On Weakly $\bar{\phi}$ −Ricci Symmetric Lightlike Hypersurfaces of Indefinite Cosymplectic Manifolds

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Abstract: The main motive of this paper is to study weakly $\bar{\varphi}$-Ricci symmetric lightlike hypersurface $(W\,\bar{\varphi}RS) - LH$ of an indefinite cosymplectic manifold $(\bar{M}, g)$ of constant curvature -1. In this paper, we acquire a relationship between 1- forms of $(W\,\bar{\varphi}RS) - LH$ of $(\bar{M}, g)$. We study η-Einstein weakly $\bar{\varphi}$- Ricci symmetric lightlike hypersurface $(W\,\bar{\varphi}RS) - LH$ of an indefinite cosymplectic manifold $(\bar{M}, g)$. Finally it is shown that the Ricci tensor S of $(\bar{M}, g)$is satisfying cyclic parallel and Codazzitype properties of Ricci tensor.

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1. Introduction

Many authors have explored the idea of φ-symmetry on both complex and contact geometry of manifolds. Locally φ-symmetric Sasakian manifolds were introduced by [**18**] as a weaker form of locally symmetric manifolds. Some examples on φ-symmetric Kenmotsu manifolds were examined by [**5**]. Later [**6**] proposed the concept of φ-Ricci symmetric Sasakian manifolds and traced out some interesting results. Authors in [**24**] studied φ-Ricci symmetric on Kenmotsu manifolds and verified its existence with some examples. In addition authors [**19**, **20**]introduced the concept of weakly symmetric and weakly Ricci symmetric manifolds as generalizations of Chaki's pseudo-symmetric and pseudo-Ricci symmetric manifolds.

Definition 1. For Levi-Civita connection$\bar{D}$, Riemannian metric g and an associated 1-forms $\ddot{\alpha}$, $\ddot{\beta}$, $\ddot{\gamma}$ if forall F, J, $L \in \Gamma(TM)$, the Ricci tensor S satisfies

(1) $$(\bar{D}_F S)(J, L) = \ddot{\alpha}(F)S(J, L) + \ddot{\beta}(J)S(F, L) + \ddot{\gamma}(L)S(J, F)$$

Thenthe non-flat Riemannian manifold is called weakly Ricci symmetric [**15**]. Where $\ddot{\alpha}(F)$=$g(F, \ddot{\rho}\})$, $\ddot{\beta}(J)$=$g(J, \ddot{\delta})$ and $\ddot{\gamma}(L)$=$g(L, \ddot{\kappa})$, corresponding to 1-forms ($\ddot{\alpha}$,$\ddot{\beta}$, $\ddot{\gamma}$)and, ($\ddot{\rho}$, $\ddot{\delta}$,$\ddot{\kappa}$) are associated vector fields respectively.

Also on Kenmotsu manifolds $\bar{M}$ ($n \geq 3$), authors [**17**] generally introduced representation of φ-Ricci symmetries on Kenmotsu manifolds. Then $\forall$ F, J, $L$the Riemannian manifold $\bar{M}$ satisfying

(2) $$\varphi^2(D_F Q)(J) = A(F)Q(J) + B(J)Q(F) + g(QF, J)\ddot{\rho}$$

is known to be as φ-Ricci symmetric. Where Q, $(A, B)$are Ricci operator and not simultaneously zero $1-$forms, such that $g(F, \ddot{\rho}) = D(F)$, fromequation (2), if

(3) $$\varphi^2(D_F Q)(J) = 0$$

then, $\overline{M}$ is locally φ-Ricci symmetric [**17**] of dimension ≥ 3. Motivated by these authors we study weakly φ-Ricci symmetric lightlike hypersurfaces of Indefnite cosymplectic manifolds of constant curvature -1.

Duggal-Bejancu **10** introduced lightlike geometry of semi-Riemannian manifolds and is completely different from Riemannian and semi-Riemannian one. To overcome this difficulty arisen due to degenerate metric authors obtained transversal bundle for such hypersurfaces. After **10** researchers across the globe studied lightlike hypersurface of manifolds by following Duggal-Bejancu approach. For degenerate hypersurfaces of manifolds we refer ([**12**], [**13**]).

In this paper, we have studied the effect of $(\boldsymbol{W\,\varphi - RSLH})$ on anindefinite cosymplectic manifolds $(\overline{\boldsymbol{M}}, \boldsymbol{g})$. In section 2, we will provide some basic concepts and terminologies used in lightlike geometry and cosymplectic manifolds. In section 3,we study $(\boldsymbol{W\,\varphi - RSLH})$ of an indefnite cosymplectic manifolds $(\overline{\boldsymbol{M}}, \boldsymbol{g})$ to obtain some results.

2. Preliminaries

Let $(\overline{\boldsymbol{M}}, \boldsymbol{g})$ be a (2n+1) dimensional differentiable manifold endowed with $(\overline{\boldsymbol{\varphi}},\ \boldsymbol{\xi},\ \boldsymbol{\eta})$ as almost contact structure. Where $\overline{\boldsymbol{\varphi}}$ is a (1-1) type tensor field, $\boldsymbol{\eta}$ is 1-form and $\boldsymbol{\xi}$ represents an associated vector field satisfying

(4) $$\overline{\boldsymbol{\varphi}}^2 = -\boldsymbol{I} + \boldsymbol{\eta} \otimes \boldsymbol{\xi}, \qquad \boldsymbol{\eta}(\boldsymbol{\xi}) = \mathbf{1}, \boldsymbol{\eta}(\overline{\boldsymbol{\varphi}}) = \mathbf{0}, \overline{\boldsymbol{\varphi}}\boldsymbol{\xi} = \mathbf{0}.$$

For any F, Jon $(\overline{M}, g)$, if the following condition is satisfied

(5) $$\boldsymbol{g}(\overline{\boldsymbol{\varphi}}\boldsymbol{F}, \overline{\boldsymbol{\varphi}}\boldsymbol{J}) = \boldsymbol{g}(\boldsymbol{F}, \boldsymbol{J}) - \boldsymbol{\eta}(\boldsymbol{F})\boldsymbol{\eta}(\boldsymbol{J})$$

then the structure $(\overline{\varphi}, \xi, \eta, g)$ is as almost contact metric structure and the manifold with this structure is known to be as almost contact metric manifold, where g is Riemannian metric. From equation (5) we acquire

$$\eta(F) = g(F, \xi).$$

Moreover, for Riemannian metric g and Livi-Civita connectionif$(\overline{D}_F\overline{\varphi})J = 0$ and $\overline{D}_F\xi = 0$, then we call $(\overline{M}, g)$, as an indefnite cosymplectic manifold [**13**].

A plane section κin a tangent space of indefinite cosymplectic manifold $(\overline{M}, g)$ if spanned by F orthogonal to $\overline{\varphi}F$ and $\xi$is called $\overline{\varphi}$-section. Where F unit tangent is vector and is non-null vector field on an indefnite cosymplectic manifold $(\overline{M}, g)$. $\overline{\varphi}$-sectional curvature is the sectional curvature with respect to κ determined by non-null vector field F. If $\overline{\varphi}$-sectional curvature c at each point in $(\overline{M}, g)$ does not depend on $\overline{\varphi}$-section, then c is constant and $(\overline{M}, g)$ is known to be an indefnite cosymplectic space form denoted by $\overline{M}(c)$. Therefore curvature tensor of indefnite cosymplectic space form $\overline{M}(c)$ is given [**13**] by

(6) $$\boldsymbol{R}(\boldsymbol{F}, \boldsymbol{J}, \boldsymbol{L}) = \frac{\boldsymbol{c}}{\mathbf{4}}\{\boldsymbol{g}(\boldsymbol{J}, \boldsymbol{L})\boldsymbol{F} - \boldsymbol{g}(\boldsymbol{F}, \boldsymbol{L})\boldsymbol{J} + \boldsymbol{\eta}(\boldsymbol{F})\boldsymbol{\eta}(\boldsymbol{L})\boldsymbol{J} - \boldsymbol{\eta}(\boldsymbol{J})\boldsymbol{\eta}(\boldsymbol{L})\boldsymbol{F} + \boldsymbol{g}(\boldsymbol{F}, \boldsymbol{L})\boldsymbol{\eta}(\boldsymbol{J})\boldsymbol{\xi} - \boldsymbol{g}(\boldsymbol{J}, \boldsymbol{L})\boldsymbol{\eta}(\boldsymbol{F})\boldsymbol{\xi} - \boldsymbol{g}(\overline{\boldsymbol{\varphi}}\boldsymbol{J}, \boldsymbol{L})\overline{\boldsymbol{\varphi}}\boldsymbol{F} - \boldsymbol{g}(\overline{\boldsymbol{\varphi}}\boldsymbol{F}, \boldsymbol{L})\overline{\boldsymbol{\varphi}}\boldsymbol{J} - \mathbf{2}\boldsymbol{g}(\overline{\boldsymbol{\varphi}}\boldsymbol{F}, \boldsymbol{J})\overline{\boldsymbol{\varphi}}\boldsymbol{L}\}$$

For all $\boldsymbol{F},\ \boldsymbol{J},\ \boldsymbol{L} \in \boldsymbol{\Gamma}(T\overline{\boldsymbol{M}})$.

Now let us recall some of the elementary and important terminologies about the geometry of lightlike (degenerate) hypersurfaces of semi-Riemannian manifolds.

Assume that $(\boldsymbol{M}, \overline{\boldsymbol{g}}, \boldsymbol{S}(\boldsymbol{TM}))$ to be a null hypersurface of $(\overline{\boldsymbol{M}}, \boldsymbol{g})$. Then over $(\boldsymbol{M}, \overline{\boldsymbol{g}}, \boldsymbol{S}(\boldsymbol{TM}))$, there exists $\boldsymbol{tr}(\boldsymbol{TM})$ a rank 1 unique vector bundle in such a way that for any $\boldsymbol{Z}$ of $\boldsymbol{TM}^{\perp}$ on $\boldsymbol{Y} \subset \boldsymbol{M}$, there exists $\boldsymbol{X}$ a unique section of $\boldsymbol{tr}(\boldsymbol{TM})$ on the coordinate neighbourhood $\boldsymbol{Y}$ known as null transversal vector field of hypersurface $(\boldsymbol{M}, \overline{\boldsymbol{g}}, \boldsymbol{S}(\boldsymbol{TM}))$. Such that

(7) $$\overline{\boldsymbol{g}}(\boldsymbol{Z}, \boldsymbol{X}) = \mathbf{1}, \qquad \overline{\boldsymbol{g}}(\boldsymbol{X}, \boldsymbol{X}) = \overline{\boldsymbol{g}}(\boldsymbol{X}, \boldsymbol{O})) = \mathbf{0}$$

$\forall\ O \in \Gamma(S(TM_{|M}))$. Then tangent bundle $T\overline{M}$ is decomposed as

(8) $$T\overline{M}_{|M} = S(TM) \oplus \left(TM \perp\oplus\ tr(TM)\right),$$

(9) $$T\overline{M}_{|M} = TM \oplus tr(TM).$$

Here $tr(TM)$ is known to be as lightlike transversal bundle of hypersurface with respect to $S(TM)$ and $tr(TM)$ is complementry but not orthogonal vector bundle to TMin $\overline{M}_{|M}$ [9].

According to equation (8) for all $F, J \in \Gamma(S(TM_{|M})$, the local Gauss and Weingarten formulas are given as

(10) $$\overline{D}_F J = D_F J + B(F, J),$$

(11) $$\overline{D}_F X = -A_X F + \tau(F) X,$$

(12) $$D_F PJ = D^*{}_F PJ + C(F, PJ)\xi,$$

(13) $$\boldsymbol{D_F Z = A^*{}_Z F + \tau(F)\, Z}$$

Here $\overline{D}$, (D, D^*) represent Livi-Civita connection of $(\overline{M}, g)$and linear connections on $(TM, S(TM))$ respectively. (B, C) and $(A_X, A^*{}_X)$ represent local fundamental forms and shape operators on TM and $\Gamma(S(TM)$respectively. Also τ and P represent 1-form and projection morphisms of $\Gamma(TM)$ on $S(TM)$ respectively. By using the fact that $B(F, J) = g\overline{D}_F J, \xi)$, we know that the local second fundamental form B is independent of choice of $S(TM)$ and hence satisfies

(14) $$\boldsymbol{B(F, Z) = 0, \quad \forall F \in \Gamma(TM).}$$

Unfortunately D on TM is not a metric connection and hence satisfies

(15) $$\boldsymbol{(D_F g)(J, L) = B(F, J)\theta(L) + B(F, L)\theta(J) \quad \vee F, J, L \in \Gamma(TM).}$$

Here θ represents 1- form defined as, $\theta(F) = g(F, X)$, for all $F \in \Gamma(TM)$. However, D* on $S(TM)$ is metric connection, and the above shape operators are related to their local second fundamental forms as

(16) $$\boldsymbol{B(F, J) = \overline{g}(A^*{}_Z F, J), \quad g(A^*{}_Z F, X) = 0}$$

(17) $$\boldsymbol{C(F, PJ) = \overline{g}(A_X F, PJ), \quad g(A_X F, X) = 0.}$$

From (16), $A^*{}_Z Z = 0$. With respect to connections $\overline{D}$ and D, the Riemannian curvature tensors of $(\overline{M}, g)$ and $(M, \overline{g}, S(TM))$ are represented by $\overline{R}$ and R respectively as given by

(18) $$\boldsymbol{g(\overline{R}(F, J)\xi, PO) = \overline{g}(R(F, J)L, PO) + B(F, L)C(J, PO) - B(J, L)C(F, PO)}$$

(19) $$\boldsymbol{\begin{aligned} g(\overline{R}(F, J)L, \xi) &= \overline{g}(R(F, J)\xi, PO) \\ &= (D_F B)(J, L) - (D_J B)(F, L) + B(J, L)\,\tau(F) - B(F, L)\tau(J) \end{aligned}}$$

(20) $$\boldsymbol{\begin{aligned} g(\overline{R}(F, J)L, X)\& &= \overline{g}(R(F, J)\xi, X) \\ &= \overline{g}(D_F(A_X J) - D_J(A_X F) - \overline{g}(A_X(F, J), L) + \overline{g}(A_X F, L)\tau(J) \\ &- \overline{g}(A_X J, L)\tau(F) + \overline{g}(A^*{}_\xi F, A_{XJ}) - \overline{g}(A^*{}_\xi J, A_X F) - 2d\tau(F, J)\theta(L) \end{aligned}}$$

(21) $$\boldsymbol{g(\overline{R}(F, J)\xi, X) = \overline{g}(R(F, J)\xi, X) = \overline{g}(A^*{}_\xi F, A\_XJ) - \overline{g}(A^*{}_\xi J, A_X F) - 2d\tau(F, J)}$$

3. Weakly φ Ricci symmetric lightlike hypersurfaces of indefinite cosymplectic manifolds

Tangent to ξ i.e., $\xi \in \Gamma(TM)$, let $(M, \bar{g})$ be degenerate (lightlike) hypersurface of $(\bar{M}, g)$, such that $g(\xi, \xi) = \varepsilon = \mp 1$. If$\bar{g}(\bar{\varphi}Z, Z) = 0$ then itmeans$\bar{\varphi}Z$is tangent to $(M, \bar{g})$. Here Z is the local section of $TM^{\perp}$. Therefore we can select a screen distribution $S(TM)$in such a manner that it contains $\varphi(TM^{\perp})$ as a vector subbundle.

Now let us consider X a local section of transversal bundle $tr(TM)$. Therefore$\bar{g}(\bar{\varphi}X, Z) = \bar{g}(X, \bar{\varphi}Z) = 0$, we found that $\bar{\varphi}X$ is also tangential to $(M, \bar{g})$. But$\bar{g}(\bar{\varphi}X, X) = 0$, implies with respect to $Z$the components of $\bar{\varphi}X$ vanishes and hence $\bar{\varphi}X \in \Gamma\, S(TM)$. From (7),$g(\bar{\varphi}X, \bar{\varphi}Z)$=1. [**2**], If $\in M$, then $\xi \in S(TM)$ implies

$$\bar{g}(\bar{\varphi}Z, \xi) = \bar{g}(\bar{\varphi}X, \xi) = 0,$$

Then, there exists D_0, of rank $2n-4$ distribution on $(M, \bar{g})$, such that $\bar{\varphi}(D_0) = D_0$, then we have the following decomposition

$$TM = (D_1 \oplus D_2) \perp < \xi > \tag{22}$$

HereD_1 and $D_2$are distributions on let $(M, \bar{g})$. Now let us assume Y and V to be local null vector fields such that $Y = -\bar{\varphi}X$ and $V = -\bar{\varphi}Z$. Let R_1 and Q_1 be projection morphisms of tangent bundle TM into D_1 and D_2respectively, then for any $F \in \Gamma(TM)$, equation (22) yields

$$F = R_1 F + Q_1 F + \eta(F)\xi, \qquad Q_1 F = u(F)Y. \tag{23}$$

Here $u(F) = g(F, V)$, is a differential 1-form. Applying $\bar{\varphi}$ to above equation (23), we obtain

$$\bar{\varphi}F = \bar{\phi}F + u(F)X \tag{24}$$

Where$\bar{\varphi}$ represents (1, 1) type tensor field on$(M, \bar{g})$, and is defined as$\bar{\phi}F = \bar{\varphi}R_1 F$. Addition to this, we have following results

$$B(F, \xi) = 0, \; C(F, \xi) = \theta(F) \tag{25}$$

$$\bar{\phi}^2 F = -F + \eta(F)\xi + u(F)\xi, \; D_F \xi = 0$$

Let us assume that$(\bar{M}, g)$, is an indefnite cosymplectic manifold of constant curvature -1, such that

$$\bar{R}(F, J)L = g(J, L)F - g(F, L)J, \tag{26}$$

for any $F, J, L \in \Gamma(TM)$.

Now to define a non-symmetric induced Ricci-tensor $R^{(0,2)}$on $(M, \bar{g})$, it is noted that D is not the Livi-Civita connection and $R^{(0,2)}$ has no physical meaning like that of symmetric Ricci tensor Ric on $(\bar{M}, g)$. Therefore by direct calculations an induced Ricci tensor $R^{(0,2)}(F, J)$ is given by

$$R^{(0,2)}(F, J) = S(F, J) = (2n-1)\bar{g}(F, J) + B(F, J)trA_X - B(A_X F, J). \tag{27}$$

From (27), we obtain

$$R^{(0,2)}(F, \xi) = S(F, \xi)) = (2n-1)\eta(F), \quad Q\xi = (2n-1)\xi \tag{28}$$

Theorem 1. Suppose $(\bar{M}, g)$ be an indefinite cosymplectic manifold of constant curvature -1 and let $(M, \bar{g})$ be weakly$\bar{\phi}$-Ricci symmetric null hypersurface of $(\bar{M}, g)$ with $\xi \in \Gamma(TM)$, thenthe sum of non-zero 1-forms is zero everywhere

$$A(\xi) + B(\xi) + D(\xi) = 0$$

Proof. We know that a lightlike hypersurface $(M, \bar{g})$ is said to be weakly $\bar{\phi}-$Ricci symmetric null hypersurface of an indefinite cosymplectic manifold $(\bar{M}, g)$, if it satisfies

$$\bar{\phi}^2(D_F Q)J = A(F)QJ + B(J)QF + S(F, J)\rho \tag{29}$$

for all $F, J \in \Gamma(TM)$. Here $A(F) = g(F, \delta), B(J) = g(J, \kappa)$ are 1-forms and δ, κ, ρ are associated vector fields. From equation (25), we obtain

(30) $-\bar{g}((D_F Q))(J), L) + \eta(D_F Q)(J)\xi + u(D_F Q(J))Y = A(F)QJ + B(J)QF + S(F,J)\rho$

We know that

$$(D_F Q)(J) = D_F QJ - QD_F J$$

Now taking inner product of equation (29) with L, we acquire

(31) $$-\bar{g}((D_F QJ, L) + \bar{g}(QD_F J, L) + \eta(D_Q J)\,\eta(L) - \eta(QD_F J)\eta(L)$$
$$+ u(D_F QJ)\bar{g}(Y,L) - u(QD_F J)\bar{g}(Y,L) = A(F)S(J,L) + B(J)S(F,L) + D(L)S(F,J)$$

Replacing J by ξ in (31), and using (25) and (27), we acquire

(32) $$(2n-1)A(F)\eta(L) + B(\xi)S(F,L) + (2n-1)D(L)\eta(F) = 0$$

Again putting $F = L = \xi$ in (32), we acquire

$$A(\xi) + B(\xi) + D(\xi) = 0$$

Theorem 2. Suppose $(\bar{M}, g)$ be an indefinite cosymplectic manifold of constant curvature -1 and let $(M, \bar{g})$ be weakly $\bar{\phi}$-Ricci symmetric null hypersurface of $(\bar{M}, g)$with $\in \Gamma(TM)$. Let $(M, \bar{g})$ and screen bundle $S(TM)$ are totally umbilical, Then $(M, \bar{g})$ is locally $\bar{\phi}$-Ricci symmetric null hypersurface if $\bar{\alpha}\bar{\beta} = (2n-1) + \bar{\alpha}trA_X$.

Proof. Assume lightlike hypersurface$(M, \bar{g})$ to be weakly $\bar{\phi}$-Ricci symmetric null hypersurface of $(\bar{M}, g)$ such that $\xi \in \Gamma(TM)$. From (29) after taking inner product with L we obtain

(33) $$\bar{g}(\bar{\phi}^2(D_F Q)\,J, L) = A(F)S(J,L) + B(J)S(F,L) + D(L)S(F,J)$$

Using (27) in (33) we acquire

(34) $$\bar{g}(\bar{\phi}^2(D_F Q)\,J, L)\& = A(F)(2n-1)\bar{g}(J,L) + B(J,L)trA_X - B(A_X\,J,L)$$
$$+B(J)(2n-1)\bar{g}(F,L) + B(F,L)trA_X - B(A_X\,F,L) + D(L)(2n-1)\bar{g}(F,J)$$
$$+ B(F,J)trA_X - B(A_X\,F,J)$$

As assumed lightlike hypersurface $(M, \bar{g})$ and screen bundle $S(TM)$ are totally umbalical. Then substituting $B(F,J) = \bar{\alpha}\bar{g}(F,J)\$\, and\;\; C(F,J) = \bar{\beta}\bar{g}(F,J)$, in above equation we get

(35) $$\bar{g}(\bar{\phi}^2(D_F Q)\,J, L) = [(2n-1) + \bar{\alpha}trA_X - \bar{\alpha}\bar{\beta}]\bar{g}(A(F)J + B(J)F + \bar{g}(F,J)\bar{g}(\rho, L).$$

Where $\bar{\alpha}$ and $\bar{\beta}$ are smooth functions. Again using the given hypothesis $\bar{\alpha}\bar{\beta} = (2n-1) + \bar{\alpha}trA_X$in (35), it yields

(36) $$\bar{\phi}^2(D_F Q)J = 0$$

Theorem 3. Suppose $(\bar{M}, g)$ be an indefinite cosymplectic manifold of constant curvature -1 and let $(M, \bar{g})$ be weakly $\bar{\phi}$-Ricci symmetric η-Einstein null hypersurface of $(\bar{M}, g)$with $\xi \in \Gamma(TM)$. If $(M, \bar{g})$ is locally $\bar{\phi}$-Ricci symmetric null hypersurface of $(\bar{M}, g)$, then either $\bar{\alpha}$=-$\bar{\beta}$ or sum of non-zero 1-forms is zero everywhere.

Proof. As assumed $(M, \bar{g})$ is locally $\bar{\phi}$-Ricci symmetric null hypersurface of $(\bar{M}, g)$, then from equation (29) we obtain

(37) $$A(F)QJ + B(J)QF + S(F,J)\rho = 0$$

Again taking inner product of the above equation (37) with L, we get

(38) $$A(F)S(J,L) + B(J)S(F,L) + D(L)S(F,J)$$

By our assumption $(M, \bar{g})$ is an η-Einstein null hypersurface of $(\bar{M}, g)$, that is

$$S(F,J) = \bar{\alpha}\bar{g}(F,J) + \bar{\beta}\eta(F)\eta(J),$$

Hence equations (38) leads us

(39) $$A(F)\,[\bar{\alpha}\bar{g}(J,L) + \bar{\beta}\eta(J)\eta(L)]\& + B(J)\big[\bar{\alpha}\bar{g}(F,L) + \bar{\beta}\eta(F)\eta(L)\big]$$
$$+ D(L)[\bar{\alpha}\bar{g}(F,J) + \bar{\beta}\eta(F)\eta(J)] = 0$$

By putting $F = J = \xi$, $F = L = \xi$ and $J = L = \xi$ in (39), by turns and then adding the resulting equations, we have

(40)
$$A(\xi)\,[\bar{\alpha}\eta(L) + \bar{\beta}\eta(L) + \bar{\alpha}\eta(J) + \bar{\beta}\,\eta(J)] + A(F)[\bar{\alpha} + \bar{\beta}]$$
$$+B(\xi)[\bar{\alpha}\eta(L) + \bar{\beta}\eta(L) + \bar{\alpha}\eta(F) + \bar{\beta}\,\eta(F)] + B(J)[\bar{\alpha} + \bar{\beta}]$$
$$+ D(\xi)[\bar{\alpha}\eta(J) + \bar{\beta}\eta(J) + \bar{\alpha}\eta(F) + \bar{\beta}\,\eta(F)] + D(L)[\bar{\alpha} + \bar{\beta}] = 0.$$

By setting $F = J = L$in equation (39), which then leads?

(41)
$$A(\xi) + B(\xi) + D(\xi)[2\bar{\alpha}\eta(F) + 2\bar{\beta}\eta(F)] + A(F) + B(F) + D(F)[\bar{\alpha} + \bar{\beta}] = 0.$$

Using theorem (1) in equation(40), we obtain our result.

Corollary 1. Suppose $(\bar{M}, g)$ be an indefinite cosymplectic manifold of constant curvature -1 and let $(M, \bar{g})$ be weakly $\bar{\phi}$ -Ricci symmetric Einstein null hypersurface of an indefnite cosymplectic manifold $(\bar{M}, g)$ with $\xi \in \Gamma(TM)$. If $(M, \bar{g})$ is locally $\bar{\phi}$-Ricci symmetric null hypersurface of $(\bar{M}, g)$, then sum of non-zero 1-forms is zero everywhere.

Theorem 4. Let $(M, \bar{g})$ be weakly $\bar{\phi}$-Ricci symmetric degenerate hypersurface of $(\bar{M}, g)$ of constant curvature -1 with $\xi \in \Gamma(TM)$. If $(M, \bar{g})$ admits Codazzi type of Ricci tensor, then $\bar{g}([F, J], \xi) = 0$ i, e. $F = J$.

Proof. We know that the Ricci tensor $R^{(0,2)} = S$, satisfies Codazzi type of Ricci tensor if

(42)
$$(D_F S)(J, L) = (D_J S)(F, L)$$

for all $F, J, L \in \Gamma(TM)$.

Fromequations (27) and (1), we obtain

(43)
$$(2n-1)[B(F,J)\theta(L) + B(F,L)\theta(J) - \bar{g}(D_F J, L) - \bar{g}(J, D_F L)] +$$
$$D_F B(J,L) tr A_X - D_F B(A_X J, L) + B(A_X D_F J, L) + B(A_X J, D_F L)$$
$$= (2n-1)\big[B(J,F)\theta(L) + B(J,L)\theta(F) - \bar{g}\big(D_J F, L\big) - \bar{g}\big(F, D_J L\big)\big] +$$
$$D_J B(F,L) tr A_X - D_J B(A_X F, L) + B(A_X D_J F, L) + B(A_X F, D_J L)$$

Replacing L by ξ and using $B(F, \xi) = 0$ in above equation, we get

(44)
$$\bar{g}(D_F J, \xi) - \bar{g}(D_J F, \xi) = 0$$
$$D_F J - D_J F = 0$$

Or

(45)
$$[F, J] = 0$$

Hence our desired result is obtained.

Theorem 5. Let $(M, \bar{g})$ be weakly $\bar{\phi}$-Ricci symmetric degenerate hypersurface of $(\bar{M}, g)$ of constant curvature -1with $\xi \in \Gamma(TM)$. If $(M, \bar{g})$ is totally geodesic and admits cyclic parallelof Ricci tensor, then L is parallel vector field.

Proof. From equation (43), we acquire

(46)
$$\big(D_J S\big)(L,F) = (2n-1)\big[B(J,L)\theta(F) + B(J,F)\theta(L) - \bar{g}\big(D_J L, F\big) - \bar{g}\big(L, D_J F\big)\big] +$$
$$D_J B(L,F) tr\, A_X - D_J B(A_X L, F) + B(A_X D_J L, F) + B(A_X L, D_J F),$$

and

(47)
$$(D_L S)(F,J) = (2n-1)[B(L,F)\theta(J) + B(L,J)\theta(F) - \bar{g}(D_L F, J) -$$
$$\bar{g}(F, D_L J)] + D_L B(F,J) tr\, A_X - D_L B(A_X F, J) + B(A_X D_L F, J) + B(A_X F, D_L J)$$

As assumed lightlike hypersurface $(M, \bar{g})$ admits cyclic parallel of Ricci tensor S , that is

(48) $$(D_F S)(J, L) + (D_J S)(L, F) + (D_L S)(F, J) = 0,$$

Putting equations (43), (46) and (47) in (48) and using $B(F, J) = 0$in the resulting equation, we get

(49) $$\bar{g}(D_F J, L) + \bar{g}(D_F L, J) + \bar{g}(D_J L, F) + \bar{g}(D_J F, L) + \bar{g}(D_L F, J) + \bar{g}(D_L J, F) = 0.$$

Replacing $F = J = \xi$ in (49), we obtain

(50) $$\bar{g}(D_\xi L, \xi) = 0.$$

Hence our result follows from (50).

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