Lévy-stable autoregressive model for the federal funds rate

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Abstract: In this paper, we will try to adjust the behaviour of the US Federal Funds interest rate to a model of autoregressive Levy-stable. We will conduct a seriesof tests after which it will offer the best model for this data series distributed in time, in this case, a linear model type AR (1) stationary whose distribution i.i.d. innovations would be a stable Lévy law and proceed thereafter to the estimation of nine parameters for this model.
Keywords: autoregressive process, Lévy stable distribution, heavy tail.

1. Introduction

In finance and economics, we always try to understand the behavior of a given index. In the light of a sample of values taken by this index in its chronology, we always try to find the right model which is as much as possible in conformity with the natural process which generated it.
In this work, we exploited a proportion of the trajectory followed by the interest rate of the American federal funds between the years 1992 and 1994 in which it seemedto us that the best model which could adjust this passage of the trajectory is that ofa Lévy-stable stationary autoregressive process.
Initially and after having presented the descriptive characteristics of our data, we recalled, in the third section, the definition of a stable Lévy law as well as the essential of its properties, especially that concerning its extreme behavior qualified as a tail. heavy which generates the loss of the second moment, contrary to the classical cases where the variance always exists.
Similarly, in the next section, we also recalled the stationary Lévy-stable autoregressive process described by one of its equations when it is of the first order involving innovations i.i.d. Lévy-stables not necessarily centered. The stationary required here is stationary in the strict sense since the first two moments do not always exist.
We have also recalled the algebraic relations which connect the parameters of the law of stable innovations i.i.d. to those of the autoregressive process.
After having analyzed the stationary character of the time series of the data thanks to the implementation of two tests on the possible existence of a unit root and, which are those of Augmented Fuller and Schmidt-Phillips which, fortunately, rather indicate the absence of such a root, and having proceeded to the examination of the ACF and the PACF as well as the information criterion of Akaike AIC, we finally decided on a model of the type AR of first order and which is described by an equation of the form:

$$X_n = \lambda X_{n-1} + Z_n, \quad n \in \mathbf{Z}$$

Where λ is the autoregression coefficient and (Z_n) denotes the sequence of innovations i.i.d. having the same Lévy-stable law denoted $S(\alpha_Z, \mu_Z, \beta_Z, \delta_Z)$ with α_Z its tail index, μ_Z its position parameter, β_Z its asymmetry parameter and δ_Z its scale parameter.
Then, comes the estimation phase where we first started to evaluate the autoregression coefficient which

reveals a value confirming once again the stationarity of the series in this chosen framework and therefore allowing us to estimate the sequence of residuals . The latter, and after having tested the aspects describing the presence of heavy tail, stationarity and especially the non-significance of the dependence between the terms of the sequence of residues, which justifies the hypothesis made on the i.i.d. innovations, will allow us to estimate the parameters of their common Lévy-stable distribution, and consequently, those of the proposed AR(1) model.

These estimates were made using two approaches, that of quantiles and that of maximum likelihood for comparison purposes. Finally, we evaluate a few extreme quantiles.

Finally and in a forecasting framework, we estimated three successive predictive values for the data series by comparing them to the actual values of the Federal Fund Rate.

The calculations, graphics and results obtained in this work were carried out under the statistical software R.

2. Presentation of data

The data used in this study are taken from the website (http://www.economagic.com) in the period between the two dates from 29/10/1992 to 31/01/1994 which extendsover almost 460 consecutive days and representing the most important money market rate in the United States, that of the federal fund. The descriptive statistics of this data is given in the following table

Table. 1: The descriptive statistics of data

n	Min	1st Qu.	Median	Mean	3rd Qu.	Max	sd
460	2.580	2.940	2.980	3.021	3.060	4.600	0.2027

At first sight, this set of indices characterizing this statistical variable does notreflect any abnormal particularity; on the contrary, we notice a fairly small apparentvariability around the central values and which is measured either by the standarddeviation which is equal to 0.2027, either by the inter-quartile difference which is worth0.12, or by the range which represents the global difference which is worth 2.02 andconsequently, everything seems to correspond to an ordinary and "normal" situation. However, this distribution cannot be adjusted by a Normal law because other indices will reveal the non-normality of the distribution of the data and which are initially: the coefficient of asymmetry and the kurtosis whose values are respectively 3.269001for the asymmetry and 21.53841 for the kurtosis and which are far from being those ofa Gaussian distribution. For such considerations and from what will follow along thisarticle, our task will be to show that this time series can be modeled by a first-orderautoregressive stationary process driven by an i.i.d. of innovations whose distribution is produced by a right-skewed stable law.

3. Lévy stable distribution

This type of distribution is part of a class of so-called "indefinitely divisible" laws associated with random variables that can be expressed as the limit of finite sums of independent and identically distributed real random variables and whose common distribution belongs to the same family of laws as that of the starting variables (see [17]) and when these sums can be renormalized and centered then their limits are this time of a particular type of indefinitely divisible laws called "stable Lévy laws" or "α- stables" or just "stables", see for example [7] and [18]. These laws find their application in several fields such as astronomy, telecommunications (in the modeling of the noise of the telephone lines), the Internet (time of appearance of a page web), Geophysics (fragmentation, earthquakes), Economics and Finance (in stock prices, interest rates).One of their specific characteristics is that they do not always admit an average and that their variance is always infinite with the exception of the only normal law considered as a very particular law of stable laws.

The Lévy-stable distributions do not have simple explicit formulas for their densitiesexcept in three particular cases of Cauchy law, normal law and the inverse of the Gaussian also known as "Lévy's law". Otherwise, in

the general case, they are known only through their characteristic functions.
The characteristic function of a stable random variable $X \sim (\alpha, \mu, \beta, \delta)$ is given by

$$\varphi(x) = \exp\left\{ i\mu t - \delta^{\alpha} |t|^{\alpha} \left(i + i\beta \frac{t}{|t|} w(t,\alpha) \right) \right\}$$

$$w(t,\alpha) = \begin{cases} tg\left(\frac{\alpha\pi}{2}\right), \alpha \neq 1 \\ \frac{2}{\pi} \ln|t|, \alpha = 1 \end{cases}$$

Where $\alpha \in \left]0, 2\right]$ is the stability index or the characteristic exponent, $\mu \in \mathbb{R}$ is the positional parameter, $\beta \in [-1,1]$ is the asymmetry parameter and $\delta \in \left]0, +\infty\right[$ is the scale parameter. The most important properties of these laws are the following (see [18]):

1. If $X_1 \sim S(\alpha, \mu_1, \beta_1, \delta_1)$ and $X_2 \sim S(\alpha, \mu_2, \beta_2, \delta_2)$ with $X_1 \perp X_2$, then :

$$X_1 + X_2 \sim S\left(\alpha, \mu_1 + \mu_2, \frac{\beta_1\delta_1^{\alpha} + \beta_2\delta_2^{\alpha}}{\delta_1^{\alpha} + \delta_2^{\alpha}}, \delta_1^{\alpha} + \delta_2^{\alpha} \right)$$

2. Lévy-stable distributions are heavy-tailed. Indeed, if $F(x)$ is the distribution function of a stable law with characteristic exponent α then, there exists a constant $C > 0$ such that:

$$\lim_{x \to \infty} \left[1 - F(x) + F(-x)\right] = Cx^{-\alpha}$$

3. - One of the consequences of the heavy tail is that the variance does not exist(except the case $\alpha = 2$) whereas the expectation can exist, that is :

- If $\alpha < 2 \Rightarrow V(X) = \infty$
- If $\alpha > 1 \Rightarrow E|X| < \infty$
- If $\alpha \leq 1 \Rightarrow E(X) = \infty$

4. Lévy-stable AR(1) process

Autoregressive linear processes play an important role in the modeling of several phenomena which present a certain chronological evolution and in particular, those of the first order noted AR(1) described by an equation of the type :

$$X_n = \lambda X_{n-1} + Z_n, \quad n \in \mathbf{Z}$$

Where λ denotes the autoregression parameter and (Z_n) represents the sequence of innovations assumed to be independent and identically distributed. The process AR(1) have an interesting characteristics such as the property of being Markov processes, of being strongly ergodic when they are stationary, strongly mixing under a certain sufficient condition [4] and that they are discrete solutions of certain stochastic differential equations.

Recall that a Lévy-stable process (X_n) is said to be stationary (in the strict sense) when the joint law of $(X_{1+h}, \ldots, X_{n+h})$ is the same as that of $(X_1, \ldots, X_n)$ **.**

The stationary AR(1) Lévy-stable process has an autoregression parameter satisfying the condition $|\lambda| < 1$.

It is easy to show that under these conditions and for all $n \in \square$, the distribution of the random variable X_n then follows a Lévy-stable law that we will denote $S_{\alpha_X}(\mu_X, \beta_X, \delta_X)$ and satisfying the following algebraic relations :

$$\begin{cases} \alpha_X = \alpha_Z := \alpha \\ \mu_X = \dfrac{\mu_Z}{1-\lambda} \\ \beta_X = \begin{cases} \beta_Z, 0 \le \lambda > 1 \\ \dfrac{1-|\lambda|^\alpha}{1+|\lambda|^\alpha}\beta_Z, -1 < \lambda < 0 \end{cases} \\ \delta_X = \dfrac{\delta_Z}{\left[1-|\lambda|^\alpha\right]^{1/\alpha}} \end{cases}$$

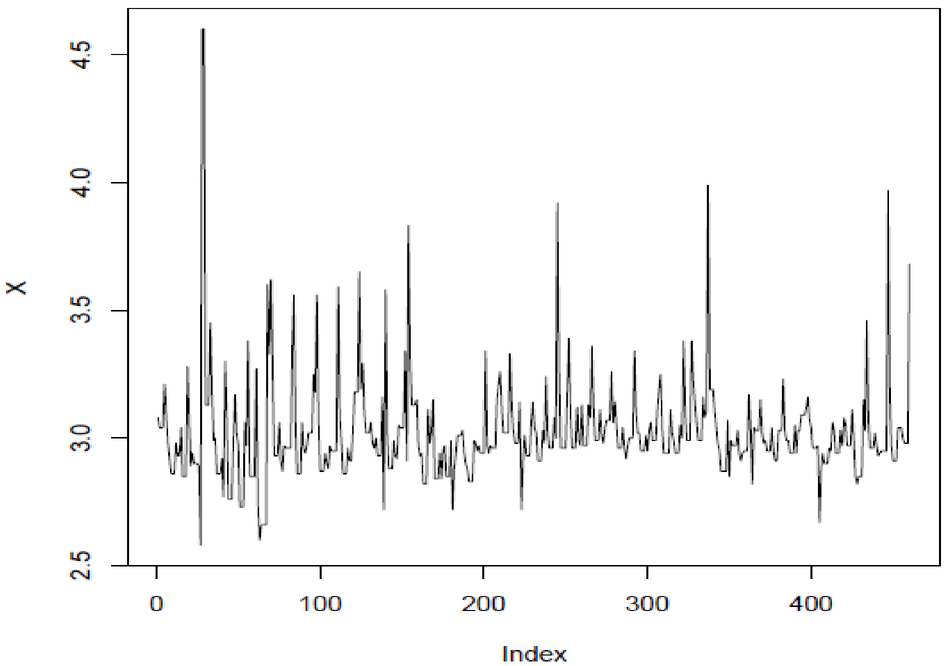
Where $\alpha_Z, \mu_Z, \beta_Z, \delta_Z$ are the parameters of the common distribution of innovations i.i.d. (Z_n).

5. The proposed model

First the diagram of the data series representing the 460 values of the federal funds interest rate in Figure 1) below shows that:

- The stationary behavior of the data series around a certain fixed value.
- The volatility of the distribution of values where we see peaks appearing. This indicate the existence of an infinite variance and an underlying heavy-tailed distribution.

Figure.1. The Federal Fund Rate data



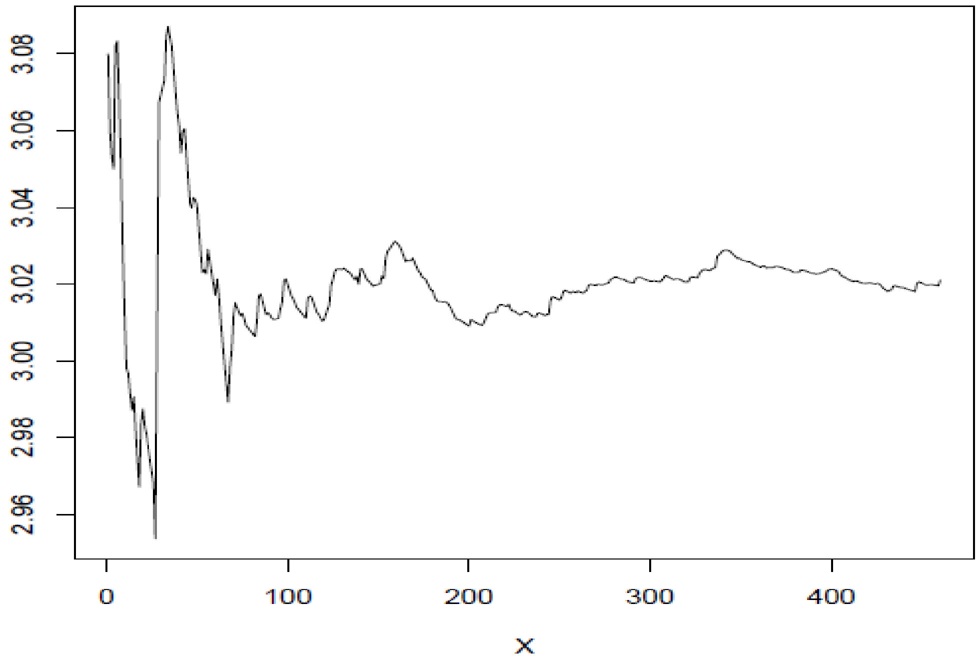
We are going to propose for our time series, a linear model of an autoregressive type and more precisely a stationary AR(1) with innovations of the same distribution α-stable $S(\alpha_Z, \mu_Z, \beta_Z, \delta_Z)$ and whose parameters will be evaluated later, and we will justify the choice of this model step by step, starting first with stationary.

5.1. Stationary of data

Applying Dickey Fuller's ADF test [3], the result gives a p-value smaller than 0.01. The Schmidt-Phillips test [9], indicating a p-value not exceeding 0.01. In conclusion, the time series is not integrated and stationary is guaranteed.

The variations of the sample mean in Figure 2 according to the progression of the sample size $n \in [1, 460]$, they fluctuate around a certain fixed value, which indicatesthe existence of the mean for this distribution.

Figure.2. Variations of the empirical mean

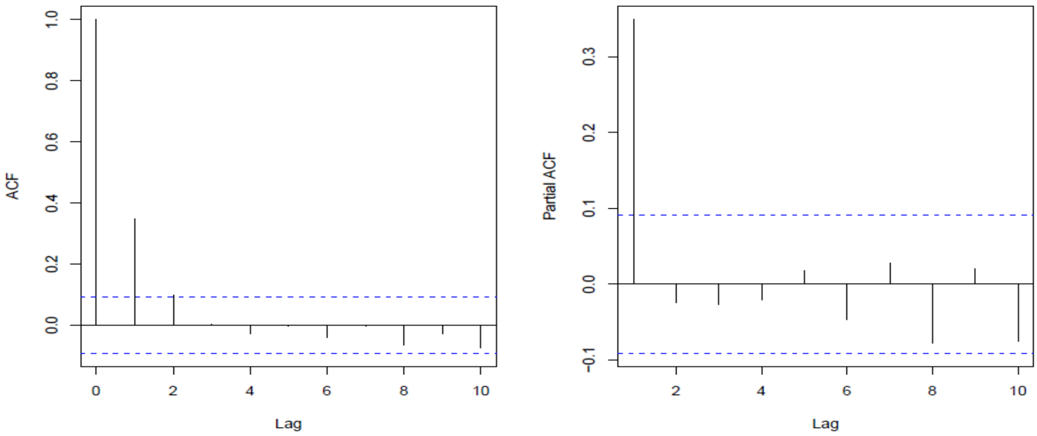


5.2 . Identifying the order of the AR linear model

In general, the autocorrelation function (ACF), as its name suggests, is used to measure the interdependence that exists between successive or time-lagged random variables defining the course of the underlying stochastic process that produces the observable data. In the case of stationary series, it must attenuate after a certain more or less long delay according to the type of process, but in a linear framework this attenuation is exponential. On the other hand, when the observed data are produced by a heavy- tailed distribution as in the case of stable distributions, the theoretical ACF does not exist; At the very least, we show in [2] that the empirical variance and the empirical autocorrelation converge towards a stable vector, which means that these empirical quantities still have a meaning, only that the confidence intervals are sometimes quite wide.

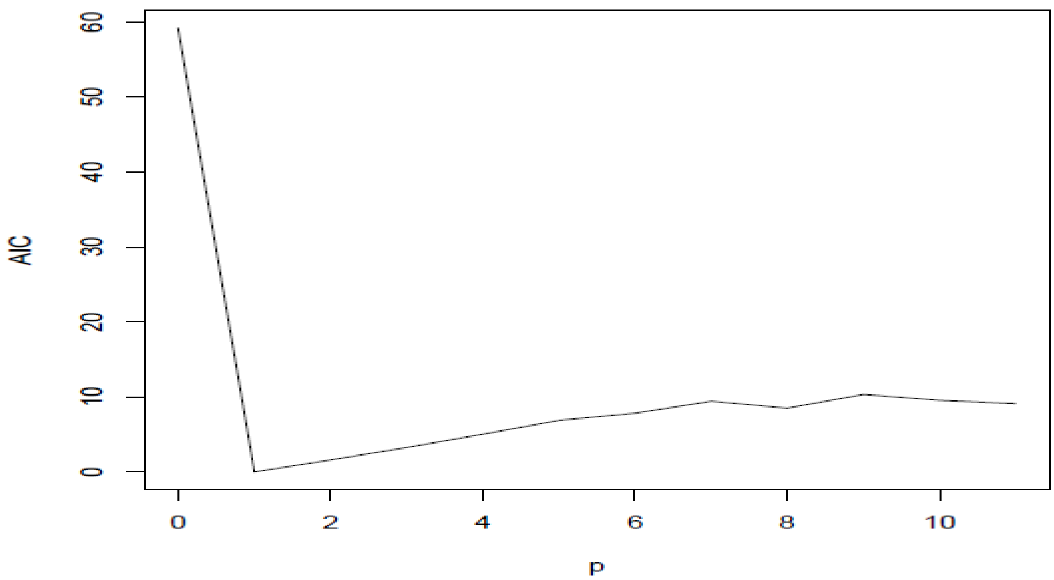
By observing the empirical autocorrelation function of the time series of our raw data in the left of Figure 3, we notice a certain dependence between the values at approximate dates at the beginning of the x-axis, this axis which represents the different delays (lags) made on the data chronology. By plotting the empirical partial autocorrelation function in the right of Figure 3, we see that in fact, the correlation is only consistent at lag 1, which leads to propose an AR autoregressive model of order 1 for this stationary series in a linear modeling context.

Figure.3. L'ACF and PACF of data



To support this hypothesis, let's look at what the information criterion *AIC* sug- gests about the order of the AR(p) model corresponding to our series. In the Figure 4, we see that the minimum is reached for a lag equal to 1 and therefore, as expected by the PACF, the order proposed by this criterion would then be $p = 1$. Which is even consistent in the case of autoregressive processes with infinite variance (see [8]).

Figure.4. AIC Criterion values of data



5.3. Estimation of the autoregression coefficient

In general, when we have a sequence of $X_1, \ldots, X_n$ which we assume satisfying the AR(1) process , the estimator provided by the method of least squares for the autoregression coefficient λ is given by the following expression :

$$\hat{\lambda}_n = \frac{\sum_{i=1}^{n-1} X_{i+1} X_i}{\sum_{i=1}^{n-1} X_i^2}$$

It is proved in [2] that this estimator is consistent for the coefficient λ, under the condition of stationary $|\lambda| < 1$ and under certain additional conditions (see [2]) concerning the heavy-tailed distributions such as stable distributions. Moreover, when the Lévy-stable distribution admits a mean, we prefer the estimator corrected by the empirical mean and which is also consistent and even more precise that the first:

$$\hat{\lambda}_n = \frac{\sum_{i=1}^{n-1}(X_{i+1} - \overline{X}_n)(X_i - \overline{X}_n)}{\sum_{i=1}^{n-1}(X_i - \overline{X}_n)^2}$$

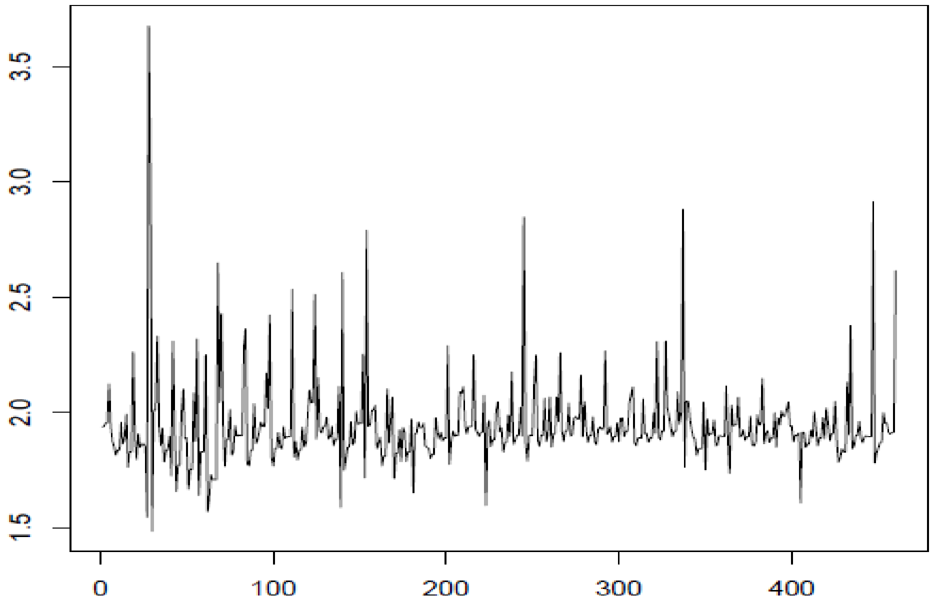
We will opt for the latter because the sample mean is convergent, we find for $\overline{X}_n = 3.021$, $\hat{\lambda}_n = 0.3573977$.

5.4. Inference on residuals

Once the autoregression coefficient has been estimated, we then form the time series of residuals represented in Figure 5 obtained using the recursive expression:

$$\hat{Z}_k = X_k - \hat{\lambda} X_{k-1}, \quad k = 2, 3, \ldots, 460$$

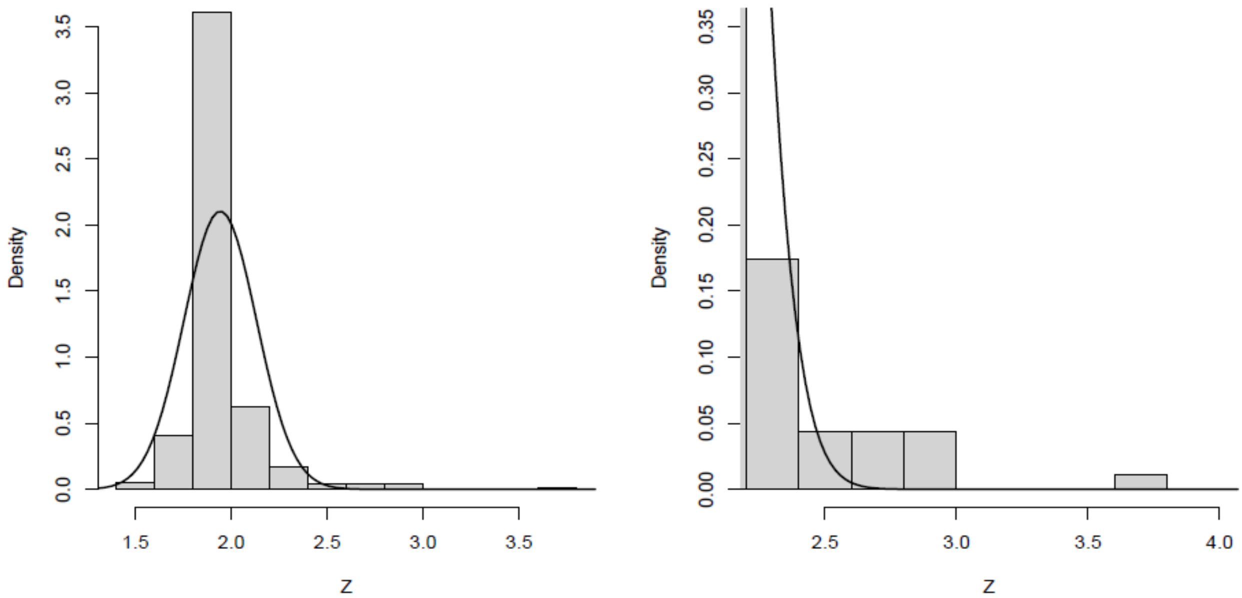
Figure.5. Residuals



An exploratory statistical analysis of this sample leads to the following results:

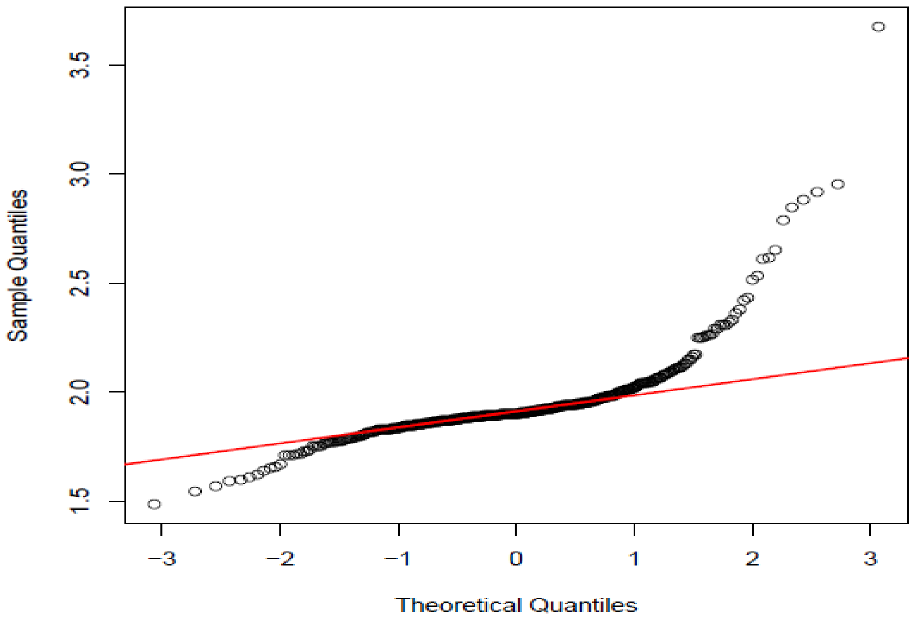
Non-normality of residual distribution: An overview on the histogram in the left of Figure 6 of the residuals shows thatthe empirical distribution cannot be fitted by a Gaussian law (kurtosis = 24.42468and asymmetry = 3.537315) and that the latter will neglect the values extremeson the right (Figure 6 on the right).

Figure.6. Histogram of residuals



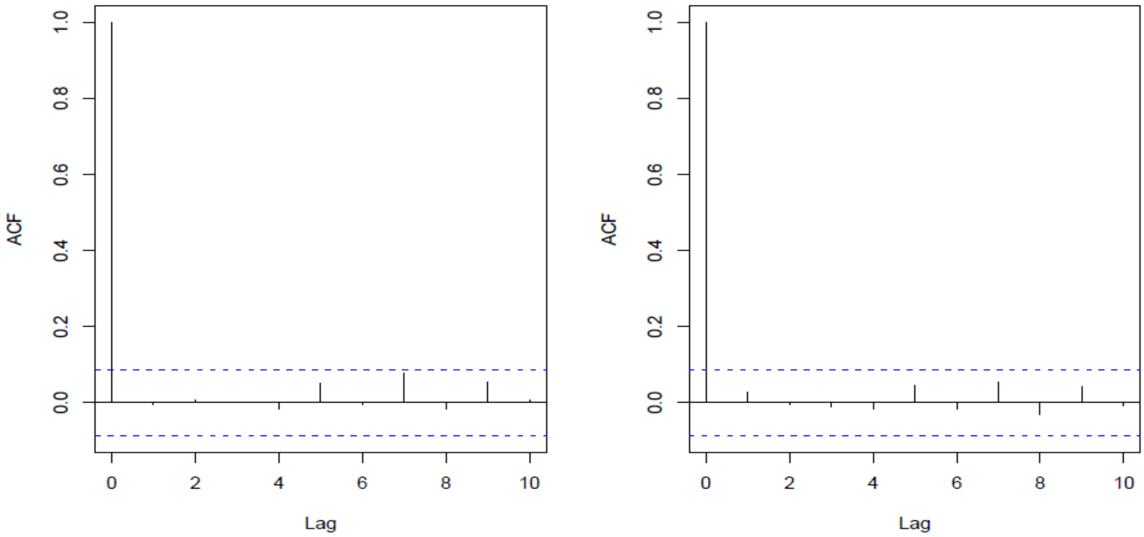
Similarly, if we look at the QQ-norm in Figure 7, we understand that the quantiles of the distribution of residuals cannot fit those of a normal distribution having the sample mean and its standard deviation as parameters and especially the right tail.

Figure.7. QQ-norm of residuals



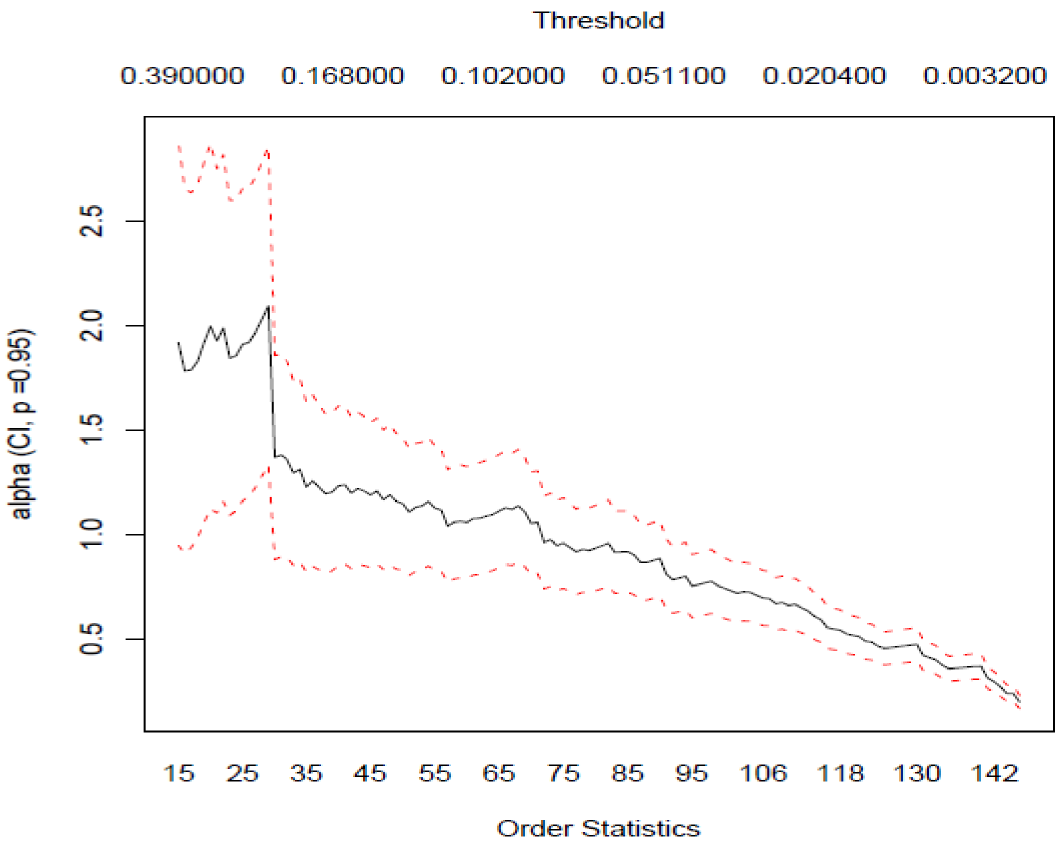
Non-significance of dependency: The respective empirical ACFs of the sam ple of residuals $\hat{Z}_n$ in the left of Figure 8 and that of their squares $\hat{Z}_n^2$ (Figure 8 on the right) show that the autocorrelations are really low.

Figure.8. ACF of residuals



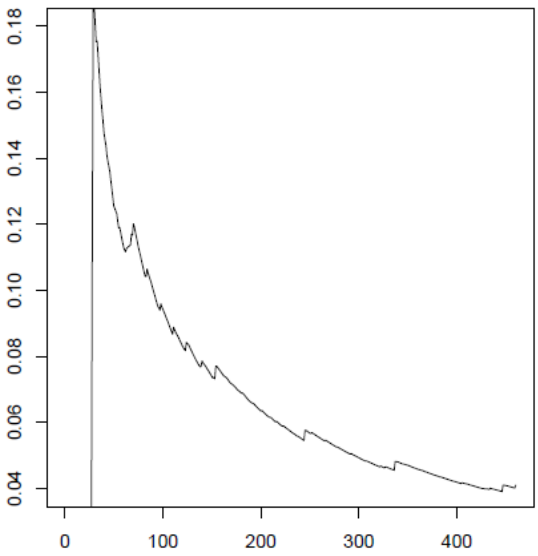
Presence of a heavy tail: The Hill plot (Figure 9) of the centered residuals shows a tail index fluctuating between the values 0 and 2, then confirming the existence of a heavy tail.

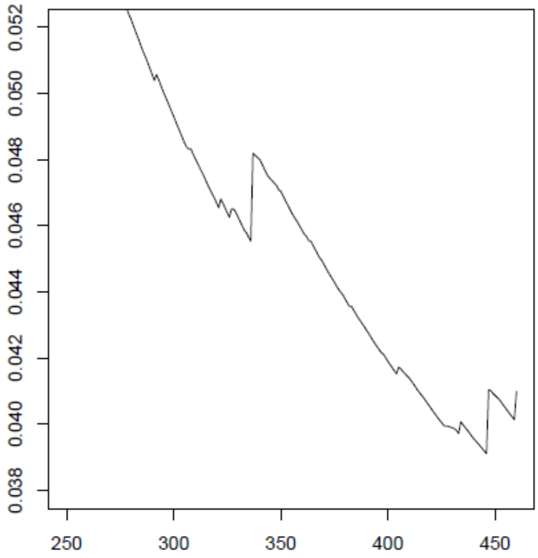
Figure. 9. Hill-plot of centered residuals



Non-convergence of variance: The fluctuations of the empirical variance (Figure 10 on the left) according to the increasing size of the sample shows the presence of jumps (see the enlargement of an intermediate zone in Figure 10 on right), which is a sign of non-convergence for the variance (see [6]).

Figure.10. Empirical variance Fluctuations





Consequently and at the end of these tests, it can be assumed that the process of innovations that generated the residuals is an i.i.d. of random variables (Z_n) with thesame Lévy-stable distribution law of parameters $\alpha_Z, \mu_Z, \beta_Z, \delta_Z$ which we will denote $S(\alpha_Z, \mu_Z, \beta_Z, \delta_Z)$ and which we will then propose to estimate from the actual sampleof residuals $(\hat{Z}_n)$.

5.5. Estimation of all model parameters

There are several procedures in the literature for estimating the parameters of an α- stable distribution from an i.i.d. including McCulloch's empirical quantile method in [12]and the MLE maximum likelihood method based on numerical methods (see [13]).

Concerning the sample of residuals $(\hat{Z}_n)$ and as comparative, we find for the two approaches what summarizes the following table:

Table.2. Parameters estimation of *Z*

	Méth. Quantiles	MLE
$\hat{\alpha}_Z$	1.07000	1.16997
$\hat{\beta}_Z$	0.43700	0.37738
$\hat{\delta}_Z$	0.04530	0.04913
$\hat{\mu}_Z$	1.89390	1.89892

Concerning AR(1), using the relations of the section (4), we will find the followingresults :

Table.3. Parameters estimation of *X*

	Méth. Quantiles	MLE
$\hat{\alpha}_X$	1.07000	1.16997
$\hat{\beta}_X$	0.43700	0.37738
$\hat{\delta}_X$	0.06787	0.07019
$\hat{\mu}_X$	2.94724	2.95506

5.6. Model adjustment

The densities estimated by these two methods provide the following graphical adjustments:

Figure. 11. Histogram adjusted by density estimated via the quantile method

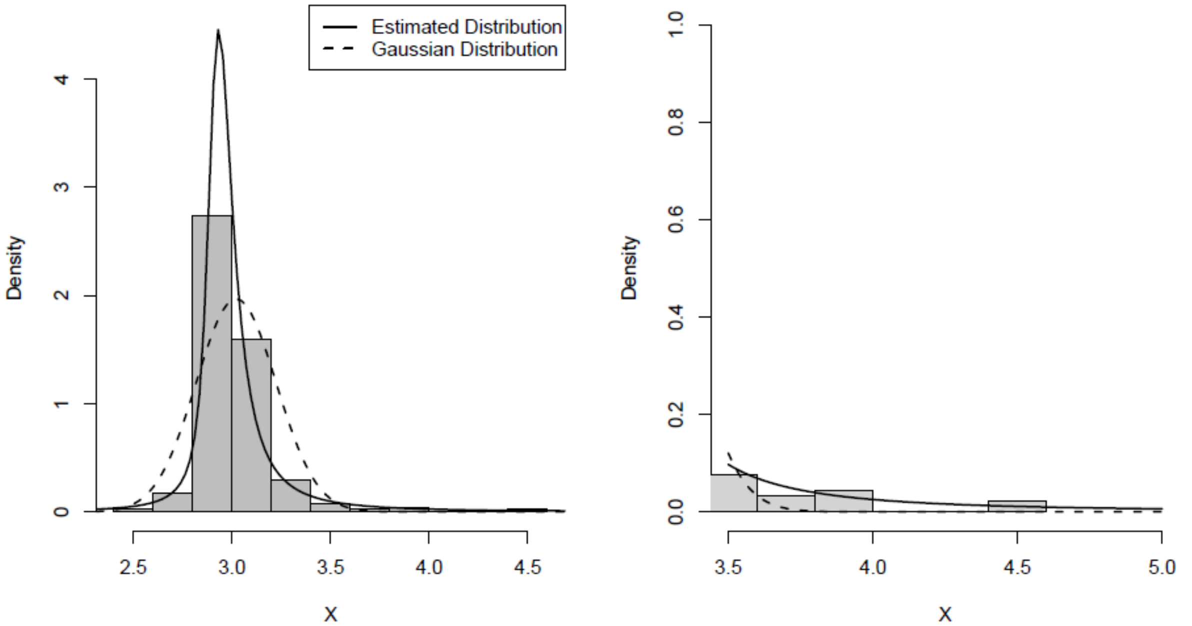
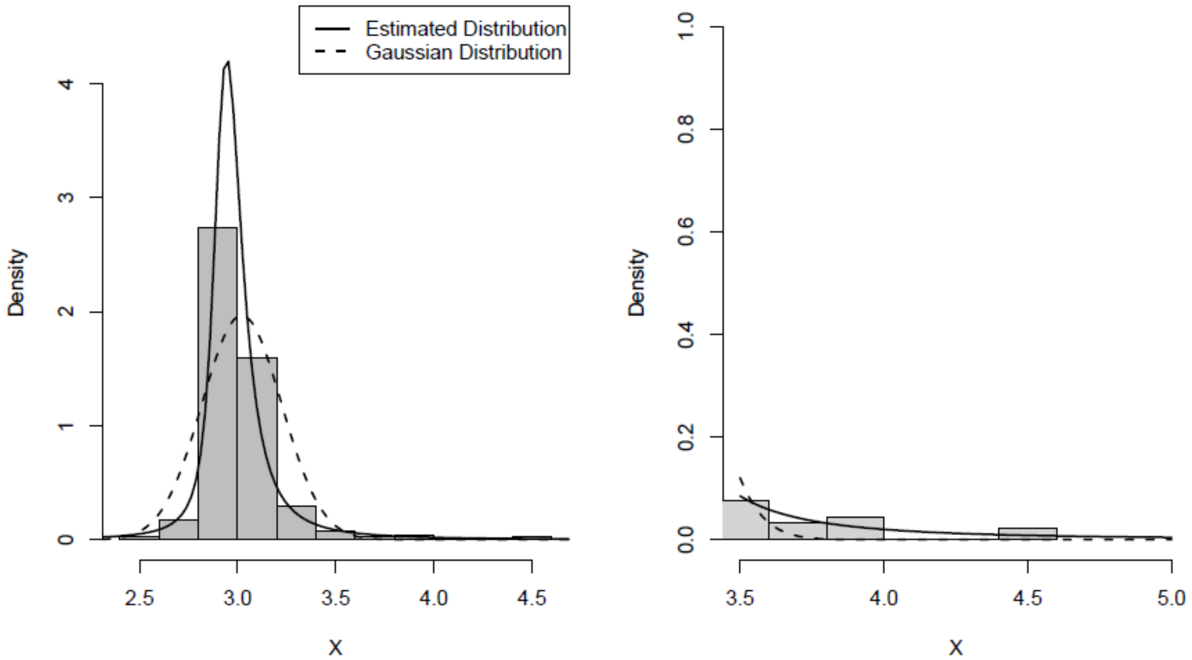
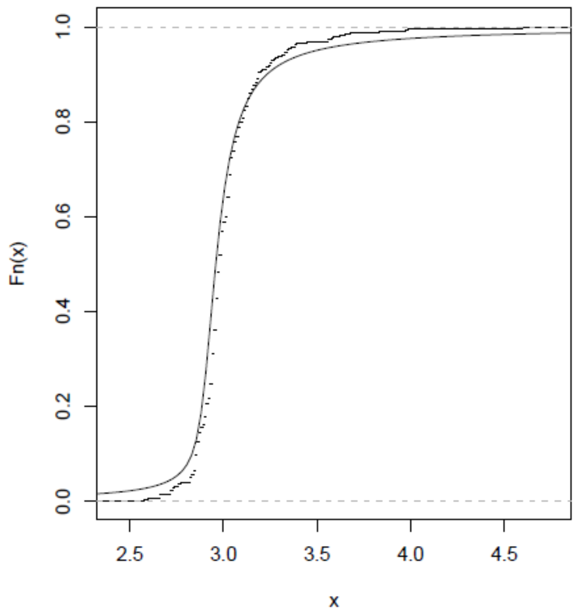


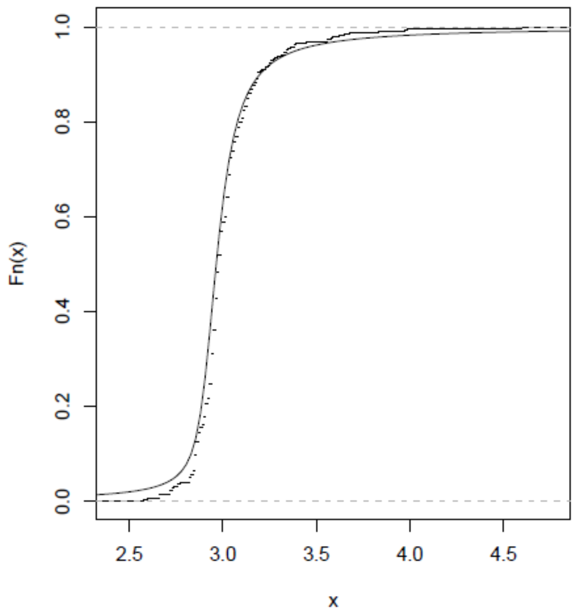
Figure. 12. Histogram adjusted by the density estimated via the MLE method



It can be observed that they give practically the same result, especially on the far right side of the sample distribution where the curves fit both in the top graphs (Figure 11, 12) and that of below (Figure 13) for their respective distribution functions and we can clearly see the similarity between them. below (Figure 13) for their respective distribution functions and we can clearly see the similarity between them.

Figure.13. The empirical distribution function and estimated via the method of McCullochon the left and MLE on the right





5.7. Extreme quantiles estimation of the model

If we also want to have an overview of the empirical and estimated extreme quantilesvia the MLE method, the following table gives a brief summary for the three quantiles

Table.4. Extreme quantiles estimation

Prob	Quant. empirical	Quant. estimated	Absolute error
1%	2.66000	2.15192	0.50808
5%	2.82950	2.74491	0.08459
10%	2.86000	2.83165	0.02835

5.8. Prediction

In the following table are indicated on the one hand, the real values $X_{461,} X_{462}, X_{463}$ of the continuation of the series of the Fund Federal as well as the prediction values $\hat{X}_{460+h}$ of the model for the horizons $h = 1, 2, 3$ and on the other hand, the relative differences between the two values.

Table. 5. The prediction values.

h	X_{460+h}	$\hat{X}_{460+h}$	Relative difference
1	3.25	3.256611	0.002034
2	3.31	3.105596	0.061753
3	3.12	3.051732	0.021880

Acknowledgements

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References

[1] A. Banerjee, J.J. Dolado, J.W. Galbraith, D.F. Hendry, Cointegration, Error Correction, and the Econometric Analysis of Non-Stationary Data, Oxford University Press, Oxford, 1993.

[2] R.A. Davis, S.I. Resnick, Limit Theory for the Sample Covariance and Correlation functions of moving averages. Ann. Statist., **14** (1986), 533--538.

[3] D.A. Dickey, W.A. Fuller, Distribution of the estimators for autoregressive time series with a unit root, Journal of the American Statistical Association, **74** (1979), 427--431.

[4] W.K.A. Donald, First Order Autoregressive Processes and Strong Mixing. Cowles Foundation Discussion Paper N° 664. Yale University, 1983.

[5] W.H. DuMouchel, Stable Distributions in Statistical Inference. PhD. thesis, Dept. of Statistics, Yale University, 1971.

[6] H. Fofack, J.P. Nolan, Distribution of parallel exchange rates in African countries, 2001.

[7] Gnedenko \& Kolmogorov, Limit Distributions for Sums of Independent Random Variables, 1954.

[8] K. Knight, Consistency of Akaike's Information Criterion for Infinite Variance Autoregressive Processes. Department of Statistics, GN22 University of Washington Seattle, Washington 98195 USA University, 1987.

[9] D. Kwiatkowski, P.C.B Phillips, P. Schmidt, Y. Shin, Testing the Null Hypothesis of Stationarity against the Alternative of a Unit Root, Journal of Econometrics, **54** (1992), 159--178.

[10] M. Loève, Probability Theory (third ed. vol 1). Princeton : D. Van Nostrand Company, 1963.

[11] J.G. MacKinnon, Numerical distribution functions for unit root and cointegration tests, Journal of Applied Econometrics, **11** (1996), 601--618.

[12] J.H. McCulloch, Simple Consistent Estimators of Stable Distribution Parameters. Commun. Statist.-Simula., **15** (4) (1986), 1109--1136.

[13] J.P. Nolan, Numerical calculation of stable densities and distribution functions. Commun. Statist. - Stochastic Models, **13** (1997), 759--774.

[14] P. Perron, Trends and Random Walks in Macroeconomic Time Series, Journal of Economic Dynamics and Control, **12** (1988), 297--332.

[15] P.C.B. Phillips, P. Perron, Testing for a unit root in time series regression, Biometrika, **75** (1988), 335--346.

[16] S.I. Resnick, C. Starica, Asymptotic Behavior of Hill's Estimator for Autoregressive Data. Comm. Statist. Stochastic Models, **13** (1997), 703--721.

[17] A.N. Shiryayev, Probability, vol. 95 of Graduate Texts in Mathematics. Springer-Verlag, 1984.

[18] V.M. Zolotarev, One-Dimensional Stable Distributions, American Mathematical Society, 1986.