SOME FIXED-POINT RESULTS FOR ASYMPTOTICALLY REGULAR MAPS IN N-FUZZY*b*-METRIC SPACE

Happy Hooda[1],
[1]Research Scholar, Department of Mathematics,
Maharshi Dayanand University, Rohtak-124001, (Hr) India

Archana Malik[2],
[2]Professor, Department of Mathematics,
Maharshi Dayanand University, Rohtak-124001, (Hr) India

Manish Vats[3],
[3]Assistant Professor, All India Jat Heroes' Memorial College,
Rohtak-124001, (Hr) India

[*]Corresponding author: Happy Hooda,
Email: happyhooda2017@gmail.com

ABSTRACT: In this paper, we define N-fuzzy *b*-metric space and then weinvestigate fixed point results on the structure of N-fuzzy *b*-metric space using asymptotically regular map and asymptotically regular sequence. These theorems generalize and improve some known fixed-point theorems in literature.

MSC: primary 47H10; secondary 54H25.

KEYWORDS: Fuzzy metric space;fixed point;N-fuzzy *b*-metric space; asymptotically regular map; asymptotically regular sequence.

1. INTRODUCTION AND PRELIMINARIES:

The foundation of fuzzy mathematics is laid by Lotfi A. Zadeh[24] in 1965. In 1975, Kramosil and Michalek [10] introduced the concept of fuzzy metric space. George and Veeramani [7] modified the concept of fuzzy metric space. In 1963, Gahler [5, 6] generalized usual notion of metric space called 2-metric space. Using the notion of 2-metric space, S. Sharma[21] and S. Kumar [11] introduced fuzzy 2-metric space without knowing each other but Ha *et.al* shows that 2-metric need not to be continuous function, further there is no easy relationship between results obtained in the two settings. In 1992, BapureDhage [4]in his PhD thesis introduced a new class of generalized metric space called D-metric space [13]. B. Singh and M. Chouhan[22] defined S-fuzzy metric space by using concept of D-metric space. However, Mustafa and Sims in [14] have pointed out that most of results claimed by Dhage and others in D-metric spaces are invalid. To overcome these fundamental flaws, they introduced a new concept of generalized metric space called G-metric space [14]. Using the concept of G-metric space, G. Sun and K Yang [23] introduced the notion of the notion of Q-Fuzzy metric space, K.P.R. Rao *et. al.*[15] proved two fixed point theorems in symmetric Q(G)-metric space Sedghi*et. al.* in [16] introduced D*-metric space which is a generalized G-metric space and gave an example which is D*-metric space but not G-metric space. Using the concept of D*-metric space, Sedghi and Shobe [17] defined M-Fuzzy metric space. Very recently, Sedghi*et. al.* in [18] defined S-metric space which is generalization of D*-metric

space and G-metric space and justified their work by various examples and definitions related to topology of S-metric space. N. Malviya [12] introduced the notion of N-fuzzy metric space, pseudo N-fuzzy metric space and describes some of their properties and examples.
In addition to fuzzy metric spaces, there are still many extensions of metric and metric space terms. Bakhtin[1] and Czerwik[3] introduced a space where, instead of triangle inequality, a weaker condition was observed, with the aim of generalization of Banach contraction principal [2]. They called these spaces *b*-metric spaces. Relation between *b*-metric and fuzzy metric spaces is considering in [9]. On the other hand, in [19] the notion of a fuzzy *b*-metric space was introduced, where the triangle inequality is replaced by a weaker one.

Now, in this paper we introduced a new space that is N-fuzzy *b*-metric space with the help of N-fuzzy metric space and *b*-Metric space and using the notion of asymptotic regularity of mappings, we prove some fixed point theorem in N-fuzzy metric space.These theorems generalize and improve some known fixed point theorems in literature.

Definition1.1:-A binary operation $*:[0,1]\times[0,1]\times[0,1]\rightarrow[0,1]$ is said to be a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b\le c*d$ for $a\le c, b\le d$.Examples of a t-norm are $a*b=\min\{a,b\}$, $a*b=ab$ and $a*b=\max\{a+b-1,0\}$.

Definition 1.2:- The 3-tuple (X, M, T) is known as fuzzy metric space (shortly, FM-space) if X is an any set, T is a continuous t-norm, and M is a fuzzy set in $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0$;

(FM-1) $M(x,y,t)>0$,

(FM-2) $M(x,y,t)=1$ iff $x=y$,

(FM-3) $M(x,y,t)=M(y,x,t)$,

(FM-4) $T(M(x,y,t),M(y,z,s))\le M(x,z,t+s)$,

(FM-5) $M(x,y,\cdot):[0,\infty)\rightarrow[0,1]$ is continuous.

Definition 1.3:-A 3-tuple $(X,N,*)$ is called an*N*-fuzzy metric space if X is an arbitrary (non-empty) set, * is a continuous t-norm, and *N*is a fuzzy set on $X^3\times(0,\infty)$, satisfying the following conditions for each $x,y,z\in X$ and $r,s,t>0$;

(N1) $N(x,y,z,t)>0$,

(N2) $N(x,y,z,t)=1$ iff $x=y=z$.

(N3) $N(x,x,a,r)*N(y,y,a,s)*N(z,z,a,t)\le N(x,y,z,r+s+t)$,

(N4) $N(x,y,z,\cdot):(0,\infty)\rightarrow[0,1]$ is continuous function.

Definition 1.4:-The 3-tuple (X,M,T) is known as fuzzy *b*-metric space if X is any set, T is a continuous *t*-norm, and M is a fuzzy set in $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0$, and a given real number $b\ge1$,

(BM-1) $M(x,y,t)>0$,

(BM-2) $M(x,y,t)=1$ if and only if $x=y$, (BM-3) $M(x,y,t)=M(y,x,t)$,

(BM-4) $T(M(x,y,\frac{t}{b}),M(y,z,\frac{s}{b}))\le M(x,z,t+s)$,

(BM-5) $M(x,y,\cdot):[0,\infty)\to[0,1]$ is continuous.

Definition 1.5:-The 3-tuple (X,N,T) is known as N-fuzzy *b*-metric space if X is any set, T is a continuous *t*-norm, and *N*is a fuzzy set in $X\times X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z,a\in X$ and *s*, *t* >0 and a given real number $b\ge 1$,

(NF-1) $N(x,y,z,t)>0$,

(NF-2) $N(x,y,z,t)=1$ if and only if $x=y=z$.

(NF-3) $T(N(x,x,a,\frac{r}{b}),N(y,y,a\frac{s}{b}),N(z,z,a\frac{t}{b}))\le N(x,y,z,r+s+t)$,

(NF-4) $N(x,y,z,\cdot):(0,\infty)\to[0,1]$ is continuous function.

Definition 1.6:-A mapping $\phi:[0,1]\to[0,1]$ is called an altering distance function if

(i) ϕ is strictly decreasing and left continuous.

(ii) $\phi(\lambda)=0$ if and only if $\lambda=1$

i.e, $\lim_{\phi\to 1-}\phi(1)=0$.

Definition 1.7:- Let p and q be two self mappings on a N-fuzzy *b*-metric space $(X,N,*)$ and $\{x_n\}$ be a sequence in X. p is said to be asymptotically regular at a point $x_n\in X$ if

$\left(\lim_{n\to\infty}N(p^n(x_0),p^n(x_0),p^{n+1}(x_0),\frac{t}{b})\right)=1,\ \forall t>0,b\ge 1.$

Also the sequence $\{x_n\}$ is said to be asymptotically regular with respect to the pair (p,q) if

$\left(\lim_{n\to\infty}N(p(x_n),p(x_n),q(x_n),\frac{t}{b})\right)=1,\quad \forall t>0,\ b\ge 1.$

Definition 1.8:-Two self mapping p and q be on a N-fuzzy *b*-metric space $(X,N,*)$are said to be compatible if

$\lim_{n\to\infty}N(pq(x_n),pq(x_n),qp(x_n),\frac{t}{b})=1,\ \forall t>0,b\ge 1.$

where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}p(x_n)=\lim_{n\to\infty}q(x_n)=x,$ for some $x\in X$.

Definition 1.9:- A function $f:\mathbb{R}\to\mathbb{R}$ is called *b*-non-decreasing if $x>by$ implies $f(x)\ge f(y)$ *for all* $x,y\in\mathbb{R}$.

2. MAIN RESULT:

Theorem 2.1:-Let $(X,N,*)$be a complete fuzzy *b*-metric space, ϕ be the altering distance function and $p:X\to X$ be such that the following condition is satisfied:

$\phi(N(p(x), p(x), p(y), \frac{t}{b}) \leq$
$b_1(x,y)\theta[\min\{\phi(N(x,x,p(x),\frac{t}{b})), \phi(N(y,y,p(y),\frac{t}{b}))\}] +$
$b_2(x,y)\psi[\min\{\phi(N(x,x,p(x),\frac{t}{b})), \phi(N(y,y,p(y),\frac{t}{b}))\}] +$
$b_3(x,y)\phi(N(x,x,y,\frac{t}{b})) +$
$b_4(x,y)\left(\phi(N(x,x,p(x),\frac{t}{b})) + \phi(N(y,y,p(y),\frac{t}{b}))\right) +$
$b_5(x,y)\left(\phi(N(x,x,p(y),\frac{t}{b})) + \phi(N(p(x),p(x),y,\frac{t}{b}))\right)$

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$\forall x, y \in X, b \geq 1$ and $t > 0$ where $b_i : X \times X \to [0,\infty),\ i = 1,2,3,4,5$ are such that for some arbitrary fixed $\lambda > 0,\ \lambda_1 > 0$ and $0 < \lambda_2 < 1$

$b_1(x,y) + b_2(x,y) \leq \lambda_1$

$b_3(x,y) + b_4(x,y) + 2b_5(x,y) \leq \lambda_2$

And $\theta, \psi : R^+ \to R^+$ are continuous functions at $\theta(0) = \psi(0) = 0$.

If p is asymptotically regular at some point $x_0 \in X$, then p has a unique fixed point in X.

Proof: Suppose that $\{x_n\}$ is a sequence in *X* where $x_0 \in X$ and $x_{n+1} = p(x)\ \forall\ n \geq 0$, Now if for some $n \geq 0,\ x_n = x_{n+1}$, then x_n is a fixed point of *f*. Suppose that $x_n \neq x_{n+1}\ \ \forall\ n$. We show that the sequence $\{x_n\}$ is Cauchy.

Suppose to the contrary $\exists\ 0 < \varepsilon < 1,\ t > 0,\ b > 1$ and two sequence of integers $\{r_n\}$ and $\{s_n\}$ such that $r_n > s_n > n$,

$$N\left(x_{r_n}, x_{r_n}, x_{s_n}, \tfrac{t}{b}\right) \leq 1 - \varepsilon$$
$$N\left(x_{r_{n-1}}, x_{r_{n-1}}, x_{s_{n-1}}, \tfrac{t}{b}\right) > 1 - \varepsilon$$
$$N\left(x_{r_{n-1}}, x_{r_{n-1}}, x_{s_n}, \tfrac{t}{b}\right) > 1 - \varepsilon,\ \ \forall\ n \in \mathbb{N} \cup \{0\} \tag{4}$$

Now, we have

$$1 - \varepsilon \geq N\left(x_{r_n}, x_{r_n}, x_{s_n}, \tfrac{t}{b}\right) \geq N\left(x_{r_n}, x_{r_n}, x_{r_{n-1}}, \frac{t}{3b}\right) * N\left(x_{r_n}, x_{r_n}, x_{r_{n-1}}, \frac{t}{3b}\right) * N\left(x_{s_n}, x_{s_n}, x_{r_{n-1}}, \frac{t}{3b}\right)$$

$$\Rightarrow\ 1 - \varepsilon \geq \lim_{n\to\infty} N\left(x_{r_n}, x_{r_n}, x_{s_n}, \tfrac{t}{b}\right) \geq (1 * 1 * 1 - \varepsilon)$$

(Since f isasymptotically regular at x_0)

$$\Rightarrow\ \lim_{n\to\infty} N\left(x_{r_n}, x_{r_n}, x_{s_n}, \tfrac{t}{b}\right) = 1 - \varepsilon \tag{5}$$

Again, $N\left(x_{r_n}, x_{r_n}, x_{s_{n-1}}, \tfrac{t}{b}\right) \geq N\left(x_{r_n}, x_{r_n}, x_{s_n}, \frac{t}{3b}\right) * N\left(x_{r_n}, x_{r_n}, x_{s_n}, \frac{t}{3b}\right) * N\left(x_{s_{n-1}}, x_{s_{n-1}}, x_{s_n}, \frac{t}{3b}\right)$

$$\Rightarrow\ \lim_{n\to\infty} N\left(x_{r_n}, x_{r_n}, x_{s_{n-1}}, \tfrac{t}{b}\right) > 1 - \varepsilon \tag{6}$$

Taking $\mathrm{x} = x_{r_{n-1}}$ and $\mathrm{y} = x_{s_{n-1}}$ in (1), we have

$$\phi\left(N\left(x_{r_n}, x_{r_n}, x_{s_n}, \tfrac{t}{b}\right)\right) \leq \mathrm{b}_1(\mathrm{x},\mathrm{y})\theta\left(\min\left\{\phi\left(\mathrm{N}\left(\mathrm{x}_{\mathrm{r}_{n-1}}, \mathrm{x}_{\mathrm{r}_{n-1}}, \mathrm{x}_{\mathrm{r}_n}, \tfrac{t}{b}\right)\right)\phi\left(\mathrm{N}\left(\mathrm{x}_{\mathrm{s}_{n-1}}, \mathrm{x}_{\mathrm{s}_{n-1}}, \mathrm{x}_{\mathrm{s}_n}, \tfrac{t}{b}\right)\right)\right\}\right)$$
$$+\mathrm{b}_2(\mathrm{x},\mathrm{y})\psi\left(\min\{\phi\left(\mathrm{N}\left(\mathrm{x}_{\mathrm{r}_{n-1}}, \mathrm{x}_{\mathrm{r}_{n-1}}, \mathrm{x}_{\mathrm{r}_n}, \tfrac{t}{b}\right)\right), \phi\left(\mathrm{N}\left(\mathrm{x}_{\mathrm{s}_{n-1}}, \mathrm{x}_{\mathrm{s}_{n-1}}, \mathrm{x}_{\mathrm{s}_n}, \tfrac{t}{b}\right)\right)\}\right)$$

$$+b_3(x,y)\phi\left(N\left(x_{r_{n-1}},x_{r_{n-1}},x_{s_{n-1}},\tfrac{t}{b}\right)\right)+b_4(x,y)\left[\phi\left(N\left(x_{r_{n-1}},x_{r_{n-1}},x_{r_n},\tfrac{t}{b}\right)\right)+\phi\left(N\left(x_{s_{n-1}},x_{s_{n-1}},x_{s_n},\tfrac{t}{b}\right)\right)\right]$$

$$+b_5(x,y)\left[\phi\left(N\left(x_{r_{n-1}},x_{r_{n-1}},x_{s_n},\tfrac{t}{b}\right)\right)+\phi\left(N\left(x_{r_n},x_{r_n},x_{s_{n-1}},\tfrac{t}{b}\right)\right)\right]$$

Taking $n\to\infty$ and by (4), (5), (6) and using the fact that p is asymptotically regular at x_0 we have, $\phi(1-\varepsilon)\le b_3(x,y)\phi(1-\varepsilon)+2b_5(x,y)\phi(1-\varepsilon)<\phi(1-\varepsilon)$ which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since $(X,N,*)$ is a complete N-fuzzy *b*-metric space, $\exists\, z\in X$ such that $x_n\to z$.

Now,

$$\phi\left(N\left(p(x_n),\left(p(x_n),p(z),\tfrac{t}{b}\right)\right)\le b_1(x,y)\theta\left(\min\left\{\phi\left(N\left(x_n,x_n,x_{n+1},\tfrac{t}{b}\right)\right),\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)\right\}\right)+$$

$$b_2(x,y)\psi\left(\min\left\{\phi\left(N\left(x_n,x_n,x_{n+1},\tfrac{t}{b}\right)\right),\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)\right\}\right)\ b_3(x,y)\phi\left(N\left(x_n,x_n,z,\tfrac{t}{b}\right)\right)+$$

$$b_4(x,y)\left[\phi\left(N\left(x_n,x_n,x_{n+1},\tfrac{t}{b}\right)\right)+\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)\right]+$$

$$b_5(x,y)\left[\phi\left(N\left(x_n,x_n,p(z),\tfrac{t}{b}\right)\right)+\phi\left(N\left(z,z,x_{n+1},\tfrac{t}{b}\right)\right)\right]$$

For $n\to\infty$

$$\lim_{n\to\infty}\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)\le\left[b_4(x,y)+b_5(x,y)\right]\lim_{n\to\infty}\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)$$

$$\Rightarrow\left[1-b_4(x,y)-b_5(x,y)\right]\lim_{n\to\infty}\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)\le 0$$

$$\Rightarrow\lim_{n\to\infty}\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)=0$$

(Since $0<b_3(x,y)+b_4(x,y)+2b_5(x,y)<1$)

$\Rightarrow\ p(z)=z$.

If u is another fixed point of f in X, then

$$\phi\left(N,p(u),p(u),p(z),\tfrac{t}{b}\right)\le b_1(x,y)\theta\left(\min\left\{\phi\left(N\left(u,u,p(u),\tfrac{t}{b}\right)\right),\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)\right\}\right)$$
$$+b_2(x,y)\psi\left(\min\left\{\phi\left(N\left(u,u,p(u),\tfrac{t}{b}\right)\right),\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)\right\}\right)$$
$$+b_3(x,y)\left(\phi\left(N\left(u,u,z,\tfrac{t}{b}\right)\right)\right.$$
$$+b_4(x,y)\left[\phi\left(N\left(u,u,p(u),\tfrac{t}{b}\right)\right)+\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)\right]$$
$$+b_5(x,y)\left[\left(\phi\left(N\left(u,u,p(z),\tfrac{t}{b}\right)\right)+\phi\left(N\left(z,z,p(u),\tfrac{t}{b}\right)\right)\right]\right.$$

$$\Rightarrow\phi\left(N\left(u,u,z,\tfrac{t}{b}\right)\right)\le b_3(x,y)\phi\left(N\left(u,u,z,\tfrac{t}{b}\right)\right)+2b_5(x,y)\phi\left(N\left(u,u,z,\tfrac{t}{b}\right)\right)$$

$$\Rightarrow\left[1-b_3(x,y)-2b_5(x,y)\right]\phi\left(N\left(u,u,z,\tfrac{t}{b}\right)\right)\le 0$$

$$\Rightarrow\phi\left(N\left(u,u,z,\tfrac{t}{b}\right)\right)=0$$

(Since $0<b_3(x,y)+b_4(x,y)+2b_5(x,y)<1$)$\Rightarrow u=z$,

COROLLARY: Let $p,q:X\to X$ be mapping on a complete fuzzy *b*-metric space $(X,N,*)$ and ϕ be the altering distance function. Let p and q be asymptotically regular at a point $x_0 \in X$ and $b>1$ and both satisfy the inequality (1). Moreover, if

$$\phi\left(N\left(p(x),p(x),q(y),\tfrac{t}{b}\right)\right)\le k\left(\phi\left(N\left(x,x,y,\tfrac{t}{b}\right)\right)+\phi\left(N\left(x,x,p(x),\tfrac{t}{b}\right)\right)+\phi\left(N\left(y,y,q(y),\tfrac{t}{b}\right)\right)\right) \tag{7}$$

where $0<k<1$ and $x,y\in X$,
Then p and q have a unique common fixed point in X.

Proof: From theorem [2.1], both f and g have unique fixed points say, z and v respectively.
Since f and g satisfy (7),

$$\phi\left(N\left(p(z),p(z),q(v),\tfrac{t}{b}\right)\right)\le k\left(\phi\left(N\left(z,z,v,\tfrac{t}{b}\right)\right)+\phi\left(N\left(z,z,p(z),\tfrac{t}{b}\right)\right)+\phi\left(N\left(v,v,q(v),\tfrac{t}{b}\right)\right)\right),$$

$$\Rightarrow \left(N\left(z,z,v,\tfrac{t}{b}\right)\right)\le k\left(\phi\left(N\left(z,z,v,\tfrac{t}{b}\right)\right)+\phi\left(N\left(z,z,z,\tfrac{t}{b}\right)\right)+\phi\left(N\left(v,v,v,\tfrac{t}{b}\right)\right)\right)$$

$$\Rightarrow (1-k)\phi\left(N\left(z,z,v,\tfrac{t}{b}\right)\right)\le 0$$

$$\Rightarrow \phi\left(N\left(z,z,v,\tfrac{t}{b}\right)\right)\le 0 \qquad \text{(Since } k<1)$$

$\Rightarrow z=v$ i.e., p and q have a unique common fixed point.

Theorem 2.2:-Let $p:X\to X$ be a mapping on a complete fuzzy b-metric space $(X,N,*)$ and ϕ is the altering distance function. If f is asymptotically regular at a point $x_0\in X$ and f satisfies,

$$\begin{aligned}&\phi(N(f(x),f(x),f(y),t))\le\\ &h_1\min\{\phi(N(x,x,y,\tfrac{t}{b})),\phi(N(f(x),(f(x),x,\tfrac{t}{b})),\phi(N(f(x),(f(x),y,\tfrac{t}{b}))\}+\\ &h_2\min\{\phi(N(x,x,y,\tfrac{t}{b})),\phi(N(f(y),(f(y),y,\tfrac{t}{b})),\phi(N(x,x,f(y),\tfrac{t}{b}))\}\end{aligned} \tag{8}$$

For all $x,y\in X, b\ge 1$ and $t>0$ where $h_1,h_2<1$,
Then f has a unique fixed point in X.

Proof: As theorem 2.1, we construct a sequence $\{x_n\}$ in X by $x_{n+1}=f(x_n)\ \forall\ n\in\mathbb{N}\cup\{0\}$, where $x_0\in X$. If there exists n with $x_n=x_{n+1}$, then x_n is a fixed point of f. Suppose that $x_n\ne x_{n+1}$ for all n.
To show that $\{x_n\}$ is a Cauchy sequence.
Let $m,n\in\mathbb{N}\cup\{0\}$. From (8)

$$\begin{aligned}&\phi\left(N\left(p(x_n),p(x_n),p(x_m),\tfrac{t}{b}\right)\right)\\ &\le h_1\min\left\{\phi\left(N\left(x_n,x_n,x_m,\tfrac{t}{b}\right)\right),\phi\left(N\left(p(x_n),p(x_n),x_n,\tfrac{t}{b}\right)\right),\phi\left(N\left(p(x_n),p(x_n),x_m,\tfrac{t}{b}\right)\right)\right\}\\ &+h_2\min\left\{\phi\left(N\left(x_n,x_n,x_m,\tfrac{t}{b}\right)\right),\phi\left(N\left(p(x_m),p(x_m),x_m,\tfrac{t}{b}\right)\right),\phi\left(N\left(x_n,x_n,p(x_m),\tfrac{t}{b}\right)\right)\right\}\end{aligned}$$

Since, f is asymptotically regular at $x_0\in X$, taking $n,m\to\infty$

$$\lim_{n,m\to\infty}\phi(N(p(x_n),p(x_n),p(x_n),\tfrac{t}{b}))=0$$

$$\Rightarrow \lim_{n,m\to\infty}N(p(x_n),p(x_n),p(x_m),\tfrac{t}{b})=1.$$

i.e. $\{x_n\}$ is a Cauchy sequence in $(X,N,*)$

since $(X,N,*)$ is a complete, therefore $x_n \to z$ (say) in X .
using (8)
$\phi(N(x_{n+1},x_{n+1},p(z),\frac{t}{b}))=$
$\phi(N(p(x_n),p(x_n),p(z),\frac{t}{b}))\le$
$h_1 \min\{\phi(N(x_n,x_n,z,\frac{t}{b})),\phi(N(p(x_n),p(x_n),x_n,\frac{t}{b})),\phi(N(p(x_n),p(x_n),z,\frac{t}{b}))$
$+h_2 \min\{\phi(N(x_n,x_n,z,\frac{t}{b})),\phi(N(p(z),p(z),z,\frac{t}{b})),\phi(N(x_n,x_n,p(z),\frac{t}{b}))\}$
$\Rightarrow \phi(N(z,z,p(z),\frac{t}{b}))=0$
$\Rightarrow p(z)=z,$ establishes that z is a fixed point for p.
Uniqueness can be shown easily.
Hence, z is the unique fixed point of p.

Theorem 2.3:- Let $(X,N,*)$ be fuzzy b-metric space, ϕ be the altering distance function and p and q be two commutative self-mappings on X such that

$$\begin{aligned}&\phi(N(p(x),p(x),p(y),t))\le\\&k_1[\phi(N(q(x),q(x),q(y),\tfrac{t}{b}))]+\\&k_2[\phi(N(q(x),q(x),q(y),\tfrac{t}{b}))+\phi(N(q(y),q(y),q(y),\tfrac{t}{b}))]\end{aligned} \tag{9}$$

where $x,y\in X$, $t>0$, $b\ge 1$ and $k_1:\square\to[0,1)$, $0<k_1,k_2<1.$ moreover if
(i) q are asymptotically regular at x_0.
(ii) $X\subseteq q(X)$,
(iii) X or $q(X)$ is a complete subspace of X,
then p and q have a unique common fixed point.

Proof: Let $x_0\in X$. since $p(X)\subseteq q(X)$, define a sequence $\{u_n\}$ by $u_{n+1}=p(x_n)=q(x_{n+1})$, $n\in\square\cup\{0\}$.
Again since p and q are asymptotically regular at x_0 ,

$$\lim_{n\to\infty}\phi(N(u_n,u_n,u_{n+1},\tfrac{t}{b}))=0 \tag{10}$$

To show that the sequence $\{u_n\}$ is Cauchy.
Suppose, there exist $0<\varepsilon<1, b\ge 1$ and two sequence of integers $\{r_n\}$ and $\{s_n\}$ such that $r_n>s_n>n$,

$$\begin{aligned}&N(u_{r_n},u_{r_n},u_{s_n},\tfrac{t}{b})\le 1-\varepsilon,\\&N(u_{r_{n-1}},u_{r_{n-1}},u_{s_{n-1}},\tfrac{t}{b})>1-\varepsilon,\\&N(u_{r_{n-1}},u_{r_{n-1}},u_{s_n},\tfrac{t}{b})>1-\varepsilon,\forall n\in\square\cup\{0\}\end{aligned} \tag{11}$$

Following the technique applied in theorem 2.1 we can show that

$$\lim_{n\to\infty}N(u_{r_n},u_{r_n},u_{s_n},\tfrac{t}{b})>1-\varepsilon,\ t>0,\ b\ge 1 \tag{12}$$

$$\begin{aligned}\phi(N(u_{r_{n+1}},u_{r_{n+1}},u_{s_{n+1}},\tfrac{t}{b}))&=\phi(N(p(x_{r_n}),p(x_{r_n}),p(x_{s_n}),\tfrac{t}{b}))\\&\le k_1[\phi(N(q(x_{r_n}),q(x_{r_n}),q(x_{s_n}),\tfrac{t}{b}))\\&\quad+k_2(\phi(N(q(x_{r_n}),q(x_{r_n}),p(x_{s_n}),\tfrac{t}{b}))\\&\quad+\phi(N(q(x_{r_n}),q(x_{r_n}),p(x_{s_n}),\tfrac{t}{b}))]\end{aligned}$$

Taking $n\to\infty$ and using (10) and (12) we have

$\phi(1-\varepsilon) \le k_1\phi\left(1-\varepsilon\right) < \phi\left(1-\varepsilon\right)$

is a contradiction. Hence $\{u_n\}$ is a Cauchy sequence.

Suppose that $q(X)$ is complete, then there exist $v \in q(X)$ such that $\lim_{n\to\infty} u_n = v.$ Also, for some $z \in X$ we have $q(z) = v.$

Now,

$$\begin{aligned}\phi(N(p(z), p(z), u_{n+1}, \tfrac{t}{b})) = \phi(N(p(z), p(z), p(x_n), \tfrac{t}{b}))\\ \le k_1[\phi(N(q(z), q(z), q(x_n), \tfrac{t}{b})\\ + k_2(\phi(N(p(z), p(z), q(z), \tfrac{t}{b}))\\ + \phi(N(p(x_n), p(x_n), q(x_n), \tfrac{t}{b})))]\end{aligned}$$

For $n \to \infty$,

$\phi(N(p(z), p(z), v, \tfrac{t}{b})) \le k_1[k_2\phi(N(p(z), p(z), v, \tfrac{t}{b}))]$

$\Rightarrow (1-k_1k_2)\phi(N(p(z), p(z), v, \tfrac{t}{b})) = 0$

$\Rightarrow \phi(N(p(z), p(z), v, \tfrac{t}{b})) = 0$

$\Rightarrow p(z) = v$

Therefore $p(z) = v = q(z)$ i.e. z is the coincident point of p and q.

Next, from (9),

$$\begin{aligned}\phi(N(p(p(z)), p(p(z)), p(z), \tfrac{t}{b})) \le k_1[\phi(N(q(p(z)), q(p(z)), p(z), \tfrac{t}{b})\\ + k_2(\phi(N(p(p(z)), p(p(z)), q(p(z)), \tfrac{t}{b}))\\ + \phi(N(p(z), p(z), q(z), \tfrac{t}{b})))]\end{aligned}$$

$= k_1[\phi(N(p(q(z)), p(q(z)), q(z), \tfrac{t}{b}))$

$+k_2(\phi(N(p(p(z)), p(p(z)), p(q(z)), \tfrac{t}{b}))$

$+\phi(N(p(z), p(z), q(z), \tfrac{t}{b})))]$

(Since $pq = qp$)

$k_1[\phi(N(p(p(z)), p(p(z)), p(z), \tfrac{t}{b})$

$+k_2(\phi(N(p(p(z)), p(p(z)), p(p(z)), \tfrac{t}{b}))$

$+\phi(N(p(z), p(z), q(z), \tfrac{t}{b})))]$

$= k_1\phi(N(p(p(z)), p(p(z)), p(z), \tfrac{t}{b}))$

$\Rightarrow (1-k_1)\phi(N(p(p(z)), p(p(z)), p(z), \tfrac{t}{b}) = 0$

$\Rightarrow \phi(N(p(p(z)), p(p(z)), p(z), \tfrac{t}{b}) = 0$

$\Rightarrow p(p(z)) = p(z) = v$

Similarly, $q(q(z)) = q(z) = v.$

Hence v is a common fixed point of p and q. If v_1 is another common point of p and q, then

$$\begin{aligned}\phi(N(p(v), p(v), p(v_1), \tfrac{t}{b})) \le k_1[\phi(N(q(v), q(v), q(v_1), \tfrac{t}{b})\\ + k_2(\phi(N(p(v), p(v)), q(v), \tfrac{t}{b}))\\ + \phi(N(p(v_1), p(v_1), q(v_1), \tfrac{t}{b})))]\end{aligned}$$

$$\begin{aligned}\phi(N(v, v, v_1, \tfrac{t}{b})) \le k_1[\phi(N(v, v, v_1, \tfrac{t}{b})\\ + k_2(\phi(N(v, v, v, \tfrac{t}{b}))\\ + \phi(N(v_1, v_1, v_1, \tfrac{t}{b})))]\end{aligned}$$

$\Rightarrow (1-k_1)\phi(N(v,v,v_1,\frac{t}{b}))=0$

$\Rightarrow v = v_1$

Hence p and q have a unique common fixed point in X.

REFERENCES:

1) Bakthin, I.A. (1989). The contraction principle in quasimetric spaces, Funct. Anal. 30, 26-37
2) Banach, S. (1922) Sur les operations dans les ensembles abstraits et leur application aux equations integrals. Fundam. Math. 3, 133-181.
3) Czerwik, S.(1993). Contraction mappings in *b*-metric spaces. Acta Math. Inform. Univ. Ostav.1(1), 5-11
4) Dhage, B.C. (1992). Generalised metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc., 84(4), pp. 329–336.
5) Gahlers, S. (1963). 2-metrische Raume and ihretopologische structure, Math. Nachr, 26, 115-148.
6) Gahlers, S. (1966). Zurgeomrtric 2-metrische Raume, Revue Roumaine Math. Pures Appl., 11 665-667.
7) George, A. and Veeramani, P. (1994). On some results in fuzzy metric spaces. Fuzzy Sets Syst. 64, 395-399.
8) Goswami, N. and Patir, B. (2019). Fixed point theorems for asymptotically regular mappings in fuzzy metric Spaces, Korean J Math 27, No.4, pp861-877.
9) Hassanzadeh, Z., Sedghi, S.(2018). Relation between *b*-metric and fuzzy metric spaces. Math. Morav. 22(1), 55-63.
10) Kramosil, O. and Michalek, J. (1975). Fuzzy metric and statistical metric spaces. Kybernetika 11, 336-344.
11) Kumar, S. (2008). Common fixed point theorem in fuzzy 2-metric spaces, Uni.Din. Bacau. Studii Si CercetiriSciintice, Serial: Mathematical, 18, 111-116.
12) Malviya, N. (2015). The N-fuzzy metric space and mappings with application DOI:1515/fascmath -0019.
13) Mustafa, Z. and Sims, B. (2003). Some remarks concerning D-metric space, International Conference on Fixed Point Theory and Applications, Valencia, Spain, 189-198.
14) Mustafa, Z. and Sims, B. (2006). A new approach to generalized metric spaces. J. Nonlinear Convex Anal. 7(2), 289-297.
15) Rao, K.P.R., Altun, I. and Bindu S.H.(2011). Common coupled fixed point theorems in generalized fuzzy metric spaces, Advances in Fuzzy Systems, Article ID 986748, 6 pages.
16) Sedghi, S., Shobe, N. and Zhou, H. (2007). A Common fixed point theorem in D*-fuzzy metric space, Fixed Point Theory and Applications, Article ID 27906, 13 pages.
17) Sedghi, S. and Shobe, N. (2007). A Common fixed point theorem in two M-fuzzy metric space, Commun, Koreaan. Math. Soc., 22(4), 513-526.
18) Sedghi, S. and Shobe, N. (2012). A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64(3), 258-266.

19) Sedghi, S. and Shobe, N. (2012). Common fixed point theorem in b-fuzzy mtric space. Nonlinear Funct. Anal. Appl. 17(3), 349-359.

20) Sedghi, S. and Shobe, N. (2014). Common fixed point theorem for R-weakly commuting maps in *b*-fuzzy mtric space. Nonlinear Funct. Anal. Appl. 19(2), 285-295.

21) Sharma, S. (2002). On fuzzy metric space, Southeast Asian Bulletin of Mathematics, 26, 133-145.

22) Singh, B., Chauhan, M.S. (1997). Generalized fuzzy metric spaces and fixed point theorems, Bull, Caal. Math. Soc., 89, 457-460.

23) Sun, G. and Yang, K. (2010). Generalized fuzzy metric spaces with properties. Research Journal of Applied Sciences, Engineering and Technology, 2(7), 673-678.

24) Zadeh, L.A. (1965). Fuzzy sets. Inf. Control 8, 338-353.