W_8 - Curvature Tensor in Lorentzian α - Sasakian Manifolds

Rajan[1], Gyanvendra Pratap Singh[2]
Pawan Prajapati[3], Anand Kumar Mishra[4]
Department of Mathematics and Statistics
Deen Dayal Upadhyaya Gorakhpur University Gorakhpur-273009
(UP) INDIA

Abstract
The object of the present paper is to study the curvature properties of Lorentzian α - Sasakian manifolds satisfying the conditions $\xi - W_8-$flatness, $\varphi - W_8-$semisymmetric, $W_8 \cdot Q = 0$, $W_8 \cdot R = 0$ and found some interesting results.

Mathematics Subject Classification 2010: 53C15, 53C25, 53D10.

Keyword and Phrases:

Almost contact manifolds, trans-Sasakian manifolds, α - Sasakian Manifolds, Lorentzian α - Sasakian Manifolds, φ- symmetric, φ-semisymmetric, W_8 curvature tensor, Einstein manifold, η-Einstein manifold.

1 Introduction

S. Tanno classified linked almost contact metric manifolds with the highest dimension automorphism groups in [10]. The sectional curvature of plane sections containing ξ is a constant, say c, for such a manifold. He demonstrated that they can be classified into three classes:

(i) homogeneous normal contact Riemannian manifolds with c >0,

[1] rajanvishwakarma497@gmail.com
[2] gpsingh.singh700@gmail.com
[3] pawanpra123@gmail.com
[4] aanandmishra1796@gmail.com

(ii) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if c = 0,

(iii) a warped product space if c <0.

It is well known that the manifolds of class (i) are characterised by the fact that they admit a Sasakian structure. There is a class of Hermitian manifolds, W_4, in the Gray-Hervella classification of almost Hermitian manifolds [2], that is closely linked to locally conformal Kaehler manifolds [9]. If the product manifold $M \times R$ belongs to the class W_4, an almost contact metric structure on a manifold M is called a trans-Sasakian structure [8, 5]. The class $C_6 \oplus C_5$ [7] corresponds to the class of type (α, β) trans-Sasakian structures. In fact, the local nature of the two subclasses of trans- Sasakian structures, namely, C_5 and C_6 structures, is well defined in [7].

We note that trans-Sasakian structures of type (0, 0), $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [5], β-Kenmotsu [13] and α-Sasakian [13], correspondingly. It is demonstrated in [12] that trans-Sasakian structures are generalized quasi-Sasakian structures [13]. As a result, trans-Sasakian structures include a large number of generalized quasi-Sasakian structures. After that, Yildiz and Murathan introduced Lorentzian α- Sasakian manifolds in [3].

An almost contact metric structure (φ, ξ, η, g) on M is called a trans-Sasakian structure [8] if $(M \times \mathrm{R}, J, G)$ belongs to the class W_4 [2], where J is the almost complex structure on $M \times \mathrm{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

for all vector fields X on M and smooth function f on $M \times \mathrm{R}$, and G is the product metric on $M \times \mathrm{R}$. This may be expressed by the condition [4]

$$(\nabla_X\varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X,Y)\xi - \eta(Y)\varphi X) \qquad (1.1)$$

for some smooth functions α and β on M, and we say that the trans- Sasakian structure is of type (α, β). From the formula (1.1) it followsthat

$$\nabla_X\xi = -\alpha\varphi X + \beta(X - \eta(X)\xi), \tag{1.2}$$

$$(\nabla_X\eta)(Y) = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y) \tag{1.3}$$

More generally one has the notation of α- Sasakian structure[13] which may be defined by

$$(\nabla_X\varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X), \tag{1.4}$$

where α is a non-zero constant. From the condition one may readily deduce that

$$\nabla_X\xi = -\alpha\varphi X \tag{1.5}$$

$$(\nabla_X\eta)(Y) = -\alpha g(\varphi X, Y) \tag{1.6}$$

Thus $\beta = 0$ and therefore a trans-Sasakian structure of type (α, β) with α a non-zero constant is always α-Sasakian [13]. If $\alpha = 1$, then α-Sasakian manifold is Sasakian manifold. Marrero [1] examined the relationship between trans-Sasakian, α-Sasakian and β-Kenmotsu structures.

In [16] Tripathi and Gupta have defined the W_8-curvature tensor, given by

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X]$$

where R and S are curvature tensor and Ricci tensor of the manifold respectively.

K. Kenmotsu [6] introduced a new class of almost contact Riemann manifold which is also known as Kenmotsu Manifold. Kenmotsu investigated fundamental properties on local structure of such manifolds. Kenmotsu manifolds are locally isometric to warped product spaces with one dimensional base and Kaehler fiber. Kenmotsu studied that if a Kenmotsu manifold satisfies the condition R(X, Y)Z = 0, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor and R(X, Y)Z is the derivative of tensor algebra at each point of the tangent space .It is well known that odd dimensional spheres admit Sasakian structures whereas odd dimensional hyperbolic spaces cannot admit Sasakian structure, so odd dimensional hyperbolic spaces admits Kenmotsu Structure. Kenmotsu manifolds are normal almost contact Riemannian Manifolds. Several properties of Kenmotsu Manifold have been studied by many authors like [11, 14, 15, 17, 18,19, 20, 21, 22, 23, 24, 25]. Motivated by all these work in this paper we study W_8-curvature tensor in Lorentzian α- Sasakian manifolds in [3].

Our paper is organized as follows: After introduction in section 2, we introduce the notion of a Lorentzian α-Sasakian manifold and also, we have given some basic definitions. In section 3, we study $\xi - W_8-$ flat in a Lorentzian α - Sasakian manifold and found to be η - Einstein manifolds. Section 4, deals with the $\varphi - W_8-$ semisymmetric condition in Lorentzian α- Sasakian manifold and found to be Einstein manifold. In section 5, we discuss a Lorentzian α - Sasakian manifold satisfying$W_8 \cdot Q = 0$ and also found to be Einstein manifold. Finally in the last section, we discuss the Lorentzian α - Sasakian manifold satisfying $W_8 \cdot R= 0$ and found to be η - Einstein Manifolds.

2 Preliminaries

A (2n+1)-dimensional smooth manifold M is called a Lorentzian α - Sasakian manifold if it admits a (1, 1)-tensor field φ, a vector field ξ, a 1-form η, and Lorentizian metric g which satisfy the conditions [3]

$$\varphi^2 = I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = -1, \varphi(\xi) = 0, \eta \, o \, \varphi = 0, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\varphi X,\varphi Y) = g(X,Y) + \eta(X)\,\eta(Y), \qquad (2.4)$$

$$(\nabla_X\varphi)Y= \alpha\{g(X,Y)\xi + \eta(Y)X\}, \qquad (2.5)$$

for all $X,Y\in TM$. Also on a Lorentzian α - Sasakian Manifold M satisfies the following [3]

$$\nabla_X\xi= \alpha\varphi X, \qquad (2.6)$$

$$(\nabla_X\eta)(Y) = \alpha g(X,\varphi Y), \qquad (2.7)$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is constant.

Also a Lorentzian α - Sasakian manifold M holds the following relations [3]:

$$\eta(R(X, Y)Z) = \alpha^2\{g(Y,Z)\eta(X) - g(X,Z)\,\eta(Y)\}, \qquad (2.8)$$

$$R(X,Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\}, \qquad (2.9)$$

$$R(\xi,X)Y = \alpha^2\{g(X,Y)\xi - \eta(Y)X\}, \qquad (2.10)$$

$$R(\xi,X)\xi = \alpha^2\{\eta(X)\xi + X\}, \qquad (2.11)$$

$$S(X,\xi) = 2n\alpha^2\eta(X) \qquad (2.12)$$

$$Q\xi= 2n\alpha^2\xi, \qquad (2.13)$$

$$S(\xi,\xi) = -2n\alpha^2, \qquad (2.14)$$

for any vector fields $X,Y,Z,$ where S is the Ricci curvature and Q is Ricci operator is given by $S(X,Y) = g(QX,Y)$.

Definition 2.1. A Lorentzian α - Sasakian Manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = \lambda_1 g(X, Y) + \lambda_2\eta(X)\,\eta(Y), \qquad (2.15)$$

for any vector fields X, Y, where λ_1,λ_2 are function on M.

If λ_1=0 then M is a special η-Einstein manifold.

3 $\xi - W_8$-flat in Lorentzian α - Sasakian Manifold

In this section, we study $\xi - W_8-$ flat in Lorentzian α - Sasakian manifold:

Definition 3.1. A Lorentzian α - Sasakian manifold is said to be $\xi - W_8-$ flat if

$$W_8(X, Y)\xi = 0 \tag{3.1}$$

for any vector fields X,Y on M. W_8-curvature tensor [16] is defined as

$$W_8(X,Y)Z=R(X,Y)Z+ \frac{1}{n-1}[S(X,Y)Z-S(Y,Z)X], \tag{3.2}$$

where R and S are curvature tensor and Ricci tensor of the manifold respectively.
Replacing Z by ξ in (3.2), we get

$$W_8(X,Y)\xi = R(X,Y)\xi + \frac{1}{n-1}[S(X,Y)\xi - S(Y,\xi) X] \tag{3.3}$$

By using (3.1) in (3.3), we get

$$R(X,Y)\xi+ \frac{1}{n-1}[S(X,Y)\xi - S(Y, \xi) X]=0 \tag{3.4}$$

By virtue of (2.9), (2.12) in (3.4) and on simplification, we obtained

$$\alpha^2\{\eta(Y)X - \eta(X)Y\}+ \frac{1}{n-1}[S(X,Y)\xi - 2n\alpha^2\eta(Y) X]=0 \tag{3.5}$$

By taking inner product with ξ in (3.5) and on simplification, we have

$$S(X,Y) = 2n\alpha^2\eta(Y)\eta(X) \tag{3.6}$$

Hence above discussion, we state that the following theorem:

Theorem 3.2 If a Lorentzian α - Sasakian manifold satisfying $\xi - W_8-$flat condition then the manifold is a special type of η-Einstein manifolds.

4 $\varphi - W_8-$ semisymmetric Condition in Lorentzian α - Sasakian Manifold

In this section, we study $\varphi - W_8-$ semisymmetric condition in Lorentzian α - Sasakian manifold:

Definition 4.1. A Lorentzian α - Sasakian manifold is said to be $\varphi - W_8-$semisymmetric if

$$W_8(X, Y) \cdot \varphi = 0 \tag{4.1}$$

for any vector field X, Y on M.
Now, (4.1) turns into

$$(W_8(X, Y) \cdot \varphi)Z = W_8(X, Y)\varphi Z - \varphi W_8(X, Y)Z = 0 \tag{4.2}$$

From equation (3.2), we get

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X] \tag{4.3}$$

Replace Z by φZ in (4.3) we obtain

$$W_8(X,Y)\varphi Z = R(X,Y)\varphi Z + \frac{1}{n-1}[S(X,Y)\varphi Z - S(Y,\varphi Z)X] \tag{4.4}$$

Making use of (4.3) and (4.4) in (4.2) and on simplification, we get

$$R(X,Y)\varphi Z - \varphi R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)\varphi X - S(Y,\varphi Z)X] = 0 \tag{4.5}$$

Putting $X = \xi$ in (4.5) and by virtue of (2.10), (2.12) and on simplification, we obtain

$$\alpha^2 g(Y,\varphi Z)\xi - \frac{1}{n-1}S(Y,\varphi Z)\xi = 0 \tag{4.6}$$

Replace φZ by Z in (4.6) and on simplication, we get

$$S(Y,Z)\xi = (n-1)\alpha^2 g(Y,Z)\xi \tag{4.7}$$

By taking inner product with ξ in (4.7) we get

$$S(Y,Z) = (n-1)\alpha^2 g(Y,Z) \tag{4.8}$$

Hence above discussion, we state the following theorem:

Theorem 4.2. If a Lorentzian α - Sasakian manifold satisfying $\varphi - W_8-$semisymmetric condition then manifold is an Einstein manifolds.

5 Lorentzian α - Sasakian Manifold Satisfying $W_8 \cdot Q = 0$

In this section, we study Lorentzian α - Sasakian manifold satisfying $W_8 \cdot Q = 0$.

Then, we have

$$W_8(X,Y)QZ - Q(W_8(X,Y)Z) = 0 \qquad (5.1)$$

Putting Y = ξ in (5.1), we obtain

$$W_8(X,\xi)QZ - Q(W_8(X,\xi)Z) = 0 \qquad (5.2)$$

By virtue of (3.2) in (5.2), we get

$$R(X,Y)QZ+\frac{1}{n-1}[S(X,\xi)QZ-S(\xi,QZ)X]-Q\{R(X,\xi)Z$$
$$+\frac{1}{n-1}[S(X,\xi)Z-S(\xi,Z)X]\}=0 \qquad (5.3)$$

By using (2.10), (2.12) in (5.3), we obtain

$$-\alpha^2[g(X,QZ)\xi-\eta(QZ)X]+\frac{1}{n-1}[2n\alpha^2\eta(X)QZ-2n\alpha^2\eta(QZ)X]\}$$
$$-Q\{-\alpha^2[g(X,Z)\xi-\eta(Z)X]$$
$$+\frac{1}{n-1}[2n\alpha^2\eta(X)Z-2n\alpha^2\eta(Z)X]\}=0 \qquad (5.4)$$

On simplifying (5.3), we have

$$-\alpha^2S(X,Z)\xi+\alpha^2Q\eta(Z)X+\frac{2n}{n-1}\alpha^2\eta(X)QZ-\frac{2n}{n-1}\alpha^2Q\eta(Z)X$$
$$+\alpha^2g(X,Z)Q\xi-\alpha^2\eta(Z)QX-\frac{2n}{n-1}\alpha^2\eta(X)QZ$$
$$+\frac{2n}{n-1}\alpha^2Q\eta(Z)X=0 \qquad (5.5)$$

Using (2.13) and simplify (5.5), we have

$$S(X,Z)\xi = 2n\alpha^2g(X,Z)\xi \qquad (5.6)$$

Taking inner product with ξ in (5.6) and on simplifaction, we have

$$S(X,Z) = 2n\alpha^2g(X,Z) \qquad (5.7)$$

Hence from above discussion, we state the following theorem:

Theorem 5.1. If a Lorentzian α - Sasakian manifold satisfying $W_8 \cdot Q = 0$, then the manifold is an Einstein manifolds.

6 Lorentzian α - Sasakian Manifold Satisfying $W_8 \cdot R = 0$

In this section, we study Lorentzian α - Sasakian manifold satisfying $W_8 \cdot R= 0$. Then, we have

$$W_8(\xi,U)R(X,Y)Z - R(W_8(\xi,U)X,Y)Z$$
$$-R(X,W_8(\xi,U)Y)Z - R(X,Y)W_8(\xi,U)Z = 0 \qquad (6.1)$$

Putting Z = ξ in (6.1), we have

$$W_8(\xi,U)R(X,Y)\xi - R(W_8(\xi,U)X,Y)\xi$$
$$-R(X,W_8(\xi,U)Y)\xi - R(X,Y)W_8(\xi,U)\xi = 0 \qquad (6.2)$$

By using (2.9) in (6.2) and on simplification, we get

$$\alpha^2\eta(W_8(\xi,U)X)Y - \alpha^2\eta(W_8(\xi,U)Y)X - R(X,Y)W_8(\xi,U)\xi = 0 \qquad (6.3)$$

By using equation (3.2) in (6.4), we get

$$\alpha^2\eta[R(\xi, U)X+ \frac{1}{n-1}\{S(\xi, U)X-S(U,X)\xi\}]Y$$
$$-\alpha^2\eta[R(\xi, U)Y+ \frac{1}{n-1}\{S(\xi, U)Y-S(U,Y)\xi\}]X$$
$$-R(X, Y)[R(\xi, U)\xi+ \frac{1}{n-1}\{S(\xi, U)\xi-S(U,\xi)\xi\}]=0 \qquad (6.4)$$

By using (2.8), (2.10), (2.12) in (6.4) and on simplification, we get

$$\alpha^2\{g(U, Y)X - g(U,X)Y\}- \frac{2n}{n-1}\alpha^2\eta(U)\eta(X)Y$$
$$-\frac{2n}{n-1}\alpha^2\eta(U)\eta(Y)X+\frac{1}{n-1}\{S(U,X)Y-S(U,Y)X\}$$
$$-R(X, Y)U=0 \qquad (6.5)$$

Putting Y = ξ in (6.5), we get

$$\alpha^2\{g(U,\xi)X - g(U,X)\xi\} - \frac{2n}{n-1}\alpha^2\eta(U)\eta(X)\xi$$

$$-\frac{2n}{n-1}\alpha^2\eta(U)\eta(\xi)X + \frac{1}{n-1}\{S(U,X)\xi - S(U,\xi)X\}$$

$$-R(X,\xi)U = 0 \qquad (6.6)$$

By using (2.3), (2.10), (2.12) in (6.6) and on simplification, we get

$$S(X,U)\xi = 2n\alpha^2\eta(U)\,\eta(X)\xi \qquad (6.7)$$

By taking inner product with ξ in (6.7), we have

$$S(X,U) = 2n\alpha^2\eta(U)\,\eta(X) \qquad (6.8)$$

Hence from above discussion, we state the following:

Theorem 6.1. If a Lorentzian α - Sasakian manifold satisfying $W_8 \cdot R = 0$, then the manifold is a special type of η - Einstein manifolds.

Acknowledgement. This work is supported by Council of Scientific and Industrial Research (CSIR), India, under Junior Research Fellowship with File. No. 09/057(0226)/ 2019-EMR-I.

References

[1] Marrero J. C., The local structure of trans-Sasakian manifolds, Ann. Mat. Pura Appl., 162(4)(1992), 77-86. J.

[2] Gray A. and Hervella L. M., The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl., 123(4)(1980), 35-58.

[3] Yildiz. and Murathan C., On Lorentzian α-Sasakian manifolds, Kyungpook Math. J., 45 (2005), 95-103.

[4] Blair D. E. and Oubina J. A., Conformal and related changes of metric on the product of two almost contact metric manifolds, Publications Matematiques, 34(1990), 199-207.

[5] Blair D. E., Riemannian geometry contact and symplectic manifolds, Progress in Mathematics in Mathematics, 203(2002). Birkhauser Boston, Inc., Boston.

[6] K. Kenmotsu , A class of almost contact Riemannian Manifold, Tohoku Math.J., 24(1972), 93-103.

[7] Marrero J. C. and Chinea D., On trans-Sasakian manifolds, Proceedings of the XIVth Spanish-Portuguese Conference on Mathematics, Vol. I-III (Spanish) (Puerto de la Cruz, 1989), 655-659, Univ. La Laguna, La Laguna, 1990.

[8] Oubina A., New classes of contact metric structures, Publ. Math. Debrecen, 32(3-4)(1985), 187-193.

[9] Dragomir S. and Ornea L., Locally conformal Kaehler geometry, Progress in Mathematics 155, Birkhauser Boston, Inc., Boston, MA, 1998.

[10] Tanno S., The automorphism groups of almost contact Riemannian manifolds, Tohoku Math. J., 21(1969), 21-38.

[11] Bagewadi C.S. and Venkatesha, Some curvature tensors on a Kenmotsu manifolds, Tensor N.S., 68(2007), 140-147.

[12] Tripathi M. M., Trans-Sasakian manifolds are generalized quasiSasakian, Nepali Math. Sci. Rep., 18(1-2)(1999-2000), 11-14.

[13] Janssens D. and Vanhacke L., Almost contact structures and curvature tensors, Kodai Math. J., 4(1981), 1-27.

[14] Blair D.E., Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, 509, Springer-Verlag, berlin New- York, (1976).

[15] S. K. Chaubey and R. H. Ojha, On the m-projective curvature tensor of a Kenmotsu manifold, Differential Geometry Dynamical Systems, 12(2010), 52-60.

[16] Tripathi M.M. and Gupta Punam, ($N(k)$, ξ)-semi-Riemannian manifolds: Semi-symmetries, arXiv:1202.6138V [math.DG] 28 (Feb2012).

[17] De U.C., On φ-symmetric Kenmotsu manifolds, International electronic Jour. of geometry, 1(2008), 33-38.

[18] Ingalahalli Gurupadavva and Bagewadi C.S. , A study on Conservative C-Bochner curvature tensor in K-contact and Kenmotsu manifolds admitting semi-symmetric metric connection, ISRN Geometry, (2012), 1-14.

[19] Hui S. K. and Chakraborty D., Infinitesimal CL-transformations on Kenmotsu manifolds, Bangmod Int. J. Mathand Comp. Sci., 3(2017), 1-9.

[20] Baishya K. K. and Chowdhury P. R., Kenmotsu Manifold with some Curvature Conditions, Annales Univ. Sci. Budapest., 59(2016), 55-65.

[21] Nagaraja H. G., Kiran Kumar D. L. and Prasad V. S., Ricci solitons on Kenmotsu manifolds under D-homothetic deformation, Khayyam J. Math., 4(2018), 102-109.

[22] Özgür C., On weakly symmetric Kenmotsu manifolds, Differential¨ Geometry-Dynamical Systems, 8(2006), 204-209.

[23] Prakasha D. G. and Hadimani B. S., On the conharmonic curvature tensor of Kenmotsu manifolds with generalized Tanaka-Webster connection, Misolc. Math. Notes, 19(2018), 491-503.

[24] Shaikh A. A. and Hui S. K., On Locally φ-Symmetric β-Kenmotsu Manifolds, ExtractaMathematicae, 24(2009), 301- 316.

[25] Sinha B. B. and Srivastava A.K., Curvature on Kenmotsu manifold, Indian J. Pure and Appl. Math., 22(1991), 23-28.