Geodesics on tangent bundles with Twisted Sasaki metric

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Abstract: Let (M^m, g) be an m-dimensional Riemannian manifold and TM be its tangent bundle. In this paper, we introduce a new class of natural metrics with respect to g non-rigid on TM, called the twisted Sasaki metric. we investigate geodesics of the Twisted Sasaki gradient metric. Afterward establish a necessary and sufficient conditions under which a curve be a geodesic respect.

Keywords: Horizontal lift, vertical lift, Cheeger-Gromoll metric, Tangent bundle, geodesic

1. Introduction

The geometry of the tangent bundle TM equipped with Sasaki metric has been studied by many authors. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM. **Musso, E**. and **Tricerri, F.(1988)** have introduced the notion of Cheeger-Gromoll metric, **Jian,W**. and **Yong,W**.**(2011)** have introduced the notion of Rescaled Metric, **Zagane, A.** and **Djaa, M**.**(2018)** have introduced the notion of Mus-Sasaki metric.

The main idea in this note consists in the modification of the Sasaki metric. First we introduce a new metric called twisted-Sasaki metric on the tangent bundle TM. This new natural metric will lead us to interesting results. First we establish the Levi-Civita connection of this metric. Afterward we establish a necessary and sufficient conditions under which a curve be a geodesic with respect to the twisted-Sasaki metric.

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ_{ij}^k the Cristoffel symbols of g.

We have two complementary distributions on TM, the vertical distribution V and the horizontal distribution H, defined by:

$$V_{(x,u)} = Ker\big(d\pi_{(x,u)}\big) = \left\{a^i \frac{\partial}{\partial y^i}\bigg|_{(x,u)}, a^i \in \mathbb{R}\right\},$$

$$V_{(x,u)} = \left\{a^i \frac{\partial}{\partial y^i}\bigg|_{(x,u)} - a^i u^j \Gamma_{ij}^{k} \frac{\partial}{\partial y^k}\bigg|_{(x,u)}, a^i \in \mathbb{R}\right\}.$$

Where $(x,u) \in TM$, such that $T_{(x,u)}TM = H_{(x,u)} \oplus V_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \left\{\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\right\}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM

If $\omega = \omega^i \frac{\partial}{\partial x^i} + \overline{\omega}^j \frac{\partial}{\partial x^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$\omega^H = \omega^i \frac{\partial}{\partial x^i} - \omega^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \in H_{(x,u)}$$

$$\omega^V = (\varpi^k + \omega^i u^j \Gamma_{ij}^k) \frac{\partial}{\partial y^k} \in V_{(x,u)}$$

2.1 Lemma

Let (M,g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X,Y \in \Gamma(TM)$ we have

1. $[X^H,Y^H]_p = [X,Y]_p^H - (R_x(X,Y)u)^V$,
2. $[X^H,Y^V]_p = (\nabla_X Y)_p^V$,
3. $[X^V,Y^V]_p = 0$

Where $p = (x,u) \in TM$.

2. Twisted Sasaki metric

2.1 Definition

Let (M,g) be a Riemannian manifold, and $f, \hbar : M \to]0;+\infty[$ be strictly positive smooth functions. The twisted Sasaki metric $G^{f,\hbar}$ on the tangent bundle TM is definied by

$$G(X^H,Y^H)(p;u) = f(p)gp(X,Y);$$
$$G(X^H,Y^V)(x;u) = 0;$$
$$G(X^V,Y^V)(x;u) = \hbar(p)gp(X;Y).$$

for all vector _elds $X,Y \in (TM)$, $(p;u) \in TM$ Note that, if $f = \hbar = 1$, then $G^{f,\hbar}$ is the Sasaki metric. If $\hbar = 1$, then $G^{f,\hbar}$ is the rescaled Sasaki metric. If $f = 1$, then $G^{f,h}$ is the vertical rescaled metric.

2.1. Levi-Civita connection of $G^{f,h}$

2.2.1 Lemma

Let (M,g) be a Riemannian manifold, ∇ (respectively $\nabla^{f,h}$) denote the Levi-Civita connection of (M,g) (respectively $(TM,G^{f,\hbar})$). Then, we have

- $G^{f,h}(\nabla_{X^H}^{f,h} Y^H, Z^H) = G^{f,h}\left(\left(\nabla_X Y + \frac{1}{2f(p)}(X(f)Y + Y(f)X - g(X,Y)grad\, f)\right)^H, Z^H\right)$
- $G^{f,h}(\nabla_{X^H}^{f,h} Y^H, Z^V) = -G^{f,h}\left((\frac{1}{2}R(X,Y)u)^V, Z^V\right)$
- $G^{f,h}(\nabla_{X^H}^{f,h} Y^V, Z^H) = G^{f,h}\left((\frac{h(p)}{2f(p)}R(u,Y)X)^H, Z^H\right)$
- $G^{f,h}(\nabla_{X^H}^{f,h} Y^V, Z^V) = G^{f,h}\left((\frac{X(h)}{2h(p)}Y + \nabla_X Y)^V, Z^V\right)$
- $G^{f,h}(\nabla_{X^V}^{f,h} Y^H, Z^H) = G^{f,h}\left((\frac{h(p)}{2f(p)}R(u,X)Y)^H, Z^H\right)$
- $G^{f,h}(\nabla_{X^V}^{f,h} Y^H, Z^V) = G^{f,h}\left((\frac{Y(h)}{2h(p)}X)^V, Z^V\right)$
- $G^{f,h}(\nabla_{X^V}^{f,h} Y^V, Z^H) = -G^{f,h}\left((\frac{1}{2f}g(X,Y)grad\, h)^H, Z^H\right)$
- $G^{f,h}(\nabla_{X^V}^{f,h} Y^V, Z^V) = 0$

For all $X,Y,Z \in \Gamma(TM)$ and $p = (x,u) \in TM$
The proof of this Lemma follow directly from Kozul formula and definition 2.2.1
As a direct consequence of Lemma 2.2.1, we get the following theorem

2.2.1 Theorem

Let (M,g) be a Riemannian manifold and $(TM;G^{f,\hbar})$ its tangent bundle equipped with the twisted Sasaki metric, $\nabla^{f,\hbar}$ be a Levi-Civita connection of the tangent bundle $(TM,G^{f,\hbar})$. Then, we have

1. $(\nabla^{f,\hbar}_{X^H} Y^H)_{(p,u)} = \left(\nabla_X Y + A_f(X,Y)\right)^H_{(p,u)} - \left(\frac{1}{2} R_p(X,Y)u\right)^V_{(p,u)}$
2. $(\nabla^{f,\hbar}_{X^H} Y^V)_{(p,u)} = \left(\frac{\hbar(p)}{2f(p)} R(u,Y)X\right)^H_{(p,u)} + \left(\frac{X(\hbar)}{2\hbar(p)} Y + \nabla_X Y\right)^V_{(p,u)}$
3. $(\nabla^{f,\hbar}_{X^V} Y^H)_{(p,u)} = \left(\frac{\hbar(p)}{2f(p)} R(u,X)Y\right)^H_{(p,u)} + \left(\frac{Y(\hbar)}{2\hbar(p)} X\right)^V_{(p,u)}$
4. $(\nabla^{f,\hbar}_{X^V} Y^V)_{(p,u)} = \left(-\frac{1}{2f(p)} G(X,Y) grad\ \hbar\right)^V_{(p,u)}$
5.

For all vector fields $X, Y \in \Gamma(TM), (p,u) \in TM$, where

$$A_f = \frac{1}{2f(p)}(X(f)Y + Y(f)X - g(X,Y) grad\ f)$$

.

3. Geodesics of Twisted Sasaki metric

Let (M,g) be a Riemannian manifold and $x: I \to M$ be a curve on M. We define a curve $C: I \to M$ by for all $t \in I, C(t) = (x(t), y(t))$ where $y(t) \in T_{x(t)}M$ i.e $y(t)$ is a vector field along $x(t)$.

3.1 Definition

Let (M,g) be a Riemannian manifold, if $x(t)$ is a curve on M. The curve $C(t) = (x(t), \dot{x}(t))$ is called the natural lift of curve $x(t)$.

3.2 Definition

Let (M,g) be a Riemannian manifold and ∇ denote the Levi-Civita connection of (M,g).

A curve $C(t) = (x(t), y(t))$ is said to be a horizontal lift of the curve $x(t)$ if and only if $\nabla_{\dot{x}} y = 0$.

3.1 Lemma

Let (M,g) be a Riemannian manifold. If $X, Y \in \Gamma(TM)$ are vector fields on M and $(x,u) \in TM$ such that $Y_x = u$, then we have

$$d_x Y(X_x) = X^H_{(x,u)} + (\nabla_X Y)^V_{(x,u)}$$

Proof.

Let (U, x^i) be a local chart on M in $x \in M$ and $(\pi^{-1}(U), x^i, y^i)$ be the induced chart on TM, if $X_x = X^i \frac{\partial}{\partial x^i}\Big|_x$ and $Y = Y^i \frac{\partial}{\partial x^i}\Big|_x = u$, then

$$d_x Y(X_x) = X^i(x) \frac{\partial}{\partial x^i}\Big|_{(x,u)} + X^i(x) \frac{\partial Y^k}{\partial x^i}(x) \frac{\partial}{\partial y^k}\Big|_{(x,u)}$$

Thus the horizontal part is given by

$$\left(d_x Y(X_x)\right)^H = X^i(x) \frac{\partial}{\partial x^i}\Big|_{(x,u)} - X^i(x) Y^j(x) \Gamma^k_{ij}(x) \frac{\partial}{\partial y^k}\Big|_{(x,u)}$$
$$= X^H_{(x,u)}$$

And the vertical part is given by

$$\left(d_x Y(X_x)\right)^V = \left\{X^j(x) \frac{\partial Y^k}{\partial x^i}(x) + X^i(x) Y^j(x) \Gamma^k_{ij}(x)\right\} \frac{\partial}{\partial y^k}\Big|_{(x,u)}$$
$$= (\nabla_X Y)^V_{(x,u)}$$

3.2 Lemma

Let (M,g) be a Riemannian manifold and ∇ denote the Levi-Civita connection of (M,g). If $x(t)$ be a curve on M and $C(t) = (x(t), y(t))$ be a curve on TM, then

$$\dot{C} = \dot{x}^H + (\nabla_{\dot{X}} Y)^V.$$

Proof. Locally, if $Y \in \Gamma(TM)$ is a vector field such $Y(x(t)) = y(t)$, then we have

$$\dot{C}(t) = dC(t) = dY(x(t)).$$

Using lemma 4.1 we obtain

$$\dot{C}(t) = dY(x(t)) = \dot{x}^H + (\nabla_{\dot{x}} y)^V$$

3.1 Theorem

Let (M, g) be a Riemannian manifold and $(TM, G^{f,h})$ its tangent bundle equipped with twisted Sasaki metric. If ∇ (resp. $\nabla^{f,h}$) denote the Levi-Civita connection of (M, g) (resp. $(TM, G^{f,h})$) and $C(t) = x(t), y(t))$ is a curve on TM such $y(t)$ is a vector field along $x(t)$, then

$$\nabla^{f,h}_{\dot{C}} \dot{C} = \left(\nabla_{\dot{x}} \dot{x} + \frac{1}{2f}(2\dot{x}(f) - \|\dot{x}\|^2 \, grad\, f) + \frac{h}{f} R(u, \nabla_{\dot{x}} y)\dot{x} - \frac{1}{2f}\|\nabla_{\dot{x}} y\|^2 grad\, h\right)^H + \left(\frac{\dot{x}(h)}{h}\nabla_{\dot{x}} y + \nabla_{\dot{x}} \nabla_{\dot{x}} y\right)^V$$

3.2 Theorem

Let (M, g) be a Riemannian manifold and $(TM, G^{f,h})$ its tangent bundle equipped with twisted Sasaki metric. If $C(t) = x(t), y(t))$ is a curve on TM such $y(t)$ is a vector field along $x(t)$, then $C(t)$ is geodesic on TM if and only if

$$\begin{cases} \nabla_{\dot{x}} \dot{x} = -\frac{1}{2f}(2\dot{x}(f) - \|\dot{x}\|^2 \, grad\, f) - \frac{h}{f} R(u, \nabla_{\dot{x}} y)\dot{x} + \frac{1}{2f}\|\nabla_{\dot{x}} y\|^2 grad\, h \\ \nabla_{\dot{x}} \nabla_{\dot{x}} y = -\frac{\dot{x}(h)}{h}\nabla_{\dot{x}} y \end{cases}$$

3.1 Corollary

Let (M, g) be a Riemannian manifold and $(TM, G^{f,h})$ its tangent bundle equipped with twisted Sasaki metric. The natural lift $C(t) = x(t), \dot{x}(t))$ of any geodesic $x(t)$ is a geodesic on TM if and only if

$$2\dot{x}(f) = \|\dot{x}\|^2 grad\, f$$

3.2 Corollary

Let (M, g) be a Riemannian manifold and $(TM, G^{f,h})$ its tangent bundle equipped with twisted Sasaki metric. The horizontal lift $C(t) = x(t), y(t))$ of the curve $x(t)$ is a geodesic on TM if and only if

$$\nabla_{\dot{x}} \dot{x} = 2\dot{x}(f) - \|\dot{x}\|^2 grad\, f$$

3.3 Corollary

Let (M, g) be a Riemannian manifold and $(TM, G^{f,h})$ its tangent bundle equipped with twisted Sasaki metric. The horizontal lift $C(t) = x(t), y(t))$ of any geodesic $x(t)$ is a geodesic on TM if and only if

$$2\dot{x}(f) = \|\dot{x}\|^2 grad\, f$$

3.1 Remark

Let (M^m, g) be an m-dimensional Riemannian manifold. If $C(t) = x(t), y(t))$ is a horizontal lift of the curve $x(t)$, locally we have:

$$\nabla_{\dot{x}} y = 0 \Leftrightarrow \frac{dy^k}{dt} + \Gamma^k_{ij}\frac{dx^j}{dt} = 0 \Leftrightarrow y'(t) = A(t).y(t),$$

Where $A(t) = [a_{kj}], a_{kj} = \sum_{i=1}^{m} -\Gamma^k_{ij}\frac{dx^j}{dt}$.

3.2 Remark

Using the remark 4.1 we can construct an infinity of examples of geodesics on $(TM, G^{f,h})$

3.1 Example We consider on $\mathbb{R}$ the metric $g = e^x dx^2$.

The Christoffel symbols of the Levi-cita connection associated with g are

$$\Gamma_{11}^1 = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right) = \frac{1}{2}$$

1) The geodesic $x(t)$ such that $x(0) = a \in \mathbb{R}, x^{'}(0) = v \in \mathbb{R}$ of g satisfy the equation,

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{n} \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{i,j}^{j} = 0 \Leftrightarrow x^{''} + \frac{1}{2}(x')^2 = 0$$

Hence $C_1(t) = (x(t), y(t)) = \left(a + 2\ln\left(1 + \frac{vt}{2}\right), \frac{2v}{2+vt}\frac{d}{dx}\right)$ is a natural lift on $T\mathbb{R}$.

2) The curve $C_2(x(t); y(t))$ such $\nabla_{\dot{x}} y = 0$ satisfy the equation

$$\frac{dy^s}{dt} + y^i \Gamma_{ij}^{s} \frac{dx^j}{dt} = 0 \Leftrightarrow y^{'} + \frac{1}{2} y x^{'} = 0$$

After that $y(t) = k.\exp\left(\frac{-v}{2+vt}\right)\frac{d}{dx},\ k \in \mathbb{R}$.

Then

$C_2(x(t), y(t)) = \left(a + 2\ln\left(1 + \frac{vt}{2}\right), k.\exp(\frac{-v}{2+vt})\frac{d}{dt}\right)$ is a horizontal lift on $T\mathbb{R}$.

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