[13] F. C. Park, J. Yu, C. Chun, B. Ravani: Design of developable surfaces using optimal control, ournal of Mechanical Design 124 (2002), pp. 602–608.

[14] D. Salomon: Curves and Surfaces for Computer Graphics, New York: Springer, 2006.

[15] G. Xu, M. Li, B. Mourrain, T. Rabczuk, J. Xu, S. Bordas: Constructing IGA-suitable planar parameterization from complex CAD boundary by domain partition and global/local optimization, Computer Methods in Applied Mechanics and Engineering 328 (2017), pp. 175– 200. doi: http://dx.doi.org/10.1016/j.cma.2017.08.052.

[16] G. Xu, T. Rabczuk, E. Güler, Q. Wu, K. Hui, G. Wang: Quasi-harmonic Bézier approximation of minimal surfaces for finding forms of structural membranes, Computers &Structures 161 (2015), pp. 55–63, doi: http://dx.doi.org/10.1016/j.compstruc.2015.09.002.