# Common Fixed Point of Mappings Satisfying Rational Inequalities in Complete Complex Valued Generalized Metric Spaces

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**Abstract** :

In this paper Some common fixed point theorems involving rationalinequalities have been proved and some consequences obtained in Complete Complex Valued Generalized Metric Spaces. Also we have extended this workperiodic point property of common fixed point problem for two rational type contractive mappings .

**Keywords:** Weakly increasing map, Common fixed point, complex valued generalized metric spaces, Partially ordered set.

## 1. Introduction and Preliminaries:

In 1922 , the polish mathematician Stefan Banach established a remarkable fixed point theorem known as the "Banach Contraction Principle " (BCP) which is one of the most important results of analysis and considered as the main source of metric fixed point theory . [ 8]

His valuable work has been elaborated via generalizing the metric conditions or by imposing conditions on the metric spaces. As a consequence of those generalizations so many metric spaces were introduced namely uniformly convex Banach spaces, strictly convex Banach spaces, cone metric spaces, pseudo metric spaces, B-metric spaces, fuzzy metric spaces etc. This paper is to introduce the concept of a complex valued generalized metric space and to study the fixed and common fixed point results for two mappings satisfying rational inequalities. The results presented in this paper substantially extend and strengthen the results given in Azam et al. [4] and Rouzkard et al. [12].

The following definitions [ 1-3] and results [ 12] will be needed in the sequel.

Let C be the set of complex numbers and let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\lesssim$ on $\mathbb{C}$as follows :$z_1 \lesssim z_2$ if and only if $\mathrm{Re}(z_1) \leq (z_2), \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.

It follows that $z_1 \lesssim z_2$ if one of the following conditions is satisfied:

(1) $\mathrm{Re}(z_1) = (z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(3) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(4) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular we will write$z_1 \lesssim z_2$and $z_1 \neq z_2$ .

If$z_1 \neq z_2$one of (1), (2) and (3) is satisfied and we will write $z_1 < z_2$ if only (3) is satisfied.

Some elementary properties of the partial order $\lesssim$ on $\mathbb{C}$are the following:

(i) If $0 \lesssim z_1 < z_2$, then $|z_1| \leq |z_2|$.

(ii) $z_1 \lesssim z_2$ is equivalent to $z_1 - z_2 \lesssim 0$.

(iii) If $z_1 \le z_2$ and $r \ge 0$ is a real number, then $rz_1 \le rz_2$.

(iv) If $0 \le z_1$ and $0 \le z_2$ with $z_1 + z_2 \ne 0$, then $\frac{z_1^2}{z_1+z_2} \le z_1$.

(v) $0 \le z_1$ and $0 \le z_2$ do not imply $0 \le z_1 z_2$.

(vi) $0 \le z_1$ does not imply $0 \le \frac{1}{z_1}$. Moreover, if $0 \le z_1$ and $0 \le \frac{1}{z_1}$, then $\mathrm{Im}(z_1) = 0$.

Now we give the definition of complex valued generalized metric space.

**Definition 1.1 [ 9]** Let X be a non-empty set. If a mapping $d: X \times X \to \mathbb{C}$ satisfies:

(a) $0 \le d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(c) $d(x,y) \le d(x,u) + d(u,v) + d(u,y)$ for all $x, y \in X$ and all distinct $u, v \in X$ each one is different from x and y.

Then $d$ is called a complex valued generalized metric on $X$ and $(X,d)$ is called a complex valued generalized metric space.

**Example** Let $\{(X_n, d_n): n \in K \subset \mathbb{N}\}$ be a family of disjoint complex valued generalized metric spaces and let $X = U\{X_n: n \in K\}$. Define, for all $x, y \in X$, a mapping $d: X \times X \to \mathbb{C}$ by

$$d(x,y) = \begin{cases} d_n(x,y), if & x, y \in X_n for some n \in K \\ 1 & if x \in X_n, y \in X_m for some m, n \in K, m \ne n. \end{cases}$$

Clearly $(X,d)$ is a complex valued generalized metric space.

**Lemma 1.2[4 ].** Let (X , d) be complex valued generalized metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

**Lemma 1.3[4].** Let (X,d) be a complex valued generalized metric space and $\{x_n\}$ be a sequence in X. then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $n \to \infty$.

The following definition is due to Altum ([3]).

**Definition 1.8[3].** Let $(X, \preceq)$ be a partially ordered set. A pair $(f, g)$ of self –map of $X$ is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$. If $f = g$, then we have $fx \preceq f^2x$ for all x in X and in this case, we say that f is a weakly increasing map.

## 2 Main Results:

**Theorem.** Let $(X, \leq)$ be a partially ordered set such that there exists a complete complex valued generalized metric d on X and (S,T) a pair of weakly increasing self – maps on X. Suppose that for every comparable $x, y \in X$ we have either

$$d(Sx,Ty) \leq a_1 \frac{[d(y,Sx)d(x,Ty)^2 + d(x,Ty)d(y,Sx)^2]}{d(x,Ty)^2 + d(y,Sx)^2}$$
$$+a_2 \frac{d(x,Ty)d(y,Sx)}{d(x,Ty) + d(y,Sx)} + a_3 \frac{d(x,Ty)^2 + d(y,Sx)^2}{d(x,Ty) + d(y,Sx)}$$
$$+a_4(x,Sx) + a_5 d(y,Ty) + a_6 d(x,y) \quad (1)$$

In case $d(x,Ty) + d(y,Sx) \neq 0$ $a_i \geq 0$ for $i = 1$ to 6 and $\sum_{i=1}^{6} a_i < 1$ or

$d(Sx,Ty) = 0$ if $d(x,Ty) + d(y,Sx) = 0$. (2)

If S or T is continuous or for any non-decreasing sequence $x_n$ with $x_n \to z$ in X we necessarily have $x_n \leq z$ for all $n \in N$. Then S and T have a common fixed point. Moreover the set of common fixed points of S and T is totally ordered iff S and T have one and only one common fixed point.

**Proof:** First we shall show that if S or T has a fixed point, then it is a common fixed point of S and T. Let u be a fixed point of S. Then from (1) with $x = y = u$ we have for $u \neq Tu$.

$$d(u,Tu) = d(Su,Tu)$$
$$\leq a_1 \frac{[d(u,Su)\{d(u,Tu)\}^2 + d(u,Tu)\{d(u,Su)^2\}]}{\{d(u,Tu)\}^2 + \{d(u,Su)\}^2}$$
$$+a_2 \frac{d(u,Tu)d(u,Su)}{d(u,Tu) + d(u,Su)} + a_3 \frac{d\{(u,Tu)\}^2 + d\{(u,Su)\}^2}{d(u,Tu) + d(u,Su)}$$
$$+a_4 d(u,Su) + a_5 d(u,Tu) + a_6 d(u,u)$$

$$\leq \frac{a_1[d(u,u)\{d(u,Tu)\}^2 + d(u,Tu)\{d(u,u)\}^2]}{\{d(u,Tu)\}^2 + \{d(u,u)\}^2}$$
$$+a_2 \frac{d(u,Tu)\, d(u,u)}{d(u,Tu) + d(u,u)} + a_3 \frac{d\{(u,Tu)\}^2 + d\{(u,u)\}^2}{d(u,Tu) + d(u,u)} + a_4 d\,(u,u) + a_5 d(u,u) + 0$$

$$\leq a_1.0 + a_2.0 + a_3\, d\,(u,Tu) + a_4.0 + a_5 d\,(u,Tu)$$
$$d\,(u,Tu) \leq (a_3 + a_5)d(u,Tu)$$

which implies that $|d(u,Tu)| \leq (a_3 + a_5)|d(u,Tu)|$

As $a_3 + a_5 < 1$ so we have $d(u,Tu) = 0$ and $u$ is a common fixed point of S and T. Similarly if u is a fixed point of T, then it is also fixed point of S.

Now let $x_0$ be an arbitrary point on X. If $Sx_0 = x_0$ then the proof is finished. Assume that $Sx_0 \neq x_0$. Construct a sequence $\{x_n\}$ $in X$ as follows:

$x_1 = Sx_0 \leq TSx_0 = Tx_1 = x_2$ and

$x_2 = Tx_1 \leq STx_1 = Sx_2 = x_3$.

Continuing this way we have $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$. Assume that $d(x_{2n}, x_{2n+1}) > 0$ for every $n \in N$. If not, then $x_{2n} = x_{2n+1}$ for some n. For all those $n, x_{2n} = x_{2n+1} = Sx_{2n}$ and the proof is finished. Assume that $d(x_{2n,}x_{2n+1}) > 0\ for n = 0, 1, 2, 3 \ldots$. $As x_{2n} and x_{2n+1}$ are comparable, so we have

$$d(x_{2n+1,}x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leq a_1 \frac{[d(x_{2n+1}, Sx_{2n})\{d(x_{2n}, Tx_{2n+1})\}^2 + d(x_{2n}, Tx_{2n+1})\{d(x_{2n+1}, Sx_{2n})\}}{\{d(x_{2n}, Tx_{2n+1})\}^2 + \{d(x_{2n+1}, Sx_{2n})\}^2}$$

$$+a_2 \frac{d(x_{2n}, Tx_{2n+1})d(x_{2n+1}, Sx_{2n})}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})}$$

$$+a_3 \frac{[\{d(x_{2n}, Tx_{2n+1})\}^2 + \{d(x_{2n+1}, Sx_{2n})\}^2}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})}$$

$$+ a_4\, d(x_{2n}, Sx_{2n}) + a_5 d(x_{2n+1}, Tx_{2n+1}) + a_6 d(x_{2n}, x_{2n+1})$$

$$= a_1 \frac{[d(x_{2n+1}, x_{2n+1})\{d(x_{2n}, x_{2n+2})\}^2 + d(x_{2n}, x_{2n+2})\{d(x_{2n+1}, x_{2n+1})\}^2}{\{d(x_{2n}, x_{2n+2})\}^2 + d(x_{2n+1}, x_{2n+1})\}}$$

$$+ a_2 \frac{d(x_{2n}, x_{2n+2})d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}$$

$$+ a_3 \frac{[\{d(x_{2n}, x_{2n+2})\}^2 + \{ d(x_{2n+1}, x_{2n+1})\}^2}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})} + a_4 d(x_{2n}, x_{2n+1})$$

$$+ a_5 d(x_{2n+1}, x_{2n+2}) + a_6 d(x_{2n}, x_{2n+1})$$

$$= a_3 d\{(x_{2n}, x_{2n+2}) + a_4\, d(x_{2n}, x_{2n+1}) + a_5\, d(x_{2n+1}, x_{2n+2}) + a_6 d(x_{2n}, x_{2n+1})$$

$$d(x_{2n+1}, x_{2n+2})(1 - a_3 - a_5) = (a_3 + a_4 + a_6)d(x_{2n}, x_{2n+1})$$

$$d\,(x_{2n+1}, x_{2n+2}) \leq \frac{(a_3 + a_4 + a_6)}{(1 - a_3 - a_5)} d(x_{2n}, x_{2n+1})$$

which implies that d $(x_{2n+1}, x_{2n+2}) \leq kd(x_{2n}, x_{2n+1})$ for all $n \geq 0,\ where\ 0 \leq k = \frac{a_3+a_4+a_6}{1-a_3-a_5} < 1$. Similarly d( $x_{2n}, x_{2n+1}) \leq kd(x_{2n-1}, x_{2n})$ for all $n \geq 0$. Hence for all $n \geq 0$. We have d$(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1})$. Consequently,

$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}) \leq \cdots \leq k^{n+1}d(x_0, x_1)$ for all $n \geq 0$. Now for $m > n$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d\,(x_{n+1}, x_m)$$

$$\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x_1)$$

$$\leq \frac{k^n}{1-k} d(x_0 x_1).$$

Therefore $d|(x_n, x_m)| \leq \frac{k^n}{1-k} |d(x_{0\,,} x_1)|. So |d(x_n, x_m)| \to 0$ as $n, m \to \infty$ gives that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete the sequence $\{x_n\}$ converges to a point u in X.

If S or T is continuous, then it is clear that *Su*=*u*=*Tu*.

If neither S nor T is continuous, then by given assumption $x_n \leq u$ for all $n \in N$. We claim that u is a fixed point of S. If not then $d(u, Su) = z > 0$ from (1), we obtain

$$z \le d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(x_{n+2},Su)$$
$$= d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(Su,Tx_{n+1})$$
$$\le d(u,x_{n+1}) + d(x_{n+1},x_{n+2}) + \frac{a_1[d(x_{n+1},Su)\{d(u,Tx_{n+1})\}^2 + d(u,Tx_{n+1})\{d(x_{n+1},Su)\}^2}{\{d(u,Tx_{n+1}\}^2 + \{d(x_{n+1},Su\}^2}$$
$$+ \frac{a_2 d(u,Tx_{n+1})d(x_{n+1},Su)}{d(u\,,Tx_{n+1}) + d(\,x_{n+1}\,,Su)} + a_3\frac{\{d(u,Tx_{n+1})\}^2 + \{d(x_{n+1},Su\}^2}{d(u,Tx_{n+1}) + d(x_{n+1},Su)}$$
$$+ a_4 d(u,Su) + a_5 d(x_{n+1},Tx_{n+1}) + a_6(u,x_{n+1})$$

And so $|z| \le d(u,x_{n+1}) + |d(x_{n+1},x_{n+2})| + a_1\frac{[|d(x_{n+1},Su)|\{|d(u,Tx_{n+1})|^2\} + |d(u,Tx_{n+1})|\{d(x_{n+1},Su)|\}^2}{\{|d(u,Tx_{n+1})|\ \ \}^2 + \{|d(x_{n+1},Su)|\ \ \}^2}$

$$+a_2\frac{|d(u,Tx_{n+1})|d(x_{n+1,}Su)|}{|d(u,Tx_{n+1})| + |d(x_{n+1},Su)|} + a_3\frac{\{|d(u,Tx_{n+1})|\}^2 + \{|d(x_{n+1},Su)|\}^2}{|d(u,Tx_{n+1})| + |d(x_{n+1},Su)|}$$
$$+ a_4\,|d(u,Su)| + a_5|d(x_{n+1},Tx_{n+1})| + a_6\,|d(u,x_{n+1})|$$

which on taking limit as $n \to \infty$ given $|z| < a_4|z|$ a contradiction and so $u = Su$. Therefore $Su = u = Su$.
Now suppose that set of common fixed points of S and T is totally ordered. We prove that common fixed point of S and T. By supposition, we can replace x by p and y by q in (1) to obtain.

$$d(p,q) = d(Sp,Tq)$$
$$\le a_1\frac{[\{d(q,Sp)\}\{d(p,Tq)\}^2 + d(p,Tq)\{d(q,Sp)\}^2]}{\{d(p,Tq)\}^2 + \{d(q,Sp)\}^2} + a_2\frac{d(p,Tq)d(q,Sp)}{d(p,Tq) + d(q,Sp)}$$
$$+ a_3\frac{d\{(p,Tq)\}^2 + d(q,Sp)\}^2}{d(p,Tq) + d(q,Sp)} + a_4 d(p,Sp) + a_5 d(q,Tq) + a_6(p,q)$$

$$= a_1\frac{[\{d(q,p)\}\{d(p,q)\}^2 + d(p,q)\{d(q,p)\}^2]}{\{d(p,q)\}^2 + \{d(p,q)\}^2} + a_2\frac{d(p,q)d(q,p)}{d(p,q) + d(q,p)} + a_3\frac{d(p,q)^2 + d(q,p)^2}{d(p,q) + d(p,q)}$$
$$+ a_4 d\,(p,p) + a_5(q,q) + a_6(p,q)$$

$$= a_1\left[\frac{2d(p,q)}{2d(p,q)^2}\right] + a_2\left[\frac{d(p,q) * d(p,q)}{2d(p,q)}\right] + a_3\left[\frac{2d(p,q)}{2d(p,q)}\right] + 0 + 0 + a_6 d(p,q)$$
$$= a_1 d(p,q) + \frac{a}{2}d(p,q) + a_3 d(p,q) + a_6 d(p,q)$$
$$= \left(a_1 + \frac{a}{2} + a_3 + a_6\right)d(p,q)$$

which implies that $|d(p,q)| \le \left(a_1 + \frac{a_2}{2} + a_3 + a_6\right)|d(p,q)|$ a contradiction. Hence $p = q$.

Conversely if S and T have only one common fixed point then the set of common fixed point of S and T being singleton is totally ordered.

Although we studied a common fixed point problem for two mapping to consider a more general result, we could use even one and yet the result would have been new. In theorem (1) take S=T, to obtain the following corollary.

**Corollary.** Let $(X, \leq)$ be a partially ordered set such that there exists a complete complex valued generalized metric d on X and let T be a weakly increasing self – map on X. Suppose that for every comparable $x, y \in X$, either

$$d(Tx,Ty) \leq a_1 \frac{[d(y,Tx)\,\{d(x,Ty)\}^2 + d(x,Ty)\,\{d(y,Tx)\}^2]}{\{d(x,Ty)\}^2 + \{d(y,Tx)\}^2}$$
$$+a_2 \frac{d(x,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)} + a_3 \frac{\{d(x,Ty)\}^2 + d\{(y,Tx)\}^2}{d(x,Ty)+d(y,Tx)}$$
$$+a_4(x,Tx)+a_5 d(y,Ty) + a_6(x,y)$$

If $d(x,Ty) + d(x,Tx) \neq 0,\ a_i \geq 0$ for $i = 1$ to 6 and $\sum_{i=1}^{6} a_i < 1$ or $d(Tx,Ty) = 0$ if $d(x,Ty) + d(y,Tx) = 0$ if T is continuous or for a non decreasing sequence $\{x_n\}$ with $x_n \to Z$ in X we necessarily have $x_n \leq Z$ for all $n \in N$ then T has a fixed point.

**Conclusion** : The concept of a complex valued generalized metric space and study the fixed point theorem and in the current work we obtain common fixed point for two mappings satisfying with rational inequalities , without exploiting any type of commutativity condition.

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