An Orthogonal Left Centralizer and Reverse Left Centralizer on Semiprime Rings

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Abstract:

Let R be a semiprime ring. Then we prove the following main result:
Let R be a 2-torsion free semiprime ring, t be a left centralizer and h be a reverse left centralizer of R, suppose that $t^2 = h^2$. Then t + h and t – h are orthogonal.

Key Words: semiprime ring , left centralizer , reverse left centralizer, orthogonal left centralizer and reverse left centralizer .

Mathematic Subject classification : 16 N 60, 42C05 , 33C45 .

1-Introduction :

A ring R is called semiprime if xRx = (0) implies = 0, such that $x \in R$ [3] .
Let R be a ring then R is called 2-torsion free if 2x = 0 implies x = 0, for all $x \in R$ [3] .
A left (resp. right) centralizer of a ring R is an additive mapping $t: R \longrightarrow R$ which satisfies the equation : t (xy) = t(x) y (resp.t (xy) = x t (y)) , for all $x, y \in R$.
t is called a centralizer of R if it is both a left and a right centralizer [4] .
A left (resp. right) Jordan centralizer of a ring R is an additive mapping $t: R \longrightarrow R$ which satisfies the equation $t(x^2) = t(x)x$ (resp. $t(x^2) = x\,t(x)$) , for all $x \in R$
t is called a Jordan centralizer of R if it is both a left and a right Jordan centralizer [4] .
Also , Jarullah and Salih introduced the concepts of a higher reverse left (resp. right) centralizer and a Jordan higher reverse left (resp. right) centralizer on rings as follows :
Let $t = (t_i)_{i \in N}$ be a family of additive mappings of a ring R into itself .Then t is called a higher reverse left (resp. right) centralizer of R if for all $x, y \in R$ and $n \in N$

$$t_n(xy) = \sum_{i=1}^{n} t_i(y)\, t_{i-1}(x) \;\left(\text{resp. } t_n(xy) = \sum_{i=1}^{n} t_{i-1}(y)\, t_i(x)\right) \quad [3] .$$

Let $t = (t_i)_{i \in N}$ be a family of additive mappings of a ring R into itself .Then t is called a Jordan higher reverse left (resp. right) centralizer of R , if the following equation holds, for all $x \in R$ and $n \in N$: $t_n(x^2) = \sum_{i=1}^{n} t_i(x)\, t_{i-1}(x)$ (resp. $t_n(x^2) = \sum_{i=1}^{n} t_{i-1}(x)\, t_i(x)$) [3] .

In this paper , we define and study the concept of orthogonal when t be a left centralizer and h be a reverse left centralizer of semiprime ring and we prove some of lemmas and theorems about orthogonally one of these theorems is :

Let R be a 2-torsion free semiprime ring , t be a left centralizer and h be a reverse left centralizer of R , such that t and h are commuting .Then t and h are orthogonal if and only if $t(x)\,h(y) + h(x)\,t(y) = 0$, for all $x, y \in R$.

In our work we need the following Lemmas :

Lemma (1.1): [2]

Let R be a 2-torsion free semiprime ring and x, y be elements of R, then the following conditions are equivalent :

(i) $xry = 0$, for all $r \in R$

(ii) $yrx = 0$, for all $r \in R$

(iii) $xry + yrx = 0$, for all $r \in R$

If one of these conditions is fulfilled ,then $xy = yx = 0$.

Lemma (1.2): [1]

Let R be a 2-torsion free semiprime ring and x , y be elements of R if $xry + yrx = 0$, for all $r \in R$,then $xry = yrx = 0$.

2-Orthogonal Left Centralizer and Reverse Left Centralizer on Semiprime Rings :

In this section we will introduce the definition of orthogonal when t be a left centralizer and h a reverse left centralizer of semiprime rings .

Definition (2.1):

Let t be a left centralizer and h be a reverse left centralizer of a ring R. Then t and h are called **orthogonal** if

$t(x)\,R\,h(y) = (0) = h(y)\,R\,t(x)$, for all $x , y \in R$.

Example (2.2):

Let R be a ring of all 2 × 2 matrices of integer numbers, such that

$$R = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} ; x, y \in Z \right\}$$

Let $t : R \to R$ be an additive mapping of a ring R into itself , such that

$$t\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \text{, for all } \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R$$

And $h : R \to R$ be an additive mapping of a ring R into itself , such that

$$h\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \text{, for all } \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R .$$

t is a left centralizer and h is a reverse left centralizer . Then t and h are orthogonal of R .

Example (2.3):

Let t is a left centralizer and h is a reverse left centralizer of a ring R , let S be a ring , such that $S = R \oplus R = \{(x,y) ; x , y \in R\}$,we define t^* and h^* on S by

$t^* ((x,y)) = (t(x) , 0)$ and $h^*((x,y)) = (0 , h(y))$, for all $(x,y) \in S$.

Then t^* and h^* are orthogonal of S.

Lemma (2.4):

Let R be a semiprime ring, suppose that t be a left centralizer and h be a reverse left centralizer of R , satisfy $t(x) R h(x) = (0)$, for all $x \in R$. Then $t(x) R h(y) = (0)$, for all $x , y \in R$.

Proof :

Suppose that $t(x) z h(x) = 0$, for all $x , y , z \in R$ …(1)

Replace x by x + y in (1) , we have that

$t(x + y) z h(x + y) = 0$

$t(x) z h(x) + t(x) z h(y) + t(y) z h(x) + t(y) z h(y) = 0$

Therefore , by our assumption and Lemma (1.1) , we get

t(x) z h(x) = 0 , for all x , y , z $\in$ R

Thus, t(x) R h(y) = (0), for all x , y $\in$ R

Lemma (2.5):

Let R be a 2-torsion free semiprime ring , t be a left centralizer and h be a reverse left centralizer of R , such that t and h are commuting .Then t and h are orthogonal if and only if t(x) h(y) + h(x) t(y) = 0, for all x, y $\in$ R.

Proof :

Suppose that t and h are orthogonal

T.P. t(x) h(y) + h(x) t(y) = 0, for all x , y $\in$ R

Since t and h are orthogonal , we have that

t(x) z h(y) = 0 = h(x) z t(y) , for all x , y , z $\in$ R

Therefore ,by Lemma (1.1) , we get the require result .

Conversely , Suppose that t(x) h(y) + h(x) t(y) = 0 , for all x , y $\in$ R

T.P. t and h are orthogonal

By our assumption , we have that

t(x) h(y) + h(x) t(y) = 0 , for all x , y $\in$ R

Left multiply by z , we have that

z t(x) h(y) + z h(x) t(y) = 0

Since t and h are commuting , we have that

t(x) z h(y) + h(x) z t(y) = 0 , for all z $\in$ R

By Lemma (1.2) , we have that

Thus , t(x) z h(y) = 0 = h(x) z t(y) , for all x , y , z $\in$ R

Hence , t and h are orthogonal

Theorem (2.6):

Let R be a 2-torsion free semiprime ring , t be a left centralizer and h be a reverse left centralizer of R , where t and h are commuting . Then the following conditions are equivalent :

(i) t and h are orthogonal

(ii) th = 0

(iii) ht = 0

(iv) th + ht = 0

Proof : **(i) ⇔ (ii)**

Suppose that t and h are orthogonal

t(x) z h(y) = 0 = h(y) z t(x) , for all x , y , z ∈ R

T.P. th = 0

By our assumption , we have that

h(y) z t(x) = 0

Replace x by h(y) , we have that

h(y) z t(h(y)) = 0

t((h(y)) z t(h(y)) = 0

t(h(y)) z t((h(y)) = 0

Since R is a semiprime ring , we have that

t(h(y)) = 0 , for all y ∈ R ⇒ th = 0

Conversely, suppose that th = 0

T.P. t and h are orthogonal

h(t(xy)) = 0

h(t(x) y) = 0

h(y) t(x) = 0 , for all x , y ∈ R

Since t and h are commuting , we have

t(x) h(y) = 0 , for all x , y ∈ R

Replace x by xz , we have that

t(xz) h(y) = 0

t(x) z h(y) = 0 , for all x , y , z ∈ R …(1)

Since t and h are commuting , we have

h(y) z t(x) = 0 , for all x , y ∈ R …(2)

Hence t and h are orthogonal

Proof : **(i) ⇔ (iii)**

By the same way in (i) ⇔ (ii) , we get (i) ⇔ (iii) .

Proof : **(i) ⇔ (iv)**

Suppose that t and h are orthogonal

T.P. $th + ht = 0$

By (ii) and (iii) , we get the require result

Conversely , suppose that $th + ht = 0$

T.P. t and h are orthogonal

$(th + ht)(yx) = 0$

$t(h(yx)) + h(t(yx)) = 0$

$t(h(x)y) + h(t(y)x) = 0$

$t(h(x))y + h(x)t(y) = 0$, for all $x , y \in R$.

Replace $t(h(x))$ by $t(x)h(y)$, we have that

$t(x)h(y) + h(x)t(y) = 0$, for all $x , y \in R$

By Lemma (2.5) , we get the require result .

Theorem(2.7):

Let R be a 2-torsion free semiprime ring , t be a left centralizer and h be a reverse left centralizer of R , suppose that $t^2 = h^2$. Then $t + h$ and $t - h$ are orthogonal .

Proof :

$((t + h)(t- h) + (t- h)(t + h))(x)$

$= (t(x) + h(x))(t(x) - h(x)) + (t(x) - h(x))(t(x) + h(x))$

$= t^2(x) - t(x)h(x) + h(x)t(x) - h^2(x) + t^2(x) + t(x)h(x) - h(x)t(x) - h^2(x)$

$= 0$

Therefore , $((t + h)(t- h) + (t- h)(t + h))(x) = 0$

By Theorem (2.6) (iv) $\Rightarrow$(i), we get the require result

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