Unsteady MHD thermally conducting thin film flow of a third grade fluid between two oscillating moving vertical belts.

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Abstract

Study about the MHD unsteady third grade thin film fluid flow in an oscillating two parallel moving belts. The moving vertical belts are oscillating and liquid drain down due to the gravity. Resulting fluid flow, thermal and mass transfer processes appearing in the third grade fluid are described by system of coupled non-linear equations are solved using suitable perturbation method. Numerically outputs are obtained graphically portrayed and the important characteristics parameters are discussed.

Keywords:

Third grade fluid, Lifting flow, Drainage flow, Magnetohydrodynamics(MHD), Heat transfer, Porous medium, Thin film, Chemical reaction, Perturbation method.

1 Introduction

Non-Newtonian third grade fluid have a large applications in industrial processes such as polymers, drilling mud, molten plastics, blood, tomato ketchup, slurries, oil recovery, gasoline, lubricants and diesel engine. In this types of fluids are highly complex and non-linear partial differential equations. Certain more important studies dealing with the non-Newtonian fluids flow are [(1−5)]. For different types of flow situations is the subclass of various type fluid such as third order fluid have received special attention. So the present analysis deals with the problem of non-Newtonian third grade fluid due to extensive uses. Third grade fluid captures the shear thinning and also normal stress or shear thinning effects of non-Newtonian. The impact of third grade non-Newtonian fluid has a important applications in the current global environment. The flow of these fluids utilized in petroleum industry, foam processing of plastic, production of polymer materials, food industry, biological diagnosis, manufacturing of paints and chemical industry etc. Third order thin layer fluid actual in various positions in science and technology. Rajagopal et.al.,[(6)] addressed the thermodynamics transfer of various non-Newtonian third order fluid in different planes. Gul et.al.,[(7)] computed a third grade unsteady fluid flow of vertical oscillating heat transfer with parallel plates. Shuaib et.al.,[(8)] reported heat transfer and unsteady MHD flow of third grade fluid past on oscillating vertical belt. Ellahi et.al.,[(9)] studied heat transfer analysis of third order fluid effect in different level with flat channel.

Third grade thin film flow problems appear in many field from specific situations in the flow polymer films ,thin sheets and lubrication oil processes in engineering which is probably one of the largest sub-field of thin film. Few other main process of applications are found in coating flows, micro-fluid engineering, bio-fluid and medicine. Many researchers have been established by the fluid flow and heat transfer inside thin films as follows, Hayat et.al.,[(10−13)] and Siddiqui et.al.,[(14−17)] made explicated their valuable contributions to the thin film flows of non-Newtonian third grade fluids. Idowu et.al.,[(18)] displayed a thermal radiation and chemical reaction effects on third grade fluid flow between stationary and oscillating plates. Baoku et.al.,[19] developed the effects of thermal radiation on heat transfer in third grade fluid over a vertical plate with suction. Perturbation method are a special class of analytical methods for determining approximate solutions of higher order systematic non-linear equations. They are very appropriate techniques for researcher demonstrating, predicting and describing phenomena in vibrating systems that are caused by non-linear effects. Followed Holmes mark [20] presented the introduction to Perturbation methods for ordinary differential equations to determine the system of non-liner higher derivatives. Nazeer et.al.,[19] analyzed perturbation and numerical solutions of non-newtonian fluid bounded with in a porous channel.

The main aim of the present work is to study of MHD heat transfer third grade thin film flow of oscillating and moving upward belts. Let us consider two types of third grade non-Newtonian thin film flows (i) lifting flow and (ii) drainage flow. The explanation of our mathematical model to decisive the equations are clarify to solve flow equations for lift and drainage. The computational with the graphical results are established and quantitatively explained with respect to existing fluid parameters entrenched in the flow.

2 Mathematical formulation

(i)Lifting flow problem:

Assuming unsteady flow of third grade fluid and heat transfer on a flat oscillating and vertically upward moving belt as shown in figure 1. Every uniform velocity of the belt denoted as V. A magnetic profile is applied uniformly to the belt. During the motion of the belt take itself a thin layer of liquid with constant thickness δ. System of axial framework selected by (x, y) in such a way that $x-$ axis perpendicular to the surface belt and $y-$ axis chosen to the parallel belt. We consider the fluid flow to be a steady, incompressible, laminar, pressure is everywhere atmospheric and balance of gravity as remain stable.

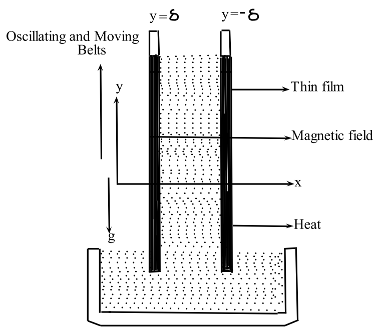

Fig.1 Physical configuration of the lifting flow problem:

Fundamental equations of third grade fluid can be expressed as,

Conservation of mass

$$\nabla.\mathbf{V}=\mathbf{0} \tag{1}$$

The fluid velocity vector denoted by $\mathbf{V}$.

Conservation of momentum

$$\rho\frac{D\mathbf{V}}{Dt}=\nabla.\mathbf{T}+\rho\mathbf{g}+\mathbf{J}\times\mathbf{B}+r \tag{2}$$

Here, $r=-\left\{\mu+\alpha_1\frac{\partial}{\partial t}\right\}\frac{\phi v}{k}$ denoted as third grade resistance fluid.

where, $\frac{D}{Dt}=\frac{\partial}{\partial t}+(\mathbf{V}.\nabla)$ is a material derivative time, ρ be a density constant, $\mathbf{T}$ is the shear stress, $\mathbf{J}$ be a Joule density, $\mathbf{B}$ is the magnetic field, $\mathbf{g}$ represented as force of gravitation, ϕ be a porous parameter, k denoted as permeability and μ be a dynamic viscosity.

Conservation of energy

$$\rho c_p\frac{D\mathbf{\Theta}}{Dt}=k\nabla^{\mathbf{2}}\mathbf{\Theta}+tr(\boldsymbol{\tau}.\mathbf{L})-\nabla.\mathbf{q_r} \tag{3}$$

Rosseland estimation of radiative heat flux is written by,

$$\mathbf{q_r}=-\frac{\partial\Theta^4}{\partial x}\frac{4\sigma'}{3\alpha'}$$

where, $\mathbf{L}=\nabla\mathbf{V}$, k be a thermal conductivity, c_p is the specific heat, $\boldsymbol{\tau}$ indicates as Cauchy tensor stress, $\mathbf{\Theta}$ be a heat transfer, σ' denoted as a Stefan-Boltzmann constant, $\mathbf{q_r}$ is the radiation heat flux, α' is a mean coefficient absorption.

Taylor series following form are expanding as Θ^4 about Θ_0 by:

$$\Theta^4 \cong 4\Theta_0^3\Theta - 3\Theta_0^4$$

Conservation of species

$$\frac{D\boldsymbol{C}}{Dt} = D\nabla^{\mathbf{2}}\mathbf{C} + tr(\boldsymbol{\tau}.\mathbf{L}) - K_c^{'}(C - C_0) \tag{4}$$

Where, $\mathbf{C}$ indicates as third grade concentration fluid, $K_c^{'}$ be a chemical rate reaction, D is the mass diffusivity coefficient.

The magnetic induction $(\mathbf{B})$ is applied to transitionally oscillating belt and external Lorentz body force are derived by,

$$\mathbf{J} \times \mathbf{B} = \left\{0, \sigma B_0^2 v[x,t], 0\right\} \tag{5}$$

The tensor of shear stress $\boldsymbol{T}$ is defined by,

$$\mathbf{T} = -p[\mathbf{I}] + \boldsymbol{\tau}, \tag{6}$$

where $-p\mathbf{I}$ is the stress of isotropic and $\boldsymbol{\tau}$ is noted below,

$$\boldsymbol{\tau} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{7}$$

In which, tensors of kinematic $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ and α_i , β_j be a material moduli constants.

$$\left.\begin{array}{l} \mathbf{A}_1 = (\mathbf{L}) + (\mathbf{L})^T \\ \mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}(\mathbf{L}) + (\mathbf{L})^T\mathbf{A}_{n-1}, n \geq 1 \end{array}\right\} \tag{8}$$

$$\mu \geq 0, |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \alpha_1 \geq 0, \beta_1, \beta_2, \beta_3 \geq 0. \tag{9}$$

The nature of the fluid flow oscillating form are,

$$\mathbf{V} = [0, v(x,t), 0] \text{ , } \Theta = \Theta(x,t) \text{ } and \text{ } C = C(x,t). \tag{10}$$

Inserting equation (10) applying in (1) and (6) - (9), the equation (1) satisfied and equation (6) becomes a tensor stress,

$$\left.\begin{aligned} T_{xx} &= -p+\alpha_2(\tfrac{\partial v}{\partial x})^2+2\beta_2(\tfrac{\partial v}{\partial x})(\tfrac{\partial^2 v}{\partial x\partial t}),\\ T_{xy} &= \mu\tfrac{\partial v}{\partial x}+\alpha_1\tfrac{\partial^2 v}{\partial x\partial t}+2(\beta_2+\beta_3)(\tfrac{\partial v}{\partial x})^3+\beta_1(\tfrac{\partial^3 v}{\partial x\partial t^2}),\\ T_{yy} &= -p+(2\alpha_1+\alpha_2)(\tfrac{\partial v}{\partial x})^2+(6\beta_1+2\beta_2)(\tfrac{\partial v}{\partial x})(\tfrac{\partial^2 v}{\partial x\partial t}),\\ T_{zz} &= -p,\\ T_{xz} &= T_{yz}=0. \end{aligned}\right\} \quad (11)$$

where, $T_{xy}=T_{yx}$, $T_{xz}=T_{zx}$, $T_{yz}=T_{zy}$

From equations (11) applying in (2), (3) and (4) with all premises are reduced, The system of third grade equation are,

$$\left.\begin{aligned} &\rho\tfrac{\partial v}{\partial t}=\mu\tfrac{\partial^2 v}{\partial x^2}+\alpha_1\tfrac{\partial}{\partial t}(\tfrac{\partial^2 v}{\partial x^2})+\beta_1\tfrac{\partial^2}{\partial t^2}(\tfrac{\partial^2 v}{\partial x^2})\\ &+6(\beta_2+\beta_3)(\tfrac{\partial v}{\partial x})^2(\tfrac{\partial^2 v}{\partial x^2})-\rho g-\sigma B_0^2 v-(\mu+2\beta_3(\tfrac{\partial v}{\partial x})^2)\tfrac{\phi v}{k}.\\ &\rho c_p\tfrac{\partial \Theta}{\partial t}=K\tfrac{\partial^2\Theta}{\partial x^2}+\mu(\tfrac{\partial v}{\partial x})^2+\alpha_1\tfrac{\partial}{\partial t}(\tfrac{\partial v}{\partial x})^2+\beta_1\tfrac{\partial^2}{\partial t^2}(\tfrac{\partial v}{\partial x})^2\\ &+2(\beta_2+\beta_3)(\tfrac{\partial v}{\partial x})^3+\tfrac{16\sigma'\Theta_0^3}{3\alpha'}\tfrac{d^2\Theta}{dx^2}.\\ &\tfrac{\partial C}{\partial t}=D\tfrac{\partial^2 C}{\partial x^2}+\mu(\tfrac{\partial v}{\partial x})^2+\alpha_1\tfrac{\partial}{\partial t}(\tfrac{\partial v}{\partial x})^2+\beta_1\tfrac{\partial^2}{\partial t^2}(\tfrac{\partial v}{\partial x})^2\\ &+2(\beta_2+\beta_3)(\tfrac{\partial v}{\partial x})^3-K_c'(C-C_0). \end{aligned}\right\} \quad (12)$$

Solve the equation (12) to introduced below boundary condition,

$$\left.\begin{aligned} v(x,t) &= V+VCos\omega t,\ \Theta(x,t)=\Theta_0,\ C(x,t)=C_0\ at\ x=\delta\\ v(-x,t) &= V+VCos\omega t,\ \Theta(-x,t)=\Theta_1,\ C(-x,t)=C_0\ at\ x=-\delta \end{aligned}\right\} (13)$$

2.1 Solution Method

(i)Determination of velocity, temperature and concentration:

Non-dimensional variables are introduced:

$$\left.\begin{array}{l} \overline{v}=\frac{v}{V},\ \overline{t}=\frac{t\mu}{\delta^2\rho},\ \overline{x}=\frac{x}{\delta},\ S_t=\frac{\rho\delta^2 g}{\nu V},\ M=\frac{\sigma B_0^2\delta^2}{\mu_0},\ \alpha=\frac{\alpha_1}{\rho\delta^2}, \\ S_c=\frac{\nu}{D},\ \beta=\frac{(\beta_2+\beta_3)V^2}{\mu\delta^2},\ \overline{\beta}_1=\frac{\beta_1\mu}{\delta^2},\ N=\frac{16\sigma'\Theta_0^3}{3\alpha' k},\ \overline{\theta}=\frac{\Theta-\Theta_0}{\Theta_1-\Theta_0},\ N_c=\frac{V^2}{c_p(\Theta_1-\Theta_0)}, \\ P_r=\frac{c_p\mu}{k},\ \overline{c}=\frac{C-C_0}{C_1-C_0},\ N_c=\frac{u^2\rho}{C_1-C_0},\ K=\frac{K'\delta^2}{\nu} \end{array}\right\}(14)$$

Using the required dimensionless quantities in (12) and dropping the symbol for simplicity, the non-dimensionless form of equations becomes,

$$\left.\begin{array}{l} \frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+\alpha\frac{\partial}{\partial t}(\frac{\partial^2 v}{\partial x^2})+6\beta(\frac{\partial v}{\partial t})^2(\frac{\partial^2 v}{\partial x^2})+\beta_1\frac{\partial^2}{\partial t^2}(\frac{\partial^2 v}{\partial x^2})-(M+\lambda)v(x) \\ -S_t-2\beta\lambda(\frac{\partial v}{\partial t})^2 v(x) \end{array}\right\}\quad(15)$$

$$\frac{\partial\theta}{\partial t}=\frac{(1+N)}{P_r}\frac{\partial^2\theta}{\partial x^2}\left\{E_c(\frac{\partial v}{\partial x})^2+\alpha(\frac{\partial v}{\partial x})(\frac{\partial^2 v}{\partial t\partial x})+\beta_1\frac{\partial^2}{\partial t^2}(\frac{\partial v}{\partial x})^2+2\beta(\frac{\partial v}{\partial x})^4\right\}\quad(16)$$

$$\left.\begin{array}{l} \frac{\partial c}{\partial t}=\frac{1}{S_c}\frac{\partial^2 c}{\partial x^2}+N_c((\frac{\partial v}{\partial x})^2+\alpha(\frac{\partial v}{\partial x})(\frac{\partial^2 v}{\partial t\partial x})+\beta_1\frac{\partial^2}{\partial t^2}(\frac{\partial v}{\partial x})^2 \\ +2\beta(\frac{\partial v}{\partial x})^4)-Kc \end{array}\right\}\quad(17)$$

where, M is a magnetic parameter, B_r be a Brinkman number, λ be a dimensionless number, β is non-Newtonian effect, P_r denoted as Prandtl number, N_c represents as concentration difference parameter, S_c denoted by schmidt number, α is second order variable and S_t is the stock number, β_1 describes the material moduli of third grade fluid.

Required dimensionless boundary conditions (13) form are:

$$\left.\begin{array}{l} v\ =\ 1+Cos\omega t,\ \theta\ =\ 1,\ c\ =\ 1\ at\ x\ =\ 1 \\ v\ =\ 1+Cos\omega t,\ \theta\ =\ 0,\ c\ =\ 0\ at\ x\ =\ -1 \end{array}\right\}\quad(18)$$

To find the velocity, thermal field and concentration profile Using perturbation techniques are,

$$\left.\begin{array}{l} v(x,t) = v_0(x) + \beta e^{nt} v_1(x) + 0(\beta^2)....\\ \theta(x,t) = \theta_0(x) + \beta e^{nt}\theta_1(x) + 0(\beta^2)....\\ c(y,t) = c_0(x) + \beta e^{nt} c_1(x) + 0(\beta^2).... \end{array}\right\} \quad (19)$$

Base part of lifting problem:

$$\frac{\partial^2 v_0}{\partial x^2} - S_t - (M+\lambda)v_0 = 0$$

Required solution of the equation are,

$$v_0(x) = C_1 e^{-\sqrt{(M+\lambda)}x} + C_2 e^{\sqrt{(M+\lambda)}x} - \frac{S_t}{(M+\lambda)}, \quad (20)$$

$$v_0(1) = 1 + Cos\omega t,\ v_0(-1) = 1 + Cos\omega t, \quad (21)$$

From the boundary condition (21) applying in (20) , the base result is,

$$v_0(x) = \left\{\frac{((1+Cos\omega t) + \frac{S_t}{(M+\lambda)})(1 - \frac{2Sinh\sqrt{(M+\lambda)}}{2Sinh2\sqrt{(M+\lambda)}})}{e^{-\sqrt{(M+\lambda)}}}\right\} e^{-\sqrt{(M+\lambda)}x} +$$

$$\left\{((1+Cos\omega t) + \frac{S_t}{(M+\lambda)})(\frac{2Sinh\sqrt{(M+\lambda)}}{2Sinh2\sqrt{(M+\lambda)}})\right\} e^{-\sqrt{(M+\lambda)}x} - \frac{S_t}{(M+\lambda)} \quad (22)$$

Perturbation part of lifting problem:

$$(1+\alpha n + n^2\beta_1)\frac{\partial^2 v_1}{\partial x^2} + (\frac{\partial v_0}{\partial x})^2 \left\{6(\frac{\partial^2 v_0}{\partial x^2}) - 2\lambda v_0\right\} - (M+\lambda+n)v_1 = 0$$

The prescribed solution of the equation are,

$$\left.\begin{array}{l} v_1(x) = C_3 e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} + C_4 e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} + a_1 l_1 e^{-3\sqrt{(M+\lambda)}x}\\ +a_1 l_2 e^{3\sqrt{(M+\lambda)}x} + a_2 l_3 e^{\sqrt{(M+\lambda)}x} + a_2 l_4 e^{-\sqrt{(M+\lambda)}x}\\ +a_3 l_5 e^{-2\sqrt{(M+\lambda)}x} + a_3 l_6 e^{2\sqrt{(M+\lambda)}x} + a_4 l_7 \end{array}\right\} \quad (23)$$

$$v_1(1) = 0, v_1(-1) = 0, \tag{24}$$

From the boundary condition (24) substitute in (23), the perturbation result are,

$$\left.\begin{array}{l} v_1(x) = \left\{ \left(\frac{f_1 e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} - f_2 e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}}{2Sinh2\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}}\right) e^{2\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}} \right. \\ \left. + f_1 e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} \right\} e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} + \left\{ \left(\frac{f_2 e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} - f_1 e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}}{2Sinh2\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}}\right) \right\} \\ e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} + a_1 l_1 e^{-3\sqrt{(M+\lambda)}x} + a_1 l_2 e^{3\sqrt{(M+\lambda)}x} + a_2 l_3 e^{\sqrt{(M+\lambda)}x} \\ + a_2 l_4 e^{-\sqrt{(M+\lambda)}x} + a_3 l_5 e^{-2\sqrt{(M+\lambda)}x} + a_3 l_6 e^{2\sqrt{(M+\lambda)}x} + a_4 l_7 \end{array}\right\} \tag{25}$$

Combined those solutions and neglecting the higher order terms (β^2) in equation (19), gives lifting velocity are,

$$\left.\begin{array}{l} v(x) = C_1 e^{-\sqrt{(M+\lambda)}x} + c_2 e^{\sqrt{(M+\lambda)}x} - \frac{S_t}{(M+\lambda)} + \beta e^{nt} \left\{ c_3 e^{-\sqrt{(\frac{M+\lambda+n}{1+\lambda n+n^2\beta_1})}x} \right. \\ + c_4 e^{\sqrt{(\frac{M+\lambda+n}{1+\lambda n+n^2\beta_1})}x} + l_1 a_1 e^{-3\sqrt{(M+\lambda)}x} + l_2 a_1 e^{3\sqrt{(M+\lambda)}x} + l_3 a_2 e^{\sqrt{(M+\lambda)}x} \\ + l_4 a_2 e^{-\sqrt{(M+\lambda)}x} + l_5 a_3 e^{-2\sqrt{(M+\lambda)}x} + l_6 a_3 e^{\sqrt{2(M+\lambda)}x} + l_7 a_4 \end{array}\right\} \tag{26}$$

2.2 Temperature analysis of lifting flow problem:

Base part of lifting problem:

$$\frac{(1+N)}{P_r}\frac{\partial^2\Theta_0}{\partial x^2} + E_c(\frac{\partial v_0}{\partial x})^2 = 0$$

The appropriate solution of above equation,

$$\Theta_0(x) = C_5 x + C_6 - \frac{P_r E_c}{(1+N)}\left\{ r_1 e^{-2\sqrt{(M+\lambda)}x} + r_2 e^{2\sqrt{(M+\lambda)}x} + r_3 x^2 \right\} \tag{27}$$

$$\theta_0(1) = 1, \theta_0(0) = -1. \tag{28}$$

From the boundary condition (28) applying in (27), the base out comes are,

$$\left.\begin{array}{l} \Theta_0(x) = (\frac{g_2 - g_1 - 2}{2}x) + (\frac{1-(g_1+g_2)}{2}) - \frac{P_r E_c}{(1+N)}\left\{ r_1 e^{-2\sqrt{(M+\lambda)}x} + \right. \\ \left. r_2 e^{2\sqrt{(M+\lambda)}x} + r_3 x^2 \right\} \end{array}\right\} \tag{29}$$

where, $g_1 = -\frac{P_r E_c}{(1+N)}(r_1 e^{-2\sqrt{(M+\lambda)}} + r_2 e^{2\sqrt{(M+\lambda)}} + r_3)$,

$g_2 = -\frac{P_r E_c}{(1+N)}(r_1 e^{2\sqrt{(M+\lambda)}} + r_2 e^{-2\sqrt{(M+\lambda)}} + r_3)$

Perturbation part of lifting problem:

$$\frac{(1+N)}{P_r}(\frac{\partial^2\Theta_1}{\partial x^2}) - n\Theta_1 + \left\{E_c(2+\alpha n + n^2\beta_1)(\frac{\partial v_0}{\partial x})(\frac{\partial v_1}{\partial x}) + 2(\frac{\partial v_0}{\partial x})^4\right\} = 0$$

Solution of prescribed equation are,

$$\Theta_1(x) = C_7 e^{-\sqrt{\frac{nP_r}{(1+N)}}x}x + C_8 e^{\sqrt{\frac{nP_r}{(1+N)}}x} - \frac{E_c P_r}{(1+N)}\left\{(2+\alpha n + n^2\beta_1)(p_1 e^{-(b_1+b)x}\right.$$

$$+p_2 e^{(b_1-b)x} + p_3 e^{-4bx} + p_4 e^{2b_1 x} + p_5 + p_6 e^{-4bx} + p_7 e^{-3bx} + p_8 e^{bx} + p_9 + p_{10} e^{(b_1+b)x}$$

$$+p_{11} e^{-2bx} + p_{12} e^{4bx} + p_{13} e^{2bx} + p_{14} + p_{15} e^{-bx} + p_{16} e^{3bx}) + p_{17} e^{-4bx}$$

$$\left.+p_{18} e^{4bx} + p_{19} e^{2bx} + p_{20} e^{-2bx} + p_{21}\right\} \quad (30)$$

$$\Theta_1(0) = 1, \Theta_1(0) = -1. \quad (31)$$

From the boundary condition (31) substitute in (30) , the perturbation result are,

$$\Theta_1(x) = \frac{-1}{e^{-\sqrt{(\frac{nP_r}{1+N})}x}}\left\{h_1 + (\frac{h_2 e^{-\sqrt{(\frac{nP_r}{1+N})}x} - h_1 e^{-\sqrt{(\frac{nP_r}{1+N})}x}}{2Sinh2\sqrt{(\frac{nP_r}{1+N})}x})\right\}e^{-\sqrt{(\frac{nP_r}{1+N})}x} +$$

$$\left\{\frac{h_2 e^{-\sqrt{(\frac{nP_r}{1+N})}x} - h_1 e^{\sqrt{(\frac{nP_r}{1+N})}x}}{2Sinh2\sqrt{(\frac{nP_r}{1+N})}x}\right\}e^{\sqrt{(\frac{nP_r}{1+N})}x} - \frac{E_c P_r}{(1+N)}$$

$$\left\{(2+\alpha n + n^2\beta_1)(p_1 e^{-(b_1+b)x} + p_2 e^{(b_1-b)x} + p_3 e^{-4bx} + p_4 e^{2b_1 x}\right.$$

$$+p_5 + p_6 e^{-4bx} + p_7 e^{-3bx} + p_8 e^{bx} + p_9 + p_{10} e^{(b_1+b)x} + p_{11} e^{-2bx} + p_{12} e^{4bx} + p_{13} e^{2bx}$$

$$\left.+p_{14} + p_{15} e^{-bx} + p_{16} e^{3bx}) + p_{17} e^{-4bx} + p_{18} e^{4bx} + p_{19} e^{2bx} + p_{20} e^{-2bx} + p_{21}\right\} \quad (32)$$

Combined those solutions and neglecting the higher order terms (β^2) in equation (19), gives lifting temperature flow,

$$\theta(x) = C_5 x + C_6 - \frac{P_r E_c}{(1+N)}\left\{r_1 e^{-2\sqrt{(M+\lambda)}x} + r_2 e^{2\sqrt{(M+\lambda)}x} + r_3 x^2\right\}$$

$$+\beta e^{nt}\left\{C_7 e^{-\sqrt{\frac{nP_r}{(1+N)}}x} x + C_8 e^{\sqrt{\frac{nP_r}{(1+N)}}x} - \frac{E_c P_r}{(1+N)}\left\{(2+\alpha n + n^2\beta_1)(p_1 e^{-(b_1+b)x}\right.\right.$$

$$+p_2 e^{(b_1-b)x} + p_3 e^{-4bx} + p_4 e^{2b_1 x} + p_5 + p_6 e^{-4bx} + p_7 e^{-3bx} + p_8 e^{bx} + p_9 + p_{10} e^{(b_1+b)x}$$

$$+p_{11} e^{-2bx} + p_{12} e^{4bx} + p_{13} e^{2bx} + p_{14} + p_{15} e^{-bx} + p_{16} e^{3bx}) + p_{17} e^{-4bx}$$

$$\left.\left. +p_{18} e^{4bx} + p_{19} e^{2bx} + p_{20} e^{-2bx} + p_{21}\right\}\right\} \tag{33}$$

where, $h_1 = -\frac{E_c P_r}{(1+N)}((2+\alpha n+n^2\beta_1)(p_1 e^{-(b_1+b)} + p_2 e^{(b_1-b)} + p_3 e^{-4b} + p_4 e^{2b_1} + p_5 + p_6 e^{-4b} + p_7 e^{-3b} + p_8 e^{b} + p_9 + p_{10} e^{(b_1+b)} + p_{11} e^{-2b} + p_{12} e^{4b} + p_{13} e^{2b} + p_{14} + p_{15} e^{-b} + p_{16} e^{3b}) + p_{17} e^{-4b} + p_{18} e^{4b} + p_{19} e^{2b} + p_{20} e^{-2b} + p_{21})$,

$h_2 = -\frac{E_c P_r}{(1+N)}((2+\alpha n+n^2\beta_1)(p_1 e^{(b_1+b)} + p_2 e^{-(b_1-b)} + p_3 e^{4b} + p_4 e^{-2b_1} + p_5 + p_6 e^{4b} + p_7 e^{3b} + p_8 e^{-b} + p_9 + p_{10} e^{-(b_1+b)} + p_{11} e^{2b} + p_{12} e^{-4b} + p_{13} e^{-2b} + p_{14} + p_{15} e^{b} + p_{16} e^{-3b}) + p_{17} e^{4b} + p_{18} e^{-4b} + p_{19} e^{-2b} + p_{20} e^{2b} + p_{21})$

2.3 Concentration analysis of lifting flow problem:

Base part of lifting problem:

$$\frac{1}{S_c}\frac{\partial^2 c_0}{\partial x^2} + \left\{N_c\left(\frac{\partial v_0}{\partial x}\right)^2\right\} - kc_0 = 0$$

The appropriate solution becomes,

$$c_0(x) = C_9 e^{-\sqrt{kS_c}x} + c_{10} e^{\sqrt{kS_c}x} - N_c S_c\left\{q_1 e^{-2\sqrt{(M+\lambda)}x} + q_2 e^{2\sqrt{(M+\lambda)}x} + q_3\right\} \tag{34}$$

$$c_0(1) = 1, c_0(0) = -1. \tag{35}$$

Substitute boundary condition (35) in (34) , base part are,

$$c_0(x) = (f_4e^{-\sqrt{kS_c}} - (\frac{(1-f_3)e^{\sqrt{kS_c}} + f_4e^{-\sqrt{kS_c}}}{2Sinh2\sqrt{kS_c}})e^{-2\sqrt{kS_c}})e^{\sqrt{kS_c}x} + (\frac{(1-f_3)e^{\sqrt{kS_c}} + f_4e^{-\sqrt{kS_c}}}{2Sinh2\sqrt{kS_c}})$$

$$e^{-\sqrt{kS_c}x} - N_cS_c\left\{q_1e^{-2\sqrt{(M+\lambda)}x} + q_2e^{2\sqrt{(M+\lambda)}x} + q_3\right\} \quad (36)$$

where, $f_3 = q_1e^{-2\sqrt{(M+\lambda)}x} + q_2e^{2\sqrt{(M+\lambda)}x} + q_3$,
$f_4 = q_1e^{2\sqrt{(M+\lambda)}x} + q_2e^{-2\sqrt{(M+\lambda)}x} + q_3$

Perturbation part of lifting problem:

$$\frac{1}{S_c}(\frac{\partial^2 c_1}{\partial x^2}) + N_c\left\{((2+\alpha n+n^2\beta_1)(\frac{\partial v_0}{\partial x})(\frac{\partial v_1}{\partial x}) + 2(\frac{\partial v_0}{\partial x})^4)\right\} - (n+k)c_1 = 0$$

solve the equation, we get

$$c_1(x) = C_{11}e^{-\sqrt{(k+n)S_c x}} + C_{12}e^{\sqrt{(k+n)S_c x}} - N_cS_c\left\{(2+\alpha n+n^2\beta_1)(p_1e^{-(b_1+b)x} + p_2e^{(b_1-b)x}\right.$$

$$+p_3e^{-4bx} + p_4e^{2bx} + p_5 + p_6e^{-2bx} + p_7e^{-3bx} + p_8e^{bx} + p_9 + p_{10}e^{(b_1+b)x} + p_{11}e^{-2bx} + p_{12}e^{4bx}$$

$$\left.+p_{13}e^{2bx} + p_{14}) + p_{15}e^{-bx} + p_{16}e^{3bx} + p_{17}e^{-4bx} + p_{18}e^{4bx} + p_{19}e^{2bx} + p_{20}e^{-2bx}\right\} (37)$$

$$c_1(0) = 1, c_1(0) = -1. \quad (38)$$

Boundary condition (38) substitute in (37) , the perturbation result is,

$$c_1(x) = \left\{(e^{\sqrt{(k+n)S_c}}(\frac{h_3e^{\sqrt{(k+n)S_c}} - h_4e^{\sqrt{(k+n)S_c}}}{2Sinh2e^{\sqrt{(k+n)S_c}}}) + h_3e^{\sqrt{(k+n)S_c}})e^{-\sqrt{(k+n)S_c x}}\right.$$

$$+(\frac{h_3e^{\sqrt{(k+n)S_c}} - h_4e^{\sqrt{(k+n)S_c}}}{2Sinh2e^{\sqrt{(k+n)S_c}}})e^{\sqrt{(k+n)S_c x}} - N_cS_c\left\{(2+\alpha n+n^2\beta_1)(p_1e^{-(b_1+b)x}\right.$$

$$+p_2e^{(b_1-b)x} + p_3e^{-4bx} + p_4e^{2bx} + p_5 + p_6e^{-2bx} + p_7e^{-3bx} + p_8e^{bx} + p_9 + p_{10}e^{(b_1+b)x} + p_{11}e^{-2bx}$$

$$\left.+p_{12}e^{4bx} + p_{13}e^{2bx} + p_{14}) + p_{15}e^{-bx} + p_{16}e^{3bx} + p_{17}e^{-4bx} + p_{18}e^{4bx} + p_{19}e^{2bx} + p_{20}e^{-2bx}\right\} (39)$$

Combined those solutions and neglecting the higher order terms (β^2) in equation (19) , gives lifting concentration flow,

$$c(x) = C_9 e^{-\sqrt{kS_c}x} + c_{10} e^{\sqrt{kS_c}x} - N_c S_c \left\{ q_1 e^{-2\sqrt{(M+\lambda)}x} + q_2 e^{2\sqrt{(M+\lambda)}x} + q_3 \right\}$$

$$+\beta e^{nt} \left\{ (e^{\sqrt{(k+n)S_c}} (\frac{h_3 e^{\sqrt{(k+n)S_c}} - h_4 e^{\sqrt{(k+n)S_c}}}{2Sinh2e^{\sqrt{(k+n)S_c}}}) + h_3 e^{\sqrt{(k+n)S_c}}) e^{-\sqrt{(k+n)S_c}x} \right.$$

$$+(\frac{h_3 e^{\sqrt{(k+n)S_c}} - h_4 e^{\sqrt{(k+n)S_c}}}{2Sinh2e^{\sqrt{(k+n)S_c}}}) e^{\sqrt{(k+n)S_c}x} - N_c S_c \left\{ (2 + \alpha n + n^2 \beta_1)(p_1 e^{-(b_1+b)x} \right.$$

$$+p_2 e^{(b_1-b)x} + p_3 e^{-4bx} + p_4 e^{2bx} + p_5 + p_6 e^{-2bx} + p_7 e^{-3bx} + p_8 e^{bx} + p_9 + p_{10} e^{(b_1+b)x} + p_{11} e^{-2bx}$$

$$\left. +p_{12} e^{4bx} + p_{13} e^{2bx} + p_{14}) + p_{15} e^{-bx} + p_{16} e^{3bx} + p_{17} e^{-4bx} + p_{18} e^{4bx} + p_{19} e^{2bx} + p_{20} e^{-2bx} \right\} \quad (40)$$

where, $h_3 = -N_c S_c \left\{ (2+\alpha n+n^2\beta_1)(p_1 e^{-(b_1+b)x} + p_2 e^{(b_1-b)x} + p_3 e^{-4bx} + p_4 e^{2bx} + p_5 + p_6 e^{-2bx} + p_7 e^{-3bx} + p_8 e^{bx} + p_9 + p_{10} e^{(b_1+b)x} + p_{11} e^{-2bx} + p_{12} e^{4bx} + p_{13} e^{2bx} + p_{14}) + p_{15} e^{-bx} + p_{16} e^{3bx} + p_{17} e^{-4bx} + p_{18} e^{4bx} + p_{19} e^{2bx} + p_{20} e^{-2bx} \right\}$

$$h_4 = -N_c S_c \left\{ (2+\alpha n+n^2\beta_1)(p_1 e^{(b_1+b)x} + p_2 e^{-(b_1-b)x} + p_3 e^{4bx} + p_4 e^{-2bx} + p_5 + p_6 e^{2bx} + p_7 e^{3bx} + p_8 e^{-bx} + p_9 + p_{10} e^{-(b_1+b)x} + p_{11} e^{2bx} + p_{12} e^{-4bx} + p_{13} e^{-2bx} + p_{14}) + p_{15} e^{bx} + p_{16} e^{-3bx} + p_{17} e^{4bx} + p_{18} e^{-4bx} + p_{19} e^{-2bx} + p_{20} e^{2bx} \right\}$$

3 (ii)Drainage flow problem:

In drainage flow problem we consider the belts is vertically not moving and oscillating only as sketched in figure 2 . Remaining axial coordinates and assumptions are same in previous lifting flow problems. Non-Newtonian third order fluid layer draining down to the belt due to gravity. since gravity balance mentioned only positive.

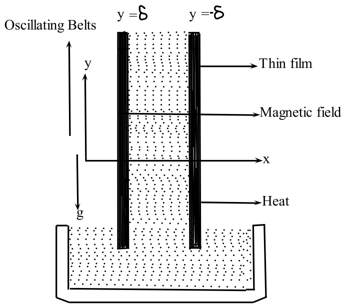

Fig.2 Physical configuration of the drainage flow problem:

Using the stress tensor component equation (11) and with all assumptions in (2), (3) and (4) of drainage flow reduced, the coupled equations are described by,

$$\left.\begin{aligned}
&\rho\frac{\partial v}{\partial t}=\mu\frac{\partial^2 v}{\partial x^2}+\alpha_1\frac{\partial}{\partial t}(\frac{\partial^2 v}{\partial x^2})+\beta_1\frac{\partial^2}{\partial t^2}(\frac{\partial^2 v}{\partial x^2})\\
&+6(\beta_2+\beta_3)(\frac{\partial v}{\partial x})^2(\frac{\partial^2 v}{\partial x^2})+\rho g-\sigma B_0^2 v-(\mu+2\beta_3(\frac{\partial v}{\partial x})^2)\frac{\phi v}{k}.\\
&\rho c_p\frac{\partial \Theta}{\partial t}=K\frac{\partial^2 \Theta}{\partial x^2}+\mu(\frac{\partial v}{\partial x})^2+\alpha_1\frac{\partial}{\partial t}(\frac{\partial v}{\partial x})^2+\beta_1\frac{\partial^2}{\partial t^2}(\frac{\partial v}{\partial x})^2\\
&+2(\beta_2+\beta_3)(\frac{\partial v}{\partial x})^3+\frac{16\sigma\Theta_0^3}{3\alpha'}\frac{d^2\Theta}{dx^2}.\\
&\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial x^2}+\mu(\frac{\partial v}{\partial x})^2+\alpha_1\frac{\partial}{\partial t}(\frac{\partial v}{\partial x})^2+\beta_1\frac{\partial^2}{\partial t^2}(\frac{\partial v}{\partial x})^2\\
&+2(\beta_2+\beta_3)(\frac{\partial v}{\partial x})^3-K_c'(C-C_0).
\end{aligned}\right\}\quad(41)$$

Appropriate boundary condition are introduced, to solve the equation (41),

$$\left.\begin{aligned}
&v(x,t)=VCos\omega t,\ \Theta(x,t)=\Theta_0,\ C(x,t)=C_0\ at\ x=\delta\\
&v(x,t)=VCos\omega t,\ \Theta(-x,t)=\Theta_1,\ C(-x,t)=C_0\ at\ x=-\delta
\end{aligned}\right\}\quad(42)$$

Applying the dimensionless quantities in equations (41) and (42) and neglecting bars we obtain,

$$\left.\begin{aligned}
&\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+\alpha\frac{\partial}{\partial t}(\frac{\partial^2 v}{\partial x^2})+6\beta(\frac{\partial v}{\partial t})^2(\frac{\partial^2 v}{\partial x^2})-(M+\lambda)v(x)\\
&+S_t+2\beta\lambda(\frac{\partial v}{\partial t})^2 v(x)
\end{aligned}\right\}\quad(43)$$

$$\frac{\partial\theta}{\partial t}=\frac{(1+N)}{P_r}\frac{\partial^2\theta}{\partial x^2}\left\{E_c(\frac{\partial v}{\partial x})^2+\alpha(\frac{\partial v}{\partial x})(\frac{\partial^2 v}{\partial t\partial x})+\beta_1\frac{\partial^2}{\partial t^2}(\frac{\partial v}{\partial x})^2+2\beta(\frac{\partial v}{\partial x})^4\right\}\quad(44)$$

$$\frac{\partial c}{\partial t}=\frac{1}{S_c}\frac{\partial^2 c}{\partial x^2}+N_c((\frac{\partial v}{\partial x})^2+\alpha(\frac{\partial v}{\partial x})(\frac{\partial^2 v}{\partial t\partial x})+\beta_1\frac{\partial^2}{\partial t^2}(\frac{\partial v}{\partial x})^2+2\beta(\frac{\partial v}{\partial x})^4)-Kc$$

$$\left.\begin{aligned} v &= Cos\omega t,\ \theta = 1,\ c = 1\ at\ x = 1 \\ v &= Cos\omega t,\ \theta = 0,\ c = 0\ at\ x = -1 \end{aligned}\right\} \tag{45}$$

Form equations $(43)-(45)$ solved analytically using perturbation techniques, The system of equations are derived by,

Base part of Drainage problem:

$$\frac{\partial^2 v_0}{\partial x^2}+S_t-(M+\lambda)v_0=0$$

The required solution of the equation are,

$$v_0(x)=C_1'e^{-\sqrt{(M+\lambda)}x}+C_2'e^{\sqrt{(M+\lambda)}x}+\frac{S_t}{(M+\lambda)} \tag{46}$$

$$v_0(1)=Cos\omega t,\ v_0(-1)=Cos\omega t, \tag{47}$$

Boundary condition (48) applying in (20), the base solution are,

$$v_0(x)=\left\{\frac{((Cos\omega t)-\frac{S_t}{(M+\lambda)})(1-\frac{2Sinh\sqrt{(M+\lambda)}}{2Sinh2\sqrt{(M+\lambda)}})}{e^{-\sqrt{(M+\lambda)}}}\right\}e^{-\sqrt{(M+\lambda)}x}$$

$$+\left\{((Cos\omega t)-\frac{S_t}{(M+\lambda)})(\frac{2Sinh\sqrt{(M+\lambda)}}{2Sinh2\sqrt{(M+\lambda)}})\right\}e^{-\sqrt{(M+\lambda)}x}+\frac{S_t}{(M+\lambda)} \tag{48}$$

Perturbation part of Drainage problem:

$$(1+\alpha n+n^2\beta_1)\frac{\partial^2 v_1}{\partial x^2}+(\frac{\partial v_0}{\partial x})^2\left\{6(\frac{\partial^2 v_0}{\partial x^2})-2\lambda v_0\right\}-(M+\lambda+n)v_1=0$$

Solution of the equations are,

$$\left.\begin{aligned} v_1(x) &= C_3'e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}+C_4'e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}+a_1'l_1'e^{-3\sqrt{(M+\lambda)}x} \\ &+a_1'l_2'e^{3\sqrt{(M+\lambda)}x}+a_2'l_3'e^{\sqrt{(M+\lambda)}x}+a_2'l_4'e^{-\sqrt{(M+\lambda)}x} \\ &+a_3'l_5'e^{-2\sqrt{(M+\lambda)}x}+a_3'l_6'e^{2\sqrt{(M+\lambda)}x}+a_4'l_7' \end{aligned}\right\} \tag{49}$$

$$v_1(1)=0, v_1(-1)=0, \tag{50}$$

From the boundary condition (51) substitute in (50), the perturbation result is,

$$\left.\begin{array}{l} v_1(x)=\left\{(\frac{f_1' e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}-f_2' e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}}{2Sinh2\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}})+e^{2\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}}\right. \\ \left.+f_1' e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}\right\}e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}+\left\{(\frac{f_2' e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}-f_1' e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}}{2Sinh2\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}})\right\} \\ e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}+a_1' l_1' e^{-3\sqrt{(M+\lambda)}x}+a_1' l_2' e^{3\sqrt{(M+\lambda)}x}+a_2' l_3' e^{\sqrt{(M+\lambda)}x} \\ +a_2' l_4' e^{-\sqrt{(M+\lambda)}x}+a_3' l_5' e^{-2\sqrt{(M+\lambda)}x}+a_3' l_6' e^{2\sqrt{(M+\lambda)}x}+a_4' l_7' \end{array}\right\} \tag{51}$$

Combined those drainage velocity and neglecting the higher order terms (β^2) in equation (19), The velocity becomes,

$$\left.\begin{array}{l} v(x)=C_1' e^{-\sqrt{(M+\alpha)}x}+C_2' e^{\sqrt{(M+\alpha)}x}+\frac{S_t}{(M+\alpha)}+\beta e^{nt}\left\{C_3' e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}\right. \\ +C_4' e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}+l_1' a_1' e^{-3\sqrt{(M+\alpha)}x}+l_2' a_1' e^{3\sqrt{(M+\alpha)}x}+l_3' a_2' e^{\sqrt{(M+\alpha)}x} \\ +l_4' a_2' e^{-\sqrt{(M+\alpha)}x}+l_5' a_3' e^{-2\sqrt{(M+\alpha)}x}+l_6' a_3' e^{\sqrt{2(M+\alpha)}x}+l_7' a_4' \end{array}\right\} \tag{52}$$

3.1 Temperature analysis of drainage flow problem:

Base part of drainage problem:

$$\frac{(1+N)}{P_r}\frac{\partial^2\Theta_0}{\partial x^2}+E_c(\frac{\partial v_0}{\partial x})^2=0$$

solve the base equation,

$$\Theta_0(x)=C_5' x+C_6'-\frac{P_r E_c}{(1+N)}\left\{r_1' e^{-2\sqrt{(M+\lambda)}x}+r_2' e^{2\sqrt{(M+\lambda)}x}+r_3' x^2\right\} \tag{53}$$

$$\theta_0(1)=1, \theta_0(0)=-1. \tag{54}$$

Applying boundary condition (55) in (54), the base solution is,

$$\Theta_0(x)=(\frac{g_2'-g_1'-2}{2}x)+(\frac{1-(g_1'+g_2')}{2})-\frac{P_r E_c}{(1+N)}\left\{r_1' e^{-2\sqrt{(M+\lambda)}x}\right.$$

$$+r_2'e^{2\sqrt{(M+\lambda)}x}+r_3'x^2\Big\} \tag{55}$$

where, $g_1'=-\frac{P_rE_c}{(1+N)}(r_1'e^{-2\sqrt{(M+\lambda)}}+r_2'e^{2\sqrt{(M+\lambda)}}+r_3')$,
$g_2'=-\frac{P_rE_c}{(1+N)}(r_1'e^{2\sqrt{(M+\lambda)}}+r_2'e^{-2\sqrt{(M+\lambda)}}+r_3')$.

Perturbation part of drainage problem:

$$\frac{(1+N)}{P_r}(\frac{\partial^2\Theta_1}{\partial x^2})-n\Theta_1+\left\{E_c(2+\alpha n+n^2\beta_1)(\frac{\partial v_0}{\partial x})(\frac{\partial v_1}{\partial x})+2(\frac{\partial v_0}{\partial x})^4\right\}=0$$

The solution of equation becomes,

$$\Theta_1(x)=C_7'e^{-\sqrt{\frac{nP_r}{(1+N)}}x}x+C_8'e^{\sqrt{\frac{nP_r}{(1+N)}}x}-\frac{E_cP_r}{(1+N)}\Big\{(2+\alpha n+n^2\beta_1)(p_1'e^{-(b_1+b)x}$$

$$+p_2'e^{(b_1-b)x}+p_3'e^{-4bx}+p_4'e^{2b_1x}+p_5'+p_6'e^{-4bx}+p_7'e^{-3bx}+p_8'e^{bx}+p_9'+p_{10}'e^{(b_1+b)x}$$

$$+p_{11}'e^{-2bx}+p_{12}'e^{4bx}+p_{13}'e^{2bx}+p_{14}'+p_{15}'e^{-bx}+p_{16}'e^{3bx})+p_{17}'e^{-4bx}$$

$$+p_{18}'e^{4bx}+p_{19}'e^{2bx}+p_{20}'e^{-2bx}+p_{21}'\Big\} \tag{56}$$

$$\Theta_1(0)=1,\Theta_1(0)=-1. \tag{57}$$

From boundary condition (58) applying in (57) , the perturbation solution is,

$$\Theta_1(x)=\frac{-1}{e^{-\sqrt{(\frac{nP_r}{1+N})}x}}\left\{h_1'+(\frac{h_2'e^{-\sqrt{(\frac{nP_r}{1+N})}x}-h_1'e^{-\sqrt{(\frac{nP_r}{1+N})}x}}{2Sinh2\sqrt{(\frac{nP_r}{1+N})}x})\right\}e^{-\sqrt{(\frac{nP_r}{1+N})}x}$$

$$+\left\{\frac{h_2'e^{-\sqrt{(\frac{nP_r}{1+N})}x}-h_1'e^{\sqrt{(\frac{nP_r}{1+N})}x}}{2Sinh2\sqrt{(\frac{nP_r}{1+N})}x}\right\}e^{\sqrt{(\frac{nP_r}{1+N})}x}-\frac{E_cP_r}{(1+N)}\Big\{(2+\alpha n+n^2\beta_1)$$

$$(p_1'e^{-(b_1+b)x}+p_2'e^{(b_1-b)x}+p_3'e^{-4bx}+p_4'e^{2b_1x}+p_5'+p_6'e^{-4bx}+p_7'e^{-3bx}+$$

$$p_8'e^{bx}+p_9'+p_{10}'e^{(b_1+b)x}+p_{11}'e^{-2bx}+p_{12}'e^{4bx}+p_{13}'e^{2bx}+p_{14}'+p_{15}'e^{-bx}$$

$$+p_{16}'e^{3bx})+p_{17}'e^{-4bx}+p_{18}'e^{4bx}+p_{19}'e^{2bx}+p_{20}'e^{-2bx}+p_{21}'\Big\} \tag{58}$$

Combined those drainage temperature and neglecting the higher order terms (β^2) in equation (19), The solution becomes,

$$\theta(x) = C_5'x + C_6' - \frac{P_rE_c}{(1+N)}\left\{r_1'e^{-2\sqrt{(M+\lambda)}x} + r_2'e^{2\sqrt{(M+\lambda)}x} + r_3'x^2\right\}$$

$$+\beta e^{nt}\left\{C_7'e^{-\sqrt{\frac{nP_r}{(1+N)}}x}x + C_8'e^{\sqrt{\frac{nP_r}{(1+N)}}x} - \frac{E_cP_r}{(1+N)}\left\{(2+\alpha n+n^2\beta_1)(p_1'e^{-(b_1+b)x}\right.\right.$$

$$+p_2'e^{(b_1-b)x} + p_3'e^{-4bx} + p_4'e^{2b_1x} + p_5' + p_6e^{-4bx} + p_7'e^{-3bx} + p_8'e^{bx} + p_9' + p_{10}'e^{(b_1+b)x}$$

$$+p_{11}'e^{-2bx} + p_{12}'e^{4bx} + p_{13}'e^{2bx} + p_{14}' + p_{15}'e^{-bx} + p_{16}'e^{3bx}) + p_{17}'e^{-4bx}$$

$$\left.\left.+p_{18}'e^{4bx} + p_{19}'e^{2bx} + p_{20}'e^{-2bx} + p_{21}'\right\}\right\} \tag{59}$$

where, $h_1' = -\frac{E_cP_r}{(1+N)}((2+\alpha n+n^2\beta_1)(p_1'e^{-(b_1+b)} + p_2'e^{(b_1-b)} + p_3'e^{-4b} + p_4'e^{2b_1} + p_5' + p_6'e^{-4b} + p_7'e^{-3b} + p_8'e^{b} + p_9' + p_{10}'e^{(b_1+b)} + p_{11}'e^{-2b} + p_{12}'e^{4b} + p_{13}'e^{2b} + p_{14}' + p_{15}'e^{-b} + p_{16}'e^{3b}) + p_{17}'e^{-4b} + p_{18}'e^{4b} + p_{19}'e^{2b} + p_{20}'e^{-2b} + p_{21}')$,

$h_2' = -\frac{E_cP_r}{(1+N)}((2+\alpha n+n^2\beta_1)(p_1'e^{(b_1+b)} + p_2'e^{-(b_1-b)} + p_3'e^{4b} + p_4'e^{-2b_1} + p_5' + p_6'e^{4b} + p_7'e^{3b} + p_8'e^{-b} + p_9' + p_{10}'e^{-(b_1+b)} + p_{11}'e^{2b} + p_{12}'e^{-4b} + p_{13}'e^{-2b} + p_{14}' + p_{15}'e^{b} + p_{16}'e^{-3b}) + p_{17}'e^{4b} + p_{18}'e^{-4b} + p_{19}'e^{-2b} + p_{20}'e^{2b} + p_{21}')$

3.2 Concentration analysis of drainage flow problem:

Base part of drainage problem:

$$\frac{1}{S_c}\frac{\partial^2 c_0}{\partial x^2} + \left\{N_c(\frac{\partial v_0}{\partial x})^2\right\} - kc_0 = 0$$

Solution of the equation as become,

$$c_0(x) = C_9'e^{-\sqrt{kS_c}x} + C_{10}'e^{\sqrt{kS_c}x} - N_cS_c\left\{q_1'e^{-2\sqrt{(M+\lambda)}x}\right.$$

$$\left.+q_2'e^{2\sqrt{(M+\lambda)}x} + q_3'\right\} \tag{60}$$

$$c_0(1) = 1, c_0(0) = -1. \tag{61}$$

From boundary condition (62) substitute in (61) , base part is,

$$c_0(x) = (f_4^{'}e^{-\sqrt{kS_c}} - (\frac{(1-f_3^{'})e^{\sqrt{kS_c}} + f_4^{'}e^{-\sqrt{kS_c}}}{2Sinh2\sqrt{kS_c}})e^{-2\sqrt{kS_c}})e^{\sqrt{kS_c}x}$$

$$+(\frac{(1-f_3^{'})e^{\sqrt{kS_c}} + f_4^{'}e^{-\sqrt{kS_c}}}{2Sinh2\sqrt{kS_c}})e^{-\sqrt{kS_c}x}$$

$$-N_cS_c\left\{q_1^{'}e^{-2\sqrt{(M+\lambda)}x} + q_2^{'}e^{2\sqrt{(M+\lambda)}x} + q_3^{'}\right\} \tag{62}$$

where, $f_3^{'} = q_1^{'}e^{-2\sqrt{(M+\lambda)}x} + q_2^{'}e^{2\sqrt{(M+\lambda)}x} + q_3^{'}$,
$f_4^{'} = q_1^{'}e^{2\sqrt{(M+\lambda)}x} + q_2^{'}e^{-2\sqrt{(M+\lambda)}x} + q_3^{'}$

Perturbation part of lifting problem:

$$\frac{1}{S_c}(\frac{\partial^2 c_1}{\partial x^2}) + N_c\left\{((2+\alpha n + n^2\beta_1)(\frac{\partial v_0}{\partial x})(\frac{\partial v_1}{\partial x}) + 2(\frac{\partial v_0}{\partial x})^4)\right\} - (n+k)c_1 = 0$$

solve the equation, we get

$$c_1(x) = C_{11}^{'}e^{-\sqrt{(k+n)S_c x}} + C_{12}^{'}e^{\sqrt{(k+n)S_c x}} - N_cS_c\left\{(2+\alpha n + n^2\beta_1)(p_1^{'}e^{-(b_1+b)x} + p_2^{'}e^{(b_1-b)x}\right.$$

$$+p_3^{'}e^{-4bx} + p_4^{'}e^{2bx} + p_5^{'} + p_6^{'}e^{-2bx} + p_7^{'}e^{-3bx} + p_8^{'}e^{bx} + p_9^{'} + p_{10}^{'}e^{(b_1+b)x} + p_{11}^{'}e^{-2bx} + p_{12}^{'}e^{4bx}$$

$$\left.+p_{13}^{'}e^{2bx} + p_{14}^{'}) + p_{15}^{'}e^{-bx} + p_{16}^{'}e^{3bx} + p_{17}^{'}e^{-4bx} + p_{18}^{'}e^{4bx} + p_{19}^{'}e^{2bx} + p_{20}^{'}e^{-2bx}\right\} \tag{63}$$

$$c_1(0) = 1, c_1(0) = -1. \tag{64}$$

Boundary condition (65) applying in (64) , the perturbation are,

$$c_1(x) = \left\{(e^{\sqrt{(k+n)S_c}}(\frac{h_3^{'}e^{\sqrt{(k+n)S_c}} - h_4^{'}e^{\sqrt{(k+n)S_c}}}{2Sinh2e^{\sqrt{(k+n)S_c}}}) + h_3^{'}e^{\sqrt{(k+n)S_c}})e^{-\sqrt{(k+n)S_c x}}\right.$$

$$+(\frac{h_3^{'}e^{\sqrt{(k+n)S_c}} - h_4^{'}e^{\sqrt{(k+n)S_c}}}{2Sinh2e^{\sqrt{(k+n)S_c}}})e^{\sqrt{(k+n)S_c x}} - N_cS_c\left\{(2+\alpha n + n^2\beta_1)(p_1^{'}e^{-(b_1+b)x}\right.$$

$$+p_2'e^{(b_1-b)x}+p_3'e^{-4bx}+p_4'e^{2bx}+p_5'+p_6'e^{-2bx}+p_7'e^{-3bx}+p_8'e^{bx}+p_9'+p_{10}'e^{(b_1+b)x}+p_{11}'e^{-2bx}$$

$$+p_{12}'e^{4bx}+p_{13}'e^{2bx}+p_{14}')+p_{15}'e^{-bx}+p_{16}'e^{3bx}+p_{17}'e^{-4bx}+p_{18}'e^{4bx}+p_{19}'e^{2bx}+p_{20}'e^{-2bx}\Big\}(65)$$

Combined those drainage concentration and neglecting the higher order terms (β^2) in equation (19), The solution becomes,

$$c(x)=C_9'e^{-\sqrt{kS_c}x}+C_{10}'e^{\sqrt{kS_c}x}-N_cS_c\Big\{q_1'e^{-2\sqrt{(M+\lambda)}x}+q_2'e^{2\sqrt{(M+\lambda)}x}+q_3'\Big\}$$

$$+\beta e^{nt}\Big\{(e^{\sqrt{(k+n)S_c}}(\frac{h_3e^{\sqrt{(k+n)S_c}}-h_4e^{\sqrt{(k+n)S_c}}}{2Sinh2e^{\sqrt{(k+n)S_c}}})+h_3e^{\sqrt{(k+n)S_c}})e^{-\sqrt{(k+n)S_c}x}$$

$$+(\frac{h_3e^{\sqrt{(k+n)S_c}}-h_4e^{\sqrt{(k+n)S_c}}}{2Sinh2e^{\sqrt{(k+n)S_c}}})e^{\sqrt{(k+n)S_c}x}-N_cS_c\Big\{(2+\alpha n+n^2\beta_1)(p_1'e^{-(b_1+b)x}$$

$$+p_2'e^{(b_1-b)x}+p_3'e^{-4bx}+p_4'e^{2bx}+p_5'+p_6'e^{-2bx}+p_7'e^{-3bx}+p_8'e^{bx}+p_9'+p_{10}'e^{(b_1+b)x}+p_{11}'e^{-2bx}$$

$$+p_{12}'e^{4bx}+p_{13}'e^{2bx}+p_{14}')+p_{15}'e^{-bx}+p_{16}'e^{3bx}+p_{17}'e^{-4bx}+p_{18}'e^{4bx}+p_{19}'e^{2bx}+p_{20}'e^{-2bx}\Big\}(66)$$

where, $h_3=-N_cS_c\Big\{(2+\alpha n+n^2\beta_1)(p_1'e^{-(b_1+b)x}+p_2'e^{(b_1-b)x}+p_3'e^{-4bx}+p_4'e^{2bx}+p_5'+p_6'e^{-2bx}+p_7'e^{-3bx}+p_8'e^{bx}+p_9'+p_{10}'e^{(b_1+b)x}+p_{11}'e^{-2bx}+p_{12}'e^{4bx}+p_{13}'e^{2bx}+p_{14}')+p_{15}'e^{-bx}+p_{16}'e^{3bx}+p_{17}'e^{-4bx}+p_{18}'e^{4bx}+p_{19}'e^{2bx}+p_{20}'e^{-2bx}\Big\}$

$h_4=-N_cS_c\Big\{(2+\alpha n+n^2\beta_1)(p_1'e^{(b_1+b)x}+p_2'e^{-(b_1-b)x}+p_3'e^{4bx}+p_4'e^{-2bx}+p_5'+p_6'e^{2bx}+p_7'e^{3bx}+p_8'e^{-bx}+p_9'+p_{10}'e^{-(b_1+b)x}+p_{11}'e^{2bx}+p_{12}'e^{-4bx}+p_{13}'e^{-2bx}+p_{14}')+p_{15}'e^{bx}+p_{16}'e^{-3bx}+p_{17}'e^{4bx}+p_{18}'e^{-4bx}+p_{19}'e^{-2bx}+p_{20}'e^{2bx}\Big\}$

4 Results and Discussion

To see the influences of emerging pertinent variables, such as third order parameter (β), magnetic induction (M), stock number S_t, material moduli third order (β_1), porous parameter (λ), second grade number (α), Eckert number (E_c), Prandtl number (P_r), Schmidt number (S_c), Chemical difference parameter (N_c), Chemical reaction rate (K), Oscillatory flow (w). Figures $(3-32)$ have been displayed for both lift and drainage problem. The solutions are discussed numerically and graphically depicted by MATHEMATICA 10.0. The effects of parameters on velocity are depicted for both flow in figures $(3-8)$ and $(18-23)$. Figures $(3,4)$ and $(18,21)$ is prepared to established the effect of magnetic parameter M, third order parameter β on fluid velocity V. One can notice that increases velocity by raises the value of both flow β. It is noted that the third grade velocity is larger when compared with the Newtonian fluid. Figures $(5,6,7,8)$ and $(19,20,22,23)$ shows the effect of stock number S_t, axis profile x, second grade number α, porous parameter λ, on lift and drainage field velocity. This figures shows that lift velocity rises when (S_t,λ) enhances while in case of drainage flow velocity descends and (S_t,λ) increases. It shows that raises stock number, porous parameter causes the motion of fluid due to opposite direction. Figures $(6,8)$ and $(19,22)$ represents the axis profile, second grade variable. We observe that the temperature is raises by rises in drainage and enhances by falling in lift. Figures $(9-12)$ and $(24-27)$ displays the effects of physical parameters E_c, N, P_r, β_1 on heat field of lift and drainage flow of third order problem respectively. Figures (11) and (21) are plotted to examine the effect of β_1 on temperature field of both flows. Raising values of non-Newtonian parameter increases the thermal for both flow system. Figures $(9,10,12)$ and $(24,25,27)$ are described to observe the influence of E_c, P_r, N enhances the thermal field in lift but declined in drainage. The prandlt number falling in lift and rises in drainage with the time series of thermal distribution. Figures $(13-16)$ and $(28-31)$ are elucidate the impact of profile axial, chemical difference parameter, reaction of chemical rate, Schmidth number on the concentration field. Chemical reaction rate parameter and axial profile are escalation on upward direction and enhances on downward direction of flow. Chemical difference parameter are raises when flow is lifting and falling in flow draining. The oscillatory flow is most high frequency of combined fluid with decreases in upward flow and increases in downward flow.

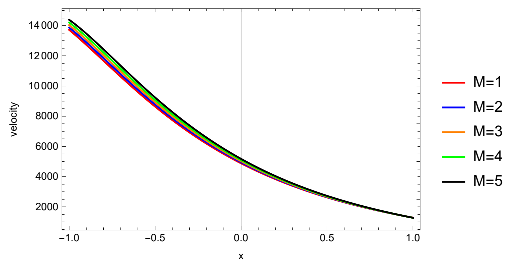

Figure-3: Effect of the "M" on the lift velocity variation.

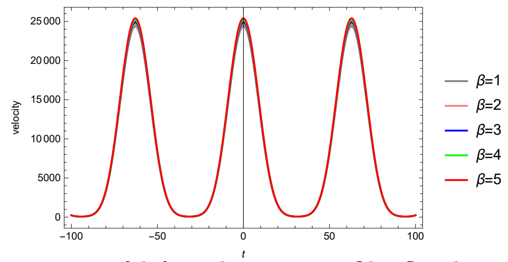

Figure-4: Time series of lift velocity profile fluid with variation "β_1".

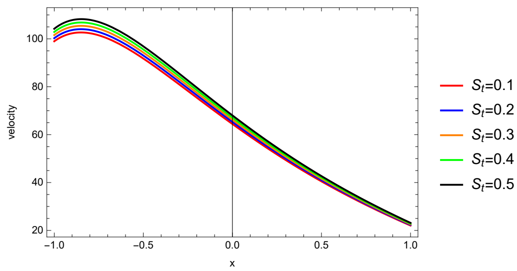

Figure-5: Effect of "S_t" with lift variation.

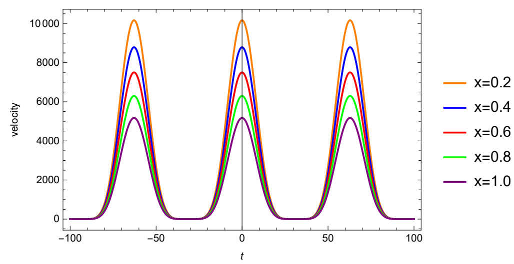

Figure-6: Time series of lift velocity profile fluid with variation "x".

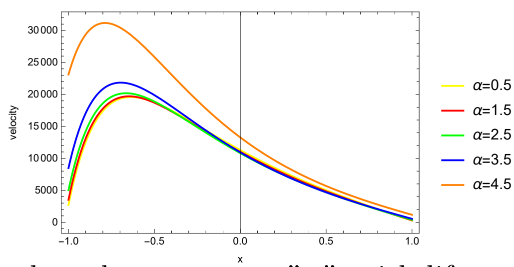

Figure-7: Second grade parameter "α" with lift variation.

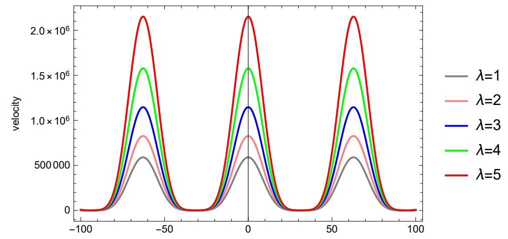

Figure-8: Time series of lift velocity profile fluid with variation "λ".

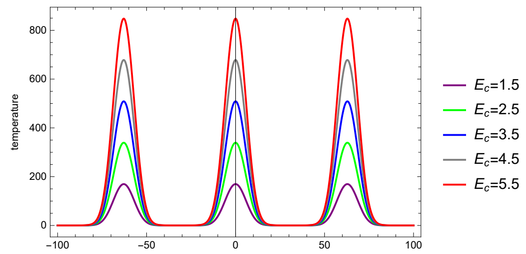

Figure-9: Time series of lift temperature field fluid with variation "E_c".

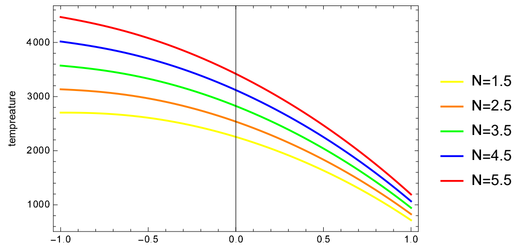

Figure-10: lift temperature field fluid with variation of "N".

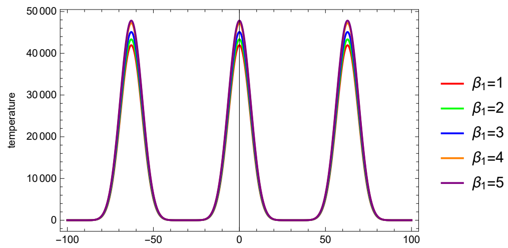

Figure-11: Time series of lift temperature field fluid with variation "β_1".

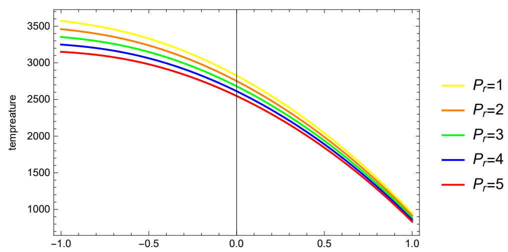

Figure-12: lift temperature field fluid with variation of "P_r".

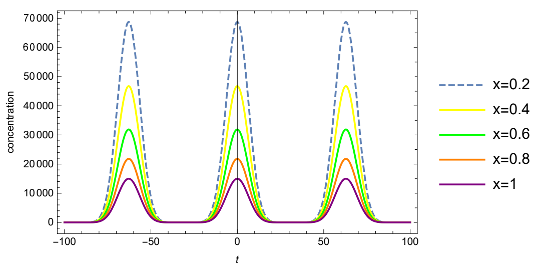

Figure-13: Time series of lift concentration fluid with variation "x".

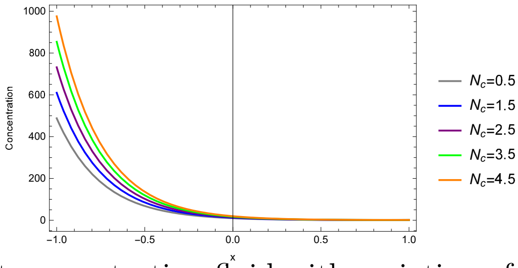

Figure-14: Lift concentration fluid with variation of "N_c".

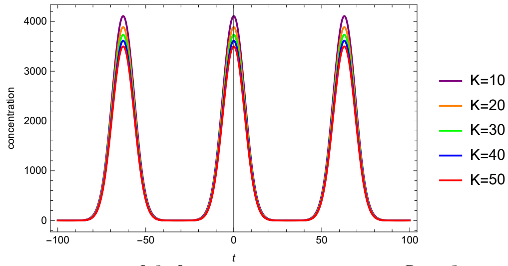

Figure-15: Time series of lift concentration fluid with variation "K".

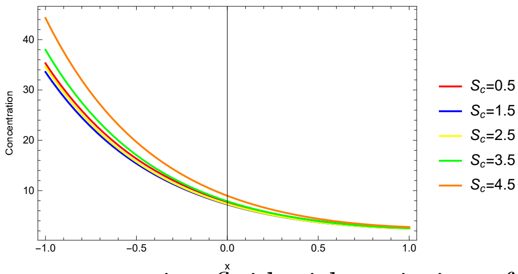

Figure-16: Lift concentration fluid with variation of "S_c".

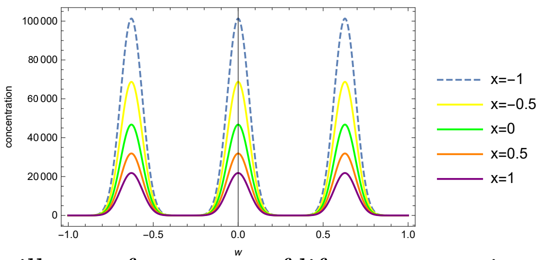

Figure-17: Oscillatory frequency of lift concentration fluid with variation "x".

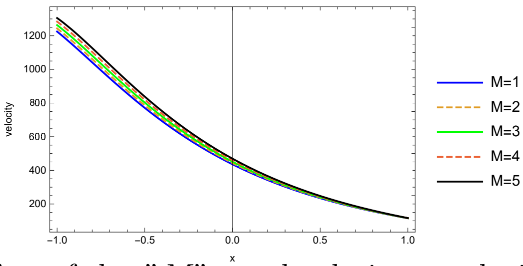

Figure-18: Effect of the "M" on the drainage velocity.

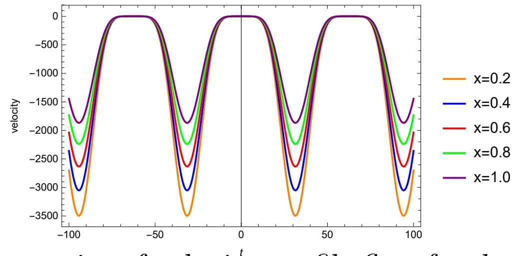

Figure-19: Time series of velocity profile flow for drainage with variation "x".

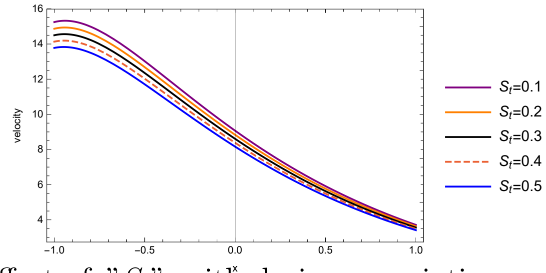

Figure-20: Effect of "S_t" with drainage variation.

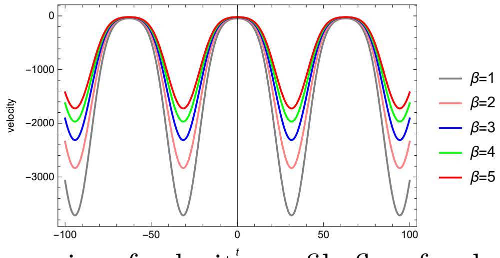

Figure-21: Time series of velocity profile flow for drainage with variation "β".

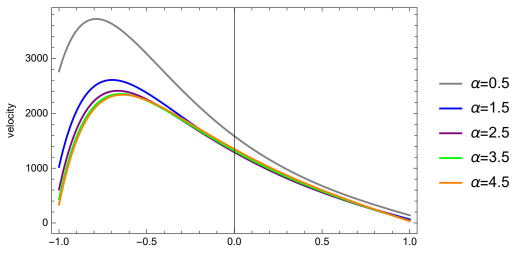

Figure-22: Second grade parameter "α" with drainage variation.

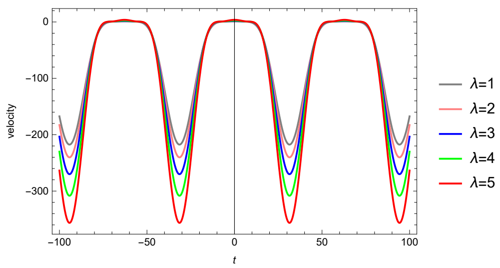

Figure-23: Time series of velocity profile flow for drainage with variation "λ".

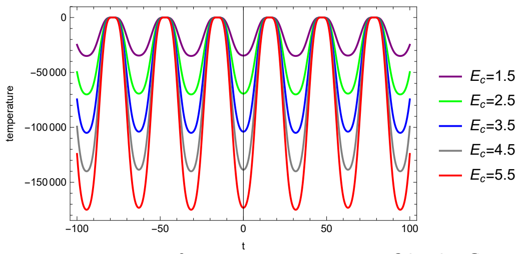

Figure-24: Time series of temperature filed flow for drainage with variation "E_c".

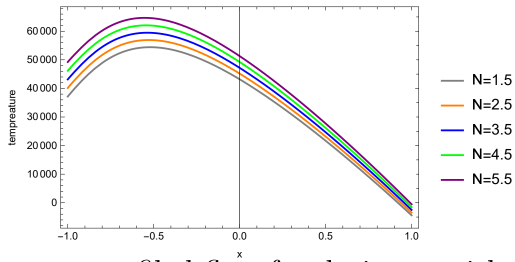

Figure-25: temperature filed flow for drainage with variation "N".

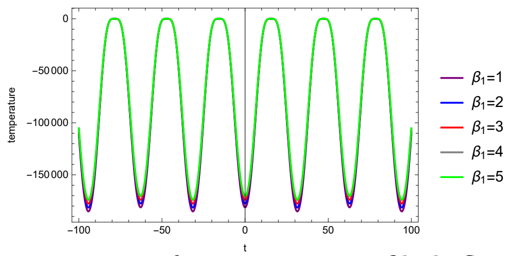

Figure-26: Time series of temperature filed flow for drainage with variation "β_1".

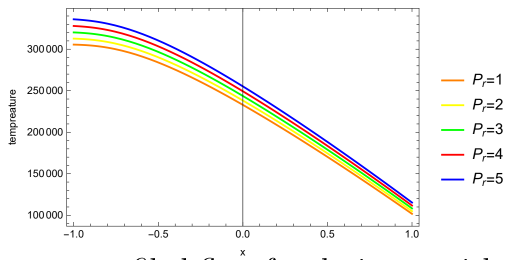

Figure-27: temperature filed flow for drainage with variation "P_r".

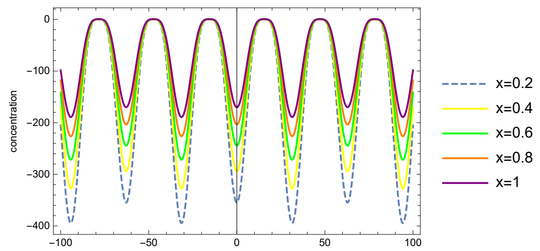

Figure-28: Time series of drainage concentration fluid with variation ”x”.

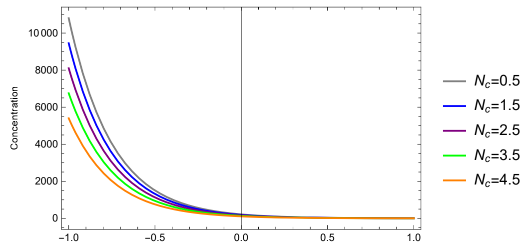

Figure-29: Drainage concentration fluid with variation of ”N_c”.

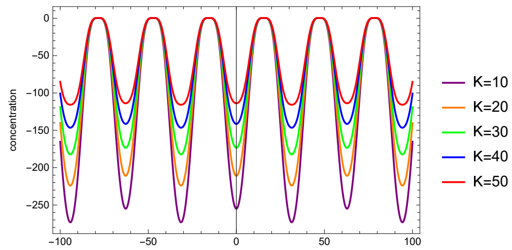

Figure-30: Time series of drainage concentration fluid with variation ”K”.

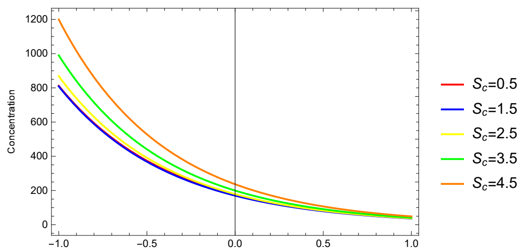

Figure-31: Drainage concentration fluid with variation ”S_c”.

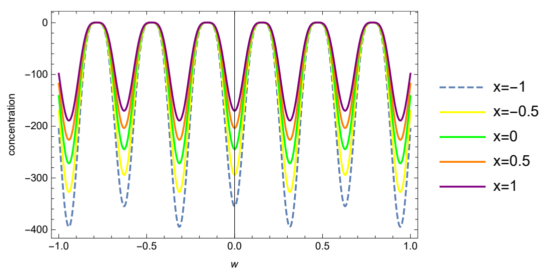

Figure-32: Oscillatory frequency of Drainage concentration fluid with variation "x" .

Its conclude that, Raises Schmidt number S_c when concentration is increases for both flow of chemical reaction.

5 Conclusion

MHD unsteady third grade thin film flow of thermally conducting oscillating vertical moving belts has been discussed with lifting and drainage flows. The flows of third grade fluids in porous medium are quite prevalent in many engineering fields such as enhanced oil recovery, drilling of mud and oil, paper and textile coating and composite manufacturing processes. In the non-Newtonian parameter, gravitational force is more dominant coefficient of the flow problem. The flow is lifting gravitational force upward direction in time series of fluid positive direction and drainage gravitational force downward direction in time series of fluid negative way.

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Appendix:

$C_1 = \frac{((1+Cos\omega t)+\frac{S_t}{(M+\lambda)})(1-\frac{2Sinh\sqrt{(M+\lambda)}}{2Sinh2\sqrt{(M+\lambda)}})}{e^{-\sqrt{(M+\lambda)}}}$;

$C_2 = ((1+Cos\omega t)+\frac{S_t}{(M+\lambda)})(\frac{2Sinh\sqrt{(M+\lambda)}}{2Sinh2\sqrt{(M+\lambda)}})$;

$C_3 = (\frac{f_1 e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} - f_2 e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}}{2Sinh2\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}}) + f_1 e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}$;

$C_4 = (\frac{f_2 e^{-\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x} - f_1 e^{\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}x}}{2Sinh2\sqrt{(\frac{M+\lambda+n}{1+\alpha n+n^2\beta_1})}})$

$a_1 = \frac{(1+\alpha n+n^2\beta_1)}{(9(M+\lambda)(1+\alpha n)-(M+\lambda+n+n^2\beta_1))}$; $a_2 = \frac{(1+\alpha n+n^2\beta_1)}{((M+\lambda)(1+\alpha n)-(M+\lambda+n+n^2\beta_1))}$;

$a_3 = \frac{(1+\alpha n+n^2\beta_1)}{4((M+\lambda)(1+\alpha n)-(M+\lambda+n+n^2\beta_1))}$; $a_4 = -\frac{(1+\alpha n+n^2\beta_1)}{(M+\lambda+n)}$;

$l_1 = -(6(M+\lambda)^2 A_1^3 + 2\lambda(M+\lambda)A_1^3)$; $l_2 = -(6(M+\lambda)^2 A_2^3 + 2\lambda(M+\lambda)A_2^3)$;

$l_3 = 6(M+\lambda)^2 A_1 A_2^2 - 2\lambda(M+\lambda)A_1 A_2^2$; $l_4 = 6(M+\lambda)^2 A_2 A_1^2 - 2\lambda(M+\lambda)A_2 A_1^2$;

$l_5 = -2\lambda S_t A_1^2$; $l_6 = -2\lambda S_t A_2^2$; $l_7 = 4\lambda S_t A_1 A_2$;

$b = \frac{\sqrt{(M+n+\lambda)}}{(1+\alpha n+n^2\beta_1)}$; $b_1 = \sqrt{M+\lambda}$; $r_1 = 4b^4 A_1^2$; $r_2 = 4b^4 A_2^2$; $r_3 = b^2 A_1 A_2$;

$b_3 = \frac{\sqrt{nP_r}}{1+N}$; $b_2 = -\frac{nP_r}{1+N}$; $b_4 = -(n+k)S_c$; $b_5 = \sqrt{(n+k)S_c}$; $p_1 = \frac{bb_1 A_1 A_3}{b_1^2+b^2+2bb_1+b_2}$;

$p_2 = \frac{bb1A1A4}{b_1^2+b^2-2bb_1+b_2}$; $p_3 \frac{3b^2 a_1 A_1 c_1}{16b^2+b_2}$; $p_4 = -\frac{3b^2 a_1 A_1 c_2}{4b^2+b_1}$;

$p_5 = \frac{b^2 a_2 A_1 c_3}{b_2}$; $p_6 = \frac{b^2 a_2 A_1 c_2}{4b^2+b_2}$; $p_7 = \frac{2b^2 a_3 A_1 c_5}{9b^2+b_2}$;

$p_8 = -\frac{2b^2 a_3 A_1 c_6}{b^2+b_2}$; $p_9 = -\frac{b_1 b A_3 c_2}{b_2}$; $p_{10} = \frac{b_1 b A_2 A_4}{b_1^2+b^2+2bb_1+b_1}$;

$p_{11} = -\frac{3b^2 a_1 A_2 c_1}{4b^2+b_2}$; $p_{12} = \frac{3b^2 a_1 A_2 c_1}{16b^2+b_2}$; $p_{13} = \frac{b^2 a_2 A_2 c_3}{16b^2+b_2}$;

$p_{14} = \frac{b^2 A_2 c_4 a_2}{b_2}$; $p_{15} = \frac{2b^2 a3c5A_2}{b^2+b_2}$; $p_{16} = \frac{2b^2 a_3 c_6 A_2}{9b^2+b_2}$;

$p_{17} = \frac{2b^4 A1^4}{16b^2+b_2}$; $p_{18} = \frac{2b^4 A_2^4}{16b^2+b_2}$; $p_{19} = -\frac{4b^4 A_1 A_2^3}{4b^2+b_2}$;

$p_{20} = -\frac{4(b^4)A2(A1^3)}{4b^2+b_2}$; $p_{21} = -\frac{6b^4 A_2^2 A_1^2}{b_2}$;

$m_3 = l_9 + l_{12}$; $m_4 = l_{10} + l_{11}$; $b_2 = -np_r$; $l_1 = \frac{bb_1 A_1 A_3}{b_1^2+b+2bb_1+b_2}$; $l_2 = -\frac{bb_1 A_1 A_4}{b_1^2+b+2bb_1+b_2}$;

$l_3 = -\frac{bb_1 A2A3}{b_1^2+b+2bb_1+b_2}$; $l_4 = \frac{bb_1 A_2 A_4}{b_1^2+b+2bb_1+b_2}$; $l_5 = \frac{3b_1^2 a_1 A1c_1}{16b_1^2+b_2}$; $l_6 = \frac{3b_1^2 a_1 A_2 c_2}{1_6 b1^2+b_2}$;

$l_7 = \frac{2b_1^2 a3A_1 c_5}{9b_1^2+b_2}$;

$l_8 = \frac{2b_1^2 a_3 A_2 c_6}{9b_1^2+b_2}$; $l_9 = -\frac{3b_1^2 a_1 A_1 c_2}{4b_1^2+b2}$; $l_{10} = \frac{3b_1^2 a_1 A_2 c_1}{4b_1^2+b_2}$; $l_{11} = \frac{b_1^2 a_2 A_1 c_4}{4b_1^2+b2}$; $l_{12} = \frac{b_1^2 a_2 A_2 c_3}{4b_1^2+b2}$;

$l_{13} = -\frac{2b_1^2 a_3 A_1 c_6}{b_1^2+b_2}$; $l_{14} = -\frac{2b_1^2 a_3 A_2 c_5}{b_1^2+b2}$; $l_{15} = \frac{b_1^2 a_2 A_1 c_3 + b_1^2 a_2 A_2 c_2}{b_2}$; $l_{16} = \frac{2b_1^4 A_1^2}{16b_1^2+b_2}$;

$l_{17} = \frac{2b_1^4 A_2^2}{16b_1^2+b_2}$;

$l_{18} = -\frac{12b_1^4 A_1^2 A_2^2}{b_2}$; $l_{19} = -\frac{8b_1^4 A_1^3 A_2}{4b_1^2+b_2}$; $l_{20} = -\frac{8b_1^4 A_2^3 A_1}{4b_1^2+b_2}$; $l_{21} = \frac{A_1 A_3}{b_1^2+b+2bb_1+b_2}$;

$l_{22} = \frac{A_2 A_3}{bb_1^2+b+2bb_1+b_2}$; $l_{23} = -\frac{S_t A_3}{(M+\lambda)(b_1^2+b2)}$; $l_{24} = \frac{A_1 A_4}{b_1^2+b+2bb_1+b_2}$; $l_{25} = \frac{A_2 A_4}{b_1^2+b+2bb_1+b_2}$;

$l_{26} = -\frac{S_t A_4}{(M+\lambda)(b_1^2+b_2)}$; $l_{27} = \frac{a_1 A_1 c_1}{16b_1^2+b_2}$; $l_{28} = \frac{a_1 A_2 c1}{4b_1^2+b_2}$; $l_{29} = -\frac{S_t a_1 c_1}{(M+\lambda)(9b_1^2+b_2)}$;
$l_{30} = \frac{a_1 A_1 c_2}{4b_1^2+b_2}$; $l_{31} = \frac{a_1 A_2 c_2}{16b_1^2+b_2}$; $l_{32} = -\frac{S_t a_1 c_2}{(M+\lambda)(9b_1^2+b_2)}$; $l_{35} = -\frac{S_t a_3 c_5}{(M+\lambda)(4b_1^2+b_2)}$;
$l_{33} = \frac{a_3 A_1 c_5}{9b_1^2+b2}$; $l_{34} = \frac{a_3 A_2 c_5}{b_1^2+b_2}$; $l_{36} = \frac{a_3 A_2 c_6}{9b_1^2+b_2}$; $l_{37} = -\frac{S_t a_3 c_6}{(M+\lambda)(4b_1^2+b_2)}$;
$l_{38} = \frac{a_4 A_1 c_7}{b_1^2+b_2}$; $l_{39} = \frac{a_4 A_2 c_7}{b_1^2+b_2}$; $l_{40} = -\frac{s_t a_4 c_7}{(M+\lambda])(b_2)}$; $l_{41} = \frac{a_2 A_1 c_4}{4b_1^2+b_2}$; $l_{42} = -\frac{s_t a_2 c_4}{(M+\lambda)(b_1^2+b_2)}$;
$l_{43} = \frac{a_2 A_2 c_3}{4b_1^2+b_2}$; $l_{44} = -\frac{s_t a_2 c_3}{(M+\lambda)(b_1^2+b2)}$; $l_{45} = -\frac{a_2 A_2 c_4 + A_1 a_2 c_3}{b_2}$; $A_5 = B_r p_r(r_1 + r_2 + r_3 + r_4)$;
$A_6 = 1 + B_r p_r((e^{-2\sqrt{(M+\lambda)-1}})r_1 + (e^{2\sqrt{(M+\lambda)-1}})r_2 + (e^{-\sqrt{(M+\lambda)-1}})r_4 + (e^{\sqrt{(M+\lambda)-1}})r_5 + r_3 + r_6)$; $r_1 = \frac{(M+1)}{4}A_1^2$;
$r_2 = \frac{(M+1)}{4}A_2^2$; $r_3 = -\lambda A_1 A_2$; $r_4 = -\frac{2S_t A_1}{(M+\lambda)^2}$; $r_5 = -\frac{2S_t A_2}{(M+\lambda)^2}$;
$r_6 = \frac{M}{2}(\frac{S_t}{(M+\lambda)})^2$; $f_3 = N_c S_c(h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7 + h_8 + h_9 + q_7 + q_8 + q_{13} + q_{14} - \frac{s_3}{2b_3})$; $f_4 = N_c S_c \frac{s_3}{2b_3}$;
$s_3 = (b_3 - (b_1 + b))h_1(e^-(b_1 + b + b_3)) + (b_3 - b_1 + b)h_2(e^{(-b_1+b-b_3)}) + (b_3 + b_1 - b)h_3(e^{(b_1-b-b_3)}) + (b_3 + b_1 + b)h_4(e^{(b_1+b-b_3)}) + (b_3 - 4b_1)h_5(e^{(-4b_1+b_3)}) + (4b_1 + b3)h_6(e^{(4b_1-b_3)}) + (b_3 - 3b_1)q_7 e^{(-3b1-b3)} + (b_3 + 3b_1)q_8(e^{(3b_1-b_3)}) + (2b_1 + b_3)h_7(e^{(2b_1-b_3)}) + (-2b_1 + b_3)h_8(e^{(-2b_1+b_3)}) + (b_1 + b_3)q_{13}(e^(b_1 - b_3)) + (-b_1 + b_3)q_{14}(e^{(-b_1-b_3)})$;
$f_2 = \frac{1}{2}(m_1 + m_2 + m_3) - (g_1 + g_2 + g_3)$; $f_1 = \frac{1}{2}(m_1 + m_2 + m_3)$; $m_1 = (g_1(e^{-b_4} - \frac{2\sqrt{(M+\lambda)}}{b_4}e^{-2\sqrt{(M+\lambda)}})$;
$m_2 = (g_2(e^{-b_4} + \frac{2\sqrt{(M+\lambda)}}{b_4}e^{2\sqrt{(M+\lambda)}})$; $m3 = g3e^-b_4$; $b_4 = \sqrt{(}ks)$; $b_3 = \sqrt{s(n+k)}$;
$A_1^{'} = (Cos(at) - \frac{m}{(M+\lambda)})(1 - \frac{e^-\sqrt{(M+\lambda)}}{2})$; $A_2^{'} = (Cos(at) - \frac{m}{(M+\lambda)})(\frac{e^-\sqrt{(M+\lambda)}}{2})$;
$A_3^{'} = (\frac{e^-b}{2} - 1)(\eta_1) + \frac{b_1}{2b}(\eta_3)$; $A_4^{'} = -\frac{1}{2}(e^- b\eta_1 + \frac{b_1}{b}\eta_2)$;
$c_5^{'} = -2\lambda m A_1^2$; $c_6^{'} = -2\lambda m A_2^2$; $c_7^{'} = 4\lambda m A_1 A_2$;