V-FUZZY *b*-METRIC SPACE AND RELATED FIXED-POINT THEOREM

Happy Hooda[1],
[1]Research Scholar, Department of Mathematics,
Maharshi Dayanand University, Rohtak-124001, (Hr) INDIA

Manjeet Rathee [2],
[2]Assistant Professor, All India Jat Heroes' Memorial College,
Rohtak-124001, (Hr) INDIA

Archana Malik[3],
[3]Professor, Department of Mathematics,
Maharshi Dayanand University, Rohtak-124001, (Hr) INDIA

[*]Corresponding author: Happy Hooda, E-mail: happyhooda2017@gmail.com

ABSTRACT: The paper concerns our sustained efforts for introduction of V-fuzzy *b*-metric spaces and to study their basic topological properties. As an application of this concept, we prove coupled common fixed-point theorems for mixed monotone maps in partially ordered V-fuzzy*b*-metric spaces.

MSC: primary 47H10; secondary 54H25.

KEYWORDS: Fuzzy metric space;partial ordered set; mixed monotone mappings; common coupled fixed point; fixed point; *G*-fuzzy metric space; *V*-fuzzy metric space.

1. INTRODUCTION AND PRELIMINARIES:

A metric space is just a non-empty set X associates with a function d of two variables enabling us to measure the distance between points. In advanced mathematics, we need to find the distance not only between numbers and vectors, but also between more complicated objects like sequences, sets and functions. In order to find appropriate concepts of a metric space, numerous approaches exist in this sphere. Thus, new notions of distance lead to new notions of convergence and continuity. A number of generalizations of a metric space have been discussed by many eminent mathematicians. Mustafa and Sims [15] introduced the notion of a G-metric space and suggested an important generalization of a metric space as follows.

Definition 1.1[15]The pair (X,G) is called a G-metric space if X is a nonempty set and d is a G-metric on X. That is, $G: X^3 \to [0,\infty)$ such that for all $x_1, x_2, x_3, a \in X$, we have

(i) $G(x_1,x_2,x_3)=0$ if and only if $x_1 = x_2 = x_3$

(ii) $G(x_1,x_1,x_3) \geq 0$ with $x_1 \neq x_2$

(iii) $G(x_1,x_1,x_2) \leq G(x_1,x_2,x_3)$ for all with $x_2 = x_3$

(iv) $G(x_1,x_2,x_3) = G(x_1,x_3,x_2) = G(x_2,x_1,x_3) = G(x_2,x_3,x_1) = G(x_3,x_1,x_2) = G(x_3,x_2,x_1)$

(v) $G(x_1,x_2,x_3) \leq G(x_1,a,a) + G(a,x_2,x_3)$

Example 1.1[15] Let (X,d) be a metric space. Define $G:X^3\to[0,\infty)$ by
$G(x_1,x_2,x_3)=\dfrac{d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_1)}{3}$, Then (X,G) is a G-metric space.

Recently, some authors studied some important fixed point theorems with application in G-metric space. In 2012, Sedghi et al. [18] introduced a new generalized metric space called an S-metric space.

Definition 1.2 [18] The pair (X,d) is called an S-metric space if X is a nonempty set and $d:X^3\to[0,\infty)$ such that for all $x_1,x_2,x_3\in X$, we have

(i) $d(x_1,x_2,x_3)\geq 0$

(ii) $d(x_1,x_2,x_3)=0\Leftrightarrow x_1=x_2=x_3$

(iii) $d(x_1,x_2,x_3)\leq d(x_1,x_1,a)+d(x_2,x_2,a)+d(x_3,x_3,a)$

Example 1.2[18] Let $\mathbb{R}$ be the real line. Then $S(x,y,z)=|x-y|+|y-z|$ for all $x,y,z\in\mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Abbas et. al. [1] established the notion of A-metric spaces, a generalization of S-metric spaces.

Definition 1.3 [1]The pair (X,d) is called an A-metric space if X is a nonempty set and $d:X^n\to[0,\infty)$ such that for all $a,x_i\in X, i=1,2,...,n,$ we have

(i) $d(x_1,x_2,...,x_{n-1},x_n)\geq 0$

(ii) $d(x_1,x_2,...,x_{n-1},x_n)=0\Leftrightarrow x_1=x_2=...=x_{n-1}=x_n$

(iii) $d(x_1,x_2,...,x_{n-1},x_n)\leq d(x_1,x_1,...,x_1,x_n)+d(x_2,x_2,...,x_2,a)+...+d(x_{n-1},x_{n-1},...,x_{n-1},a)+d(x_n,x_n,...,x_n,a$

Example 1.3[1] Let $X=\mathbb{R}$. Define the function $A:X^n\to[0,\infty)$ by
$A(x_1,x_2,x_3,...,x_n)=\sum_{i=1}^{n}\sum_{i<j}|x_i-y_i|$, then (X,A) is called then usual A-metric space.

Fixed point theorems have been studied in many contexts, one of which is the fuzzy setting. The concepts of fuzzy sets were initially introduced by Zadeh[22] in 1965. To use this concept in topology and analysis, the theory of fuzzy sets and its applications have been developed by eminent authors. It is well known that a fuzzy metric space is an important generalization of a metric space.Many authors have introduced fuzzy metric spaces in different ways. For instance, George and Veeramani [7] modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [12] and define the Hausdorff topology of a fuzzy metric space.

Definition 1.4 [17]A t-norm $*$ is a function $*:[0,1]\times[0,1]\to[0,1]$ such that for all $a,b,c,d\in[0,1]$, the following are satisfied:

(i) $a*1=a$ (1 acts as the identity element)

(ii) $a*b=b*a$ (symmetry)

(iii) $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ (non-deceasing)

(iv) $a*(b*c)=(a*b)*c$ (associative).

Definition 1.5 [12]The 3-tuple $\left(X,\, M,T \right)$ is known as fuzzy metric space (shortly, FM-space) if X is an any set, T is a continuous t-norm, and M is a fuzzy set in $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0$; (i)
$M\left(x,y,0 \right)=0,$ (ii)
$M\left(x,y,t \right)=1$ if and only if $x=y,$
(iii) $M\left(x,y,t\right)=M\left(y,x,t\right),$ (iv)
$T\left(M\left(x,y,t\right),M\left(y,z,s\right)\right)\le M\left(x,z,t+s\right),$ (v)
$M(x,y,\cdot):[0,\infty)\to[0,1]$ is continuous. Informally, we can think of $M(x,y,t)$ as the degree of nearness between x and y with respect to t.

Example 1.4[7]Let (X,d) be a metric space. Define the t-norm $a*b=ab$ or $a*b=\min\{a,b\}$. For all $x,y\in X, t>0,$ let $M(x,y,z)=\dfrac{t}{t+d(x,y)}$. Then $(X,M,*)$ is a fuzzy metric space.

Lemma 1.1 [14]Let $(X,M,*)$ be a fuzzy metric space. If there exist $k\in(0,1)$ such that, for all $x,y\in X$ and $t>0,\ M(x,y,kt)\ge M(x,y,t)$ then $x=y.$
In the process of generalization of fuzzy metric spaces, Sun and Yang [21] coined the notion of G-fuzzy metric space and established common fixed point theorems for four mappings.

Definition 1.6 [21] The 3-tuple $\left(X,V,T \right)$ is known as fuzzy metric space (shortly, GF-space) if X is an any set, T is a continuous t-norm, and G is a fuzzy set in $X\times X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0$; (i)
$G\left(x,x,y,t \right)>0,$ with $x\neq y,$ (ii)
$G\left(x,x,y,t \right)\ge G\left(x,y,z,t \right)$ with $y\neq z,$ (iii)
$G(x,y,z,t)=1$ if and only if $x=y=z,$ (iv)
$G(x,y,z,t)=G(x,z,y,t)=G(y,x,z,t)=G(z,x,y,t)=G(y,z,x,t)=G(z,y,x,t),$ (v)
$T\left(G\left(x,a,a,t\right),G\left(a,y,z,s\right)\right)\le G\left(x,y,z,t+s\right),$ (vi)
$M(x,y,z,\cdot):[0,\infty)\to[0,1]$ is left continuous.

Example 1.5 [21]Let G be a G-metric on a nonempty set X. Define the t-norm $a*b=ab$ or $a*b=\min\{a,b\}$. For all $x,y\in X, t>0,$ let $G(x,y,z,t)=\dfrac{t}{t+G(x,y,z)}$. Then $(X,G,*)$ is a G-fuzzy metric space.

Lemma 1.2[21] Let $(X,G,*)$ be a GF-space. Then $G(x,y,z,t)$ is non-decreasing with respect to t for all $x,y,z\in X.$

On the other hand, the concept of coupled fixed points and mixed monotone property of a fuzzy metric space are established by Bhaskar and Lakshmikantam [4]. Lakshmikantam and Ciric [13] discussed the mixed monotone mappings and gave some coupled fixed point theorems, which can be used to discuss the existence and uniqueness of a solution for a periodic boundary value problem.

Definition 1.7 [13]An element $(x,y)\in X\times X$ is called a coupled fixed point of a mapping $P:X\times X\to X$ if $P(x,y)=x$ and $P(y,x)=y.$

Definition 1.8 [13]An element $(x,y)\in X\times X$ is called a coupled coincidence point of a mapping $P:X\times X\to X$ and $Q:X\to X$ if $P(x,y)=Q(x)$ and $P(y,x)=Q(y).$

Definition 1.9 [13]An element $(x,y)\in X\times X$ is called a common coupled fixed point of a mapping $P:X\times X\to X$ and $Q:X\to X$ if $x=P(x,y)=Q(x)$ and $y=P(y,x)=Q(y).$

Example 1.6 Let $X=[0,1]$. Define $P:X\times X\to X$ and $Q:X\to X$ as $P(x,y)=\left|\frac{x-y}{2}+\frac{1}{4}\right|,$ $Q(x)=x.$ For $x=0$ and $y=\frac{1}{2}$ we have,
$P(x,y)=\left|\frac{0-\frac{1}{2}}{2}+\frac{1}{4}\right|=0,$ $Q(0)=0$ and $P(x,y)=\left|\frac{\frac{1}{2}-0}{2}+\frac{1}{4}\right|=\frac{1}{2},$ $Q(\frac{1}{2})=\frac{1}{2}.$
So, $(x,y)=(0,\frac{1}{2})$ is a common fixed point of the mappings $P:X\times X\to X$ and $Q:X\to X$.

Definition 1.10 [13]An element $x\in X$ is called a common fixed point of a mapping $P:X\times X\to X$ and $Q:X\to X$ if $x=P(x,x)=Q(x).$

Example 1.7 Let $X=[0,1]$. Define $P:X\times X\to X$ and $Q:X\to X$ as $P(x,y)=x-\frac{y}{2}+\frac{xy}{2},$ $Q(x)=x.$ Then we have, $P(x,x)=\frac{x+x^2}{2},$ $Q(x)=x.$ For $x=1,$ we get $P(1,1)=1=Q(1).$ So, $(x,y)=(1,1)$ is a common fixed point of the pair $(P,Q).$

Definition 1.11 [4]Let $(X,\le)$ be a partially ordered set. The mapping (x,y) is said to have the mined monotone property if P is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument; that is, for any $x,y\in X,$
$x_1,x_2\in X,\ x_1\le x_2\ \Rightarrow P(x_1,y)\le P(x_2,y)$ and
$y_1,y_2\in X,\ y_1\le y_2\ \Rightarrow P(x,y_1)\ge P(x,y_2).$

Example 1.8 Let $P:X\times X\to X$ where $X=[0,1]$ be defined by

$$P(x,y)=\begin{cases} x-y & if \quad x\ge y, \\ 0 & if \quad x<y. \end{cases}$$

For $x_1,x_2\in X,\ x_1\le x_2,$ we have
$x_1-y\le x_2-y\Rightarrow P(x_1,y)\le P(x_2,y)$ and $P(x,y_1)\ge P(x,y_2).$
Therefore, the mappings P has the mixed monotone property.

Definition 1.12 [4] Let $(X,\le)$ be a partially ordered set, and $P:X\times X\to X$ and $Q:X\to X.$ We say that P has the mixed Q-monotone property if P is monotone Q-non-decreasing in its first argument and is monotone Q-non-increasing in second argument, that is, for any $x,y\in X,$ $x_1,x_2\in X,$ $Q(x_1)\le Q(x_2)\ \Rightarrow P(x_1,y)\le P(x_2,y)$ and $y_1,y_2\in X,$ $Q(y_1)\le Q(y_2)\ \Rightarrow P(x,y_1)\ge P(x,y_2).$

Example 1.9 Let $P: X\times X \to X$ where $X=[-1,1]$ be two functions given by $P(x,y)=x-y^2, \quad Q(x)=x^4.$ Therefore, the map P has the mixed Q-monotone property.

Definition 1.13 [5]The mapping $P: X\times X \to X$ and $Q: X\to X$ are said to be W-compatible if $Q(P(x,y))=P(Qx,Qy)$ and $Q(P(y,x))=P(Qy,Qx)$ whenever $P(x,y)=Q(x)$ and $P(y,x)=Q(y)$ for some $(x,y)\in X\times X$.

Example 1.10 Define $P: X\times X \to X$ and $Q: X\to X$ where $X=[-1,1]$ as $P(x,y)=\frac{x^2+y^2}{2}, \quad Q(x)=x^4.$ which satisfies. Therefore, the map P has the mixed Q-monotone property. $Q(P(x,y))=P(Qx,Qy)$ and $Q(P(y,x))=P(Qy,Qx)$. For $x=1$ and $y=1,$ we get $P(x,y)=Q(x)$ and $P(y,x)=Q(y).$ This implies that the mappings $P: X\times X\to X$ and $Q: X\to X$ are W-compatible mappings.

Many other eminent authors proved significant results which contributed in the area of fixed point theory.

Definition 2.1[8]**:-** The 3-tuple $(X, V, *)$ is called a V-fuzzy metric space if $*$ is a continuous *t*-norm, and *V* is a fuzzy set in $X^n\times(0,\infty)$ satisfying the following conditions for all $x_i, y, a\in X$ and $s,t>0$;

(i) $V(x,x,...,x,y,t)>0,\ x\neq y$

(ii) $V(x_1,x_1,...,x_1,x_2,t)\geq V(x_1,x_2,x_3,...,x_n,t),\ x_2\neq x_3\neq ...\neq x_n$

(iii) $V(x_1,x_2,x_3,...,x_n,t)=1 \Leftrightarrow x_1=x_2=x_3=...=x_n$

(iv) $V(x_1,x_2,x_3,...,x_n,t)= V(p(x_1,x_2,x_3,...,x_n),t)$ where $p(x_1,x_2,x_3,...,x_n)$ is permutation on $x_1,x_2,x_3,...,x_n$

(v) $V(x_1,x_2,x_3,...,x_{n-1},a,t)*V(a,a,a,...,a,x_n,s)\leq V(x_1,x_2,x_3,...,x_{n-1},x_n,t+s)$

(vi) $V(x_1,x_2,x_3,...,x_n,t)=1$ as $t\to\infty$

(vii) $V(x_1,x_2,x_3,...,x_{n-1},\cdot):(0,\infty)\to(0,1]$ is continuous.

Example 2.1 [8]Let (X,A) be a A-metric space. Define the *t*-norm $a*b=ab$ or $a*b=\min\{a,b\}.$ For all $x_1,x_2,x_3,...,x_n\in X, t>0,$ denote $V(x_1,x_2,x_3,...,x_n,t)=\frac{t}{t+A(x_1,x_2,x_3,...,x_n)}$. Then $(X,V,*)$ is a V-fuzzy metric space.

In addition to fuzzy metric spaces, there are still many extensions of metric and metric space terms. Bakhtin [2] and Czerwik [6] introduced a space where, instead of triangle inequality, a weaker condition was observed, with the aim of generalization of Banach contraction principal [3]. They called these spaces *b*-metric spaces. Relation between *b*-metric and fuzzy metric spaces is considering in [9]. On the other hand, in [18] the notion of a fuzzy *b*-metric space was introduced, where the triangle inequality is replaced by a weaker one.

Definition 2.2 [16]:The 3-tuple (X, M, T) is known as fuzzy *b*-metric space if X is any set, T is a continuous *t*-norm, and M is a fuzzy set in $X \times X \times (0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$, and a given real number $b \geq 1$, (i) $M(x, y, t) > 0$,

(ii) $M(x, y, t) = 1$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $T(M(x, y, \frac{t}{b}), M(y, z, \frac{s}{b})) \leq M(x, z, t+s)$,

(v) $M(x, y, \cdot): [0,\infty) \to [0,1]$ is continuous.

Now, in this paper we introduced a new space that is V-fuzzy *b*-metric space with the help of V-fuzzy metric space and *b*-Metric space and by using concept of a set-valued or multi-valued quasi-contraction mapping a fixed point theorem is established. This theorem generalizes and improves some known fixed point theorems in literature.

2. V-Fuzzy *b*-metric space

Definition 2.1:- The 3-tuple $(X, V, *)$ is called a V-fuzzy *b*-metric space if $*$ is a continuous *t*-norm, and V is a fuzzy set in $X^n \times (0,\infty)$ satisfying the following conditions for all $x_i, y, a \in X, b \geq 1$ and $s, t > 0$;

(i) $V(x, x, ..., x, y, \frac{t}{b}) > 0,\ x \neq y$

(ii) $V(x_1, x_1, ..., x_1, x_2, \frac{t}{b}) \geq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b}),\ x_2 \neq x_3 \neq ... \neq x_n$

(iii) $V(x_1, x_2, x_3, ..., x_n, \frac{t}{b}) = 1 \Leftrightarrow x_1 = x_2 = x_3 = ... = x_n$

(iv) $V(x_1, x_2, x_3, ..., x_n, \frac{t}{b}) = V(p(x_1, x_2, x_3, ..., x_n), \frac{t}{b})$ where $p(x_1, x_2, x_3, ..., x_n)$ is permutation on $x_1, x_2, x_3, ..., x_n$

(v) $V(x_1, x_2, x_3, ..., x_{n-1}, a, \frac{t}{b}) * V(a, a, a, ..., a, x_n, \frac{t}{b}) \leq V(x_1, x_2, x_3, ..., x_{n-1}, x_n, t+s)$

(vi) $V(x_1, x_2, x_3, ..., x_n, \frac{t}{b}) = 1$ as $t \to \infty$

(vii) $V(x_1, x_2, x_3, ..., x_{n-1}, \cdot): (0,\infty) \to (0,1]$ is continuous.

Lemma 2.1: Let $(X, V, *)$ be a V-fuzzy *b*-metric space. Then $V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$ isnon-decreasing with respect to t.

Proof: Since $t > 0, b \geq 1$ and $t + s > t$ for $s > 0$, by letting $l = x_n$ in condition (v) of V-fuzzy *b*-metric space we get, $V(x_1, x_2, x_3, ..., x_n, \frac{t+s}{b}) \geq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b}) * V(x_n, x_n, x_n, ..., x_n, \frac{s}{b})$. This implies that $V(x_1, x_2, x_3, ..., x_n, \frac{t+s}{b}) \geq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$. So, $V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$ is non-decreasing with respect to t.

Lemma 2.2 Let $(X, V, *)$ be a V-fuzzy metric space such that $V(x_1, x_2, x_3, ..., x_n, \frac{kt}{b}) \geq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$ with $b \geq 1$, $k \in (0,1)$. Then $x_1 = x_2 = x_3 = ... = x_n$.

Proof: By assumption $V(x_1, x_2, x_3, ..., x_n, \frac{kt}{b}) \geq V(x_1, x_2, x_3, ..., x_n, \frac{t}{b})$ (1)

For $t > 0, b \geq 1$, since $\frac{kt}{b} < \frac{t}{b}$, by lemma 2.1 we have

$V(x_1,x_2,x_3,...,x_n,\frac{kt}{b}) \le V(x_1,x_2,x_3,...,x_n,\frac{t}{b}).$ (2)

From (1) and (2), and the definition V-fuzzy metric space we get $x_1 = x_2 = x_3 = ... = x_n.$

Definition 2.2 Let $(X,V,*)$ is said to be V-Fuzzy *b*-metric space. A sequence $\{x_r\}$ is said to converge to a point $x \in X$ if $V(x_r,x_r,x_r,...,x_r,x,\frac{t}{b}) \to 1$ *as* $r \to \infty$ *for all* $t > 0, b \ge 1,$ that is, for each $\varepsilon > 0,$ there exist $n \in N$ such that for all $r \ge N,$ we have $V(x_r,x_r,x_r,...,x_r,x,\frac{t}{b}) \to 1-\varepsilon,$ and we write $\lim_{r\to\infty} x_r = x.$

Definition 2.3 Let $(X,V,*)$ is said to be V-Fuzzy *b*-metric space. A sequence $\{x_r\}$ is said to Cauchy sequence if $V(x_r,x_r,x_r,...,x_r,x_q,\frac{t}{b}) \to 1$ *as* $r,q \to \infty$ *for all* $t > 0, b \ge 1,$ that is, for each $\varepsilon > 0,$ there exist $n_0 \in N$ such that for all $r,q \ge N,$ we have $V(x_r,x_r,x_r,...,x_r,x_q,\frac{t}{b}) \to 1-\varepsilon.$

Definition 2.4 The V-Fuzzy *b*-metric space $(X,V,*)$ is said to be complete if every Cauchy sequence in *X* is convergent.

Definition 2.5 The mappings $P: X \times X \to X$ *and* $Q: X \to X$ are said to be compatible on V-Fuzzy *b*-metric space if $\lim_{r\to\infty} V(QP(x_r,y_r),QP(x_r,y_r),...,QP(x_r,y_r),P(Qx_r,Qy_r),\frac{t}{b}) = 1$

And $\lim_{r\to\infty} V(QP(y_r,x_r),QP(y_r,x_r),...,QP(y_r,x_r),P(Qy_r,Qx_r),\frac{t}{b}) = 1$

Whenever $\{x_r\}$ and $\{y_r\}$ are sequences in *X* such that

$\lim_{r\to\infty} Q(x_r) = \lim_{r\to\infty} P(x_r,y_r) = x$ *and* $\lim_{r\to\infty} Q(y_r) = \lim_{r\to\infty} P(y_r,x_r) = y$ for all $x,y \in X$ *and* $t > 0, b \ge 1.$

3. MAIN RESULTS

In this section, we prove fixed point theorems for coupled maps on partially ordered V-fuzzy*b*-metric spaces.

Theorem 3.1 Let $(X,V,*)$ be a complete V-fuzzy *b*-metric spaces, and $(X,\le)$ be a partially orders set. Let $P: X \times X \to X$ be a mapping such that *P* has the mixed monotone property and $Q: X \to X$ be two mapping such that (T1) $P(X \times X) \subseteq Q(X);$ (T2) P has the mixed Q-monotone property; (T3) there exists $k \in (0,1)$ such that

$V(P(x,y),P(x,y),...,P(x,y),P(u,v),\frac{kt}{b})$

$\ge V(Qx,Qx,...,Qx,Qu,\frac{t}{b})$

$*V(Qx,Qx,...,Qx,P(x,y),\frac{t}{b})$

$*V(Qu,Qu,...,Qu,P(u,v),\frac{t}{b})$

for all $x,y,u,v, \in X, t > 0, b \ge 1,$ for which $Q(x) \le Q(u)$ and $Q(y) \le Q(v)$ or $Q(x) \ge Q(u)$ and $Q(y) \ge Q(v);$ (T4) Q is continuous and P and Q are compatible.

Also suppose that (a) P is continuous or (b) X has the following properties:

(i) If $\{x_n\}$ is a non decreasingsequence $x_r \to x,$ then $x_r \leq x$ $for\, all$ $r \in N,$

(ii) If $\{y_n\}$ is a non decreasing sequence $y_r \to y,$ then $y_r \geq y$ $for\, all$ $r \in N.$

If there exist $x_0, y_0 \in X$ such that $Q(x_0) \leq P(x_0, y_0)$ and $Q(y_0) \geq P(y_0, x_0),$ then $P$and Qhave a coupled coincidence point in X.

Proof: Let (x_0, y_0) be a given point in $X \times X$ such that

$Q(x_0) \leq P(x_0, y_0)\, and\, Q(y_0) \geq P(y_0, x_0),$ Using (T1), Choose x_1, y_1 such that

$$P(x_0, y_0) = Q(x_1) \text{ and } P(y_0, x_0) = Q(y_1) \tag{3}$$

Construct two sequences $\{x_r\}$ *and* $\{y_r\}$ in X such that

$$P(x_r, y_r) = Q(x_{r+1}) \text{ and } P(y_r, x_r) = Q(y_{r+1}) \text{ for all } r \geq 0 \tag{4}$$

Now we shall prove that

$$Q(x_r) \leq Q(x_{r+1}) \text{ and } Q(y_r) \geq Q(y_{r+1}) \tag{5}$$

We use mathematical induction.

Step 1: Let $r = 0$. Since $Q(x_0) \leq P(x_0, y_0)\, and\, Q(y_0) \geq P(y_0, x_0),$

Using condition (3) we have $Q(x_0) \leq Q(x_1)$ *and* $Q(y_0) \geq Q(y_1)$ So inequality (5) holds for $r = 0$

Step 2: Now suppose that (5) holds for some fixed $s \geq 0$. We get

$Q(x_s) \leq Q(x_{s+1})$ *and* $Q(y_s) \geq Q(y_{s+1})$

Step 3: Since P has the mixed Q-monotone property using (4), we have

$$Q(x_{r+1}) = P(x_r, y_r) \leq P(x_{r+1}, y_r) \text{ and } Q(y_{r+1}) = P(y_r, x_r) \geq P(y_{r+1}, x_r) \tag{6}$$

Also, $Q(x_{r+2}) = P(x_{r+1}, y_{r+1}) \geq P(x_{r+1}, y_r)$ *and* $Q(y_{r+2}) = P(y_{r+1}, x_{r+1}) \leq P(y_{r+1}, x_r).$ (7)

From (6) and (7) we get

$$Q(x_r) \leq Q(x_{r+1}) \text{ and } Q(y_r) \geq Q(y_{r+1}). \tag{8}$$

From (T1) and (4) we get

$$\begin{aligned} V(P(x_{r-1}, y_{r-1}), P(x_{r-1}, y_{r-1}), ..., P(x_{r-1}, y_{r-1}), P(x_r, y_r), \tfrac{kt}{b}) &\geq V(Qx_{r-1}, Qx_{r-1}, ..., Qx_{r-1}, Qx_r, \tfrac{t}{b}) \\ &* V(Qx_{r-1}, Qx_{r-1}, ..., Qx_{r-1}, P(x_{r-1}, y_{r-1}), \tfrac{t}{b}) \\ &* V(Qx_r, Qx_r, ..., Qx_r, P(x_r, y_r), \tfrac{t}{b}), \end{aligned}$$

$$\begin{aligned} V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, \tfrac{kt}{b}) &\geq V(Qx_{r-1}, Qx_{r-1}, ..., Qx_{r-1}, Qx_r, \tfrac{t}{b}) \\ &* V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, \tfrac{kt}{b}). \end{aligned}$$

Now, two cases arise.

Case 1: If $V(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,\frac{t}{b}) < V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},\frac{kt}{b})$, then

$$\begin{aligned} V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},\tfrac{kt}{b}) &\ge V(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,\tfrac{t}{b}) \\ &\ge V(Qx_{r-2},Qx_{r-2},...,Qx_{r-2},Qx_{r-1},\tfrac{t}{bk}) \\ &\ge V(Qx_{r-3},Qx_{r-3},...,Qx_{r-3},Qx_{r-2},\tfrac{t}{bk^2}) \\ &\dots \\ &\ge V(Qx_0,Qx_0,...,Qx_0,Qx_1,\tfrac{t}{bk^{r-1}}). \end{aligned}$$

Then by simple induction we have that, for all $t \ge 0, b \ge 1$ and $r = 1,2,...,\infty$, $V(x_r,x_r,...,x_r,x_{r+1},\frac{kt}{b}) \ge V(x_0,x_0,...,x_0,x_1,\frac{t}{bk^{r-1}})$. Thus, by condition (VF-5) of the definition of a V-fuzzy metric space, for any positive integer p and real number $t > 0$, we have

$$\begin{aligned} V(Qx_r,Qx_r,...,Qx_r,Qx_{r+p},\tfrac{kt}{b}) &\ge V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},\tfrac{t}{bp}) \\ &\quad * V(Qx_{r+1},Qx_{r+1},...,Qx_{r+1},Qx_{r+2},\tfrac{t}{bp}) \\ &\quad * ...p\ times... \\ &\quad * V(Qx_{r+p-1},Qx_{r+p-1},...,Qx_{r+p-1},Qx_{r+p},\tfrac{t}{bp}). \\ &\ge V(Qx_0,Qx_0,...,Qx_0,Qx_1,\tfrac{t}{bpk^{r-1}}) \\ &\quad * ...p\ times... \\ &\quad * V(Qx_0,Qx_0,...,Qx_0,Qx_1,\tfrac{t}{bpk^{r+p-2}}). \end{aligned}$$

Therefore, taking $r \to \infty$, by definition (vi) of V-fuzzy *b*-metric space we get
$V(Qx_r,Qx_r,...,Qx_r,Qx_{r+p},\frac{t}{b}) \ge 1 * ...p\ times... * 1$,
Which implies that $\{x_n\}$ is a Cauchy sequence in *X*.

Case 2: If $V(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,\frac{t}{b}) > V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},\frac{t}{b})$, then $V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},\frac{kt}{b}) \ge V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},\frac{t}{b})$. By Lemma 2.2 we get $Q(x_r) = Q(x_{r-1})$.

Thus, there exists a positive integer *m* such that $r \ge m$ implies $Q(x_r) = Q(x_m), \forall r$, which shows that $\{Qx_n\}$ is a convergent sequence and so a Cauchy sequence in *X*. Taking $x = y_r, y = x_r, u = y_{r-1}, v = x_{r-1}$ in (T3), we get

$$\begin{aligned} V(P(y_r,x_r),P(y_r,x_r),...,P(y_r,x_r),P(y_{r-1},x_{r-1}),\tfrac{kt}{b}) &\ge V(Qy_r,Qy_r,...,Qy_r,Qu,\tfrac{t}{b}) \\ &\quad * V(Qy_r,Qy_r,...,Qy_r,P(y_r,x_r),\tfrac{t}{b}) \\ &\quad * V(Qy_{r-1},Qy_{r-1},...,Qy_{r-1},P(y_{r-1},x_{r-1}),\tfrac{t}{b}). \end{aligned}$$

So from equation (4) we have

$$\begin{aligned} V(Qy_r,Qy_r,...,Qy_r,Qy_{r+1},\tfrac{kt}{b}) &\ge V(Qy_{r-1},Qy_{r-1},...,Qy_{r-1},Qy_r,\tfrac{t}{b}) \\ &\quad * V(Qy_r,Qy_r,...,Qy_r,Qy_{r+1},\tfrac{t}{b}). \end{aligned}$$

In the same way (discussed before), $\{Qy_n\}$ is a Cauchy sequence in *X* .Since *X* is a complete space, there exist $x, y \in X$ such that

$$\lim_{r\to\infty} P(x_r,y_r) = \lim_{r\to\infty} Q(x_r) = x, \quad \lim_{r\to\infty} P(y_r,x_r) = \lim_{r\to\infty} Q(y_r) = y. \tag{9}$$

By considering condition (T4) and $r \to \infty$ we have
$V(Q(P(x_r, y_r)), Q(P(x_r, y_r)), ..., Q(P(x_r, y_r)), P(Q(x_r), Q(y_r), \frac{t}{b})) \to 1$ and
$V(Q(P(y_r, x_r)), Q(P(y_r, x_r)), ..., Q(P(y_r, x_r)), P(Q(y_r), Q(x_r), \frac{t}{b})) \to 1$ (10) as $r \to \infty$.

By condition (T4) and (a) since *P* and *Q* are continuous, form (10) we have
$V(Qx, Qx, ..., Qx, P(x, y), \frac{t}{b}) = 1$ And $V(Qy, Qy, ..., Qy, P(y, x), \frac{t}{b}) = 1$.

This implies that $P(x, y) = Q(x)$ *and* $P(y, x) = Q(y)$, and thus, we have proved that *P* and *Q*have a coupled coincidence point in *X*.Now, suppose that conditions (T4) and (b) hold. Since *Q* is continuous and *P*,*Q* are compatible mappings, we have

$$\lim_{r\to\infty} P(Q(x_r), Q(y_r)) = \lim_{r\to\infty} Q(P(x_r, y_r)) = \lim_{r\to\infty} Q(Qx_r) = Q(x) \quad and$$
$$\lim_{r\to\infty} P(Q(y_r), Q(x_r)) = \lim_{r\to\infty} Q(P(y_r, x_r)) = \lim_{r\to\infty} Q(Qy_r) = Q(y). \qquad (11)$$

By condition (v) of a V-fuzzy *b*-metric space, as $r \to \infty$, we get

$$\begin{aligned} V(Qx, Qx, ..., Qx, P(x, y), \tfrac{t}{b}) &\geq V(Qx, Qx, ..., Qx, Q(Qx_{r+1}), \tfrac{t-kt}{b}) \\ &*V(Q(Qx_{r+1}), Q(Qx_{r+1}), ..., Q(Qx_{r+1}), P(x, y), \tfrac{kt}{b}) \\ &= V(Qx, Qx, ..., Qx, Q(P(x_r, y_r)), \tfrac{t-kt}{b}) \\ &*V(Q(P(x_r, y_r)), Q(P(x_r, y_r)), ..., Q(P(x_r, y_r)), P(x, y), \tfrac{kt}{b}) \\ &\geq V(Q(P(x_r, y_r)), Q(P(x_r, y_r)), ..., Q(P(x_r, y_r)), P(x, y), \tfrac{kt}{b}). \end{aligned}$$

We get $V(Qx, Qx, ..., Qx, P(x, y), \frac{t}{b}) \geq V(P(Qx_r, Qy_r), P(Qx_r, Qy_r), ..., P(Qx_r, Qy_r), P(Qx, Qy), \frac{kt}{b})$.
(12)Using condition (T3)and equation (11),(12), we get

$$\begin{aligned} V(Qx, Qx, ..., Qx, P(x, y), \tfrac{t}{b}) &\geq V(Q(Qx_r), Q(Qx_r), ..., Q(Qx_r), Qx, \tfrac{t}{b}) \\ &*V(Q(Qx_r), Q(Qx_r), ..., P(Qx_r, Qy_r), \tfrac{t}{b}) \\ &*V(Qx, Qx, ..., Qx, P(x, y), \tfrac{t}{b}) \\ &\geq V(Qx, Qx, ..., Qx, P(x, y), \tfrac{t}{b}). \end{aligned}$$

By Lemma 2.2 we have $P(x, y) = Q(x)$. Similarly, we get $P(y, x) = Q(y)$. Hence; we proved that *P* and *Q* have a coupled coincidence point in *X*.

REFERENCES:

1) Abbas, M., Ali, B. and Suleiman, Y. I.(2015). Generalized coupled common fixed point results in partially ordered a-metric spaces, Fixed point Theory and Applications, vol. 2015, no.1, p. 64.
2) Bakthin, I.A. (1989). The contraction principle in quasimetric spaces, Funct. Anal. 30, 26-37
3) Banach, S. (1922) Sur les operations dans les ensembles abstraits et leur application aux equations integrals. Fundam. Math. 3, 133-181.
4) Bhaskar, TG, Lakshmikantham, V. (2006). Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. TMA 65, 1379-1393.
5) Choudhurya, BS, Kundu, A. (2010). A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. 73, 2524-2531.

6) Czerwik, S.(1993). Contraction mappings in b-metric spaces. Acta Math. Inform. Univ. Ostav.1(1), 5-11
7) George, A. and Veeramani, P. (1994). On some results in fuzzy metric spaces. Fuzzy Sets Syst. 64, 395-399.
8) Gupta, V., Kanwar, A., (2016). V-Fuzzy metric space and related fixed point theorems. Fixed Point Theory and Applications. Vol.1, p.51, pp.1–17, DOI 10.1186/s13663-016-0536-1.
9) Hassanzadeh, Z., Sedghi, S.(2018). Relation between b-metric and fuzzy metric spaces. Math. Morav. 22(1), 55-63.
10) Hooda, H. and Malik, A. (2019) Analytic Approach to Fixed Point Theorem on G-Metric Space Mappings, Research Review International Journal of Multidisciplinary [UGC Listed Journal] Volume-04| Issue-03| Mar 2019, pg. 1689-1693 (e-ISSN: 2455-3085, Impact Factor: 5.214).
11) Hooda, H., Malik, A., et. al. (2020). Some fixed-point results for asymptotically regular maps in N-fuzzy *b*-metric space. Turkish Journal of Computer and Mathematics Education (TURCOMAT), Vol. 11 No. 1, 647-656.(ISSN: 1309-4653)
12) Kramosil, I. and Michalek, J. (1975). Fuzzy metric and statistical metric spaces, Kybernetika, 11, 336–344.
13) Lakshmikantham, V., Ciric, L. B. (2009). Coupled fixed point theorems for nonlinear contractions in partially ordered metric space. Nonlinear Anal. TMA 70, 4341-4349.
14) Mishra, S.N., Sharma, N., et. al. (1994). Common fixed points of maps on fuzzy metric spaces. Int. J. Math. Sci. 17, 253-258.
15) Mustafa, Z. and Sims, B. (2006). A new approach to generalized metric spaces. J. Nonlinear Convex Anal. 7(2), 289-297.
16) Rakić, D., et al. (2020). On some new fixed point results in fuzzy b-metric spaces. J Inequal Appl 2020, 99 https://doi.org/10.1186/s13660-020-02371-3
17) Schweizer, B., Sklar, A. (1960). Statistical metric spaces. Pac. J. Math. 10, 314-334.
18) Sedghi, S. and Shobe, N. (2012). A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64(3), 258-266.
19) Sedghi, S. and Shobe, N. (2012). Common fixed point theorem in b-fuzzy mtric space. Nonlinear Funct. Anal. Appl. 17(3), 349-359.
20) Sookoo, M.B., Gunakala, S.R.(2020). Fixed point theorems for multivalued quasi-contraction maps in a V-fuzzy metric space, Abstract and Applied Analysis, vol. 2020 Article ID 2609316, pages 7. https://doi.org/10.1155/2020/2609316
21) Sun, G. and Yang, K. (2010). Generalized fuzzy metric spaces with properties. Research Journal of Applied Sciences, Engineering and Technology, 2(7), 673-678.
22) Zadeh, L.A. (1965). Fuzzy sets. Inf. Control 8, 338-353.