A New Generalization Of Extending Modules Via Δ-Small Sub modules

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Abstract.
In this paper, I introduce a new generalization of essential submodules, namely δ-essential submodules. A submodule *A* of an *R*- module *M* is called δ-essential in *M* provided that for every nonzero δ-small submodule *B* of *M* has a nonzero intersection with *A*. Appling this concept, I define a generalization of extending module entitled δ-extending moduels and investigate their some general properties.
Keywords. δ-essential submododules, δ-closed submodules, δ-extending modules.

1. Introduction.

Let *R* be an associative ring with identity and let *M* be a unitary left *R*- module . A submodule *A* of *M* is said to be essential in *M*, (denoted by $A\leq_e M$), if for any submodule *B* of *M*, $A\cap B=0$ implies $B = 0$ [1], and a submodule *A* of *M* is said to be closed in *M* if *A* has no proper essential extension in *M*, that is if $A\leq_e B\leq M$, then $A=B$ [1]. An *R*- module *M* is called extending (or CS- module), if every submodule of *M* is essential in a direct summand of *M*. Equivalently, *M* is extending module if every closed submodule of *M* is a direct summand [2].

In last decades, essential submodules and relevant concepts were widely studied and investigated. Many researchers tried to introduce and consider some notions in module theory closely related to essential submodules. Undoubted, one of the most famous concept in the theory of rings and modules are extending modules. Maybe firstly, this concept introduced in the 1994s. After that we have a large number of works which their main subjects were extending modules and their various generalizations (for example [3]).

Zhou in [4] introduces a generalization of small submodules namely δ-small submodules via the concept of singular modules. In fact, he called a submodule *A* of a module *M* a δ-small submodule if $M\neq A+B$ for every proper submodule *B* of *M* with $\frac{M}{B}$ singular. General properties of δ-small submodules and a nice characterization of them are also provided in [4]. By the way, I call a submodule *A* of a module *M*, δ-essential provided $A\cap B\neq 0$ for all nonzero δ-small submodules *B* of *M*. I try to study some natural and general properties of δ-essential submodules. δ-closed submodules are introduced and their some natural properties are studied. As an application, I define δ-extending modules. The module *M* is δ-extending if every submodule of *M* is δ-essential in a direct summand of *M*.

In section 2, I define and study δ-essential submodules, δ-closed submodules and δ-uniform modules.

In section 3, I introduce δ-extending modules with some examples and basic properties , In section 4, I present various characterizations of δ-extending modules and study the direct sum of δ-extending modules.

2. δ-essential and δ-closed submodules.

In this section, I introduce δ-essential submodules and δ-uniform modules as a generalization of essential submodules and uniform modules respectively. Also, I define a δ-closed submodules which is stronger that closed submodules. I study the basic properties of them that are relevant to the work.

Definition (2.1): Let A be a submodule of an R-module M, M is said to be δ-essential extension to A or A is δ-essential in M if for any nonzero δ-small submodule B of M, we have $A \cap B \neq 0$. It will be denoted by $A \leq_{\delta e} M$.

Examples and Remarks (2.2).

(1) It is clear that δ-essential submodule is a generalization of essential submodule, but not conversely. For example: Consider Z_6 as Z- module. Since 0 is the only δ-mall in Z_6, then $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ are δ-essential in Z_6 which are not essential in Z_6.
(2) Every nonzero submodule of Q as Z-module is δ-essential in Q.
(3) Every nonzero submodule of Z as Z-module is δ-essential in Z.
(4) Consider Z_6 as Z_6-module, $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ are not δ-essential in Z_6.

The following proposition, consider a condition under which δ-essential submodules versus essential submodules.

Proposition (2.3): Let M be δ-hollow R-module and let A be a submodule of M, then $A \leq_{\delta e} M$ if and only if $A \leq_{e} M$.

Next, I give characterizations of δ-essential submodules.

Proposition (2.4): Let M be an R-module and let A be a submodule of M, then $A \leq_{\delta e} M$ if and only if for any nonzero cyclic δ-small submodule K of M, $A \cap K \neq 0$.

Proof: Let K be a nonzero cyclic δ-small submodule of M and let $0 \neq x \in K$. By our assumption, $0 \neq \langle x \rangle \cap A \leq A \cap K$. Hence $A \cap K \neq 0$. The converse is obvious.

Proposition (2.5): Let M be an R-module and let A be a submodule of M, then $A \leq_{\delta e} M$ if and only if for any nonzero element x in M with Rx is δ-small has a nonzero multiple in A.

Proof: Let $0 \neq x \in M$ with $Rx \ll_{\delta} M$. By Proposition. (2.4) $Rx \cap A \neq 0$. Hence there is $r \in R$ such that $0 \neq rx \in A$. For the converse, let Rx be a nonzero cyclic δ-small submodule of M. By our assumption, there is $r \in R$ such that $0 \neq rx \in A$, hence $Rx \cap A \neq 0$. Thus $A \leq_{\delta e} M$.

The next proposition gives properties of δ-essential submodule which are needed in my work.

Proposition (2.6): Let M be any R-module. The following are hold.

(1) Let $A\leq B\leq M$. Then $A\leq_{\delta e} M$ if and only if $A\leq_{\delta e} B$ and $B\leq_{\delta e} M$.
(2) Let $A_1\leq_{\delta e} B_1\leq M$ and $A_2\leq_{\delta e} B_2 \leq M$, then $A_1\cap A_2 \leq_{\delta e} B_1\cap B_2$.
(3) If $f: M_1\to M_2$ is an R-homomorphism and $A\leq_{\delta e} M_2$, then $f^{-1}(A) \leq_{\delta e} M_1$.
(4) Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an independent family of submodules of M and $A_\alpha\leq_{\delta e} B_\alpha$, $\forall\alpha\in\Lambda$, then $\bigoplus_{\alpha\in\Lambda} A_\alpha\leq_{\delta e} \bigoplus_{\alpha\in\Lambda} B_\alpha$.

Proof. (1) Suppose that $A\leq_{\delta e} M$ and let $0\neq L\ll_\delta B$, then $0\neq L \ll_\delta M$, by [4, Lemma 1.3]. Since $A\leq_{\delta e} M$, then $A\cap L\neq 0$. Hence $A\leq_{\delta e} B$. Now let $0\neq K\ll_\delta M$, then $0\neq A\cap K\leq B\cap K$. Thus $B\leq_{\delta e} M$. Conversely, assume that $A\leq_{\delta e} B\leq_{\delta e} M$ and let $0\neq L\ll_\delta M$, then $0\neq B\cap L \ll_\delta B$. But $A\leq_{\delta e} B$, therefore $A\cap B\cap L = A\cap L\neq 0$. Thus, we get the result.

(2) Assume that $A_1\leq_{\delta e} B_1\leq M$ and $A_2\leq_{\delta e} B_2 \leq M$ and let $0\neq L \ll_\delta B_1\cap B_2 \leq B_1$, hence $L\ll_\delta B_1$ and $L\ll_\delta B_2$. Since $A_1\leq_{\delta e} B_1$, then $A_1\cap L\neq 0$ and hence $0\neq A_1\cap L\ll_\delta B_2$. But $A_2\leq_{\delta e} B_2$, therefore $A_1\cap A_2\cap L\neq 0$. Thus $A_1\cap A_2 \leq_{\delta e} B_1\cap B_2$.

(3) Let $f: M_1\to M_2$ be an R-homomorphism and let $A\leq_{\delta e} M_2$. To show that $f^{-1}(A) \leq_{\delta e} M_1$, let $0\neq x\in M_1$ with $Rx \ll_\delta M_1$, then $f(Rx) \ll_\delta M_2$. Consider the following two cases: if $x\in f^{-1}(A)$, we are done. If $x\notin f^{-1}(A)$, then $0\neq f(x) \in M_2$. Since $A\leq_{\delta e} M_2$, then there is $r\in R$ such that $0\neq r f(x)\in A$, hence $0\neq rx\in f^{-1}(A)$. Thus $f^{-1}(A) \leq_{\delta e} M_1$.

(4) We use the induction on the number of elements of Λ. Suppose that the family has only two elements. i.e., $\{A_1, A_2\}$ is independent family in M, $A_1\leq_{\delta e} B_1$ and $A_2\leq_{\delta e} B_2$. Let $\pi_1 : B_1\oplus B_2\to B_1$ and $\pi_2 : B_1\oplus B_2\to B_2$ be the projection maps. Since $A_1\leq_{\delta e} B_1$ and $A_2\leq_{\delta e} B_2$, then $\pi_1^{-1}(A_1) = A_1\oplus B_2 \leq_{\delta e} B_1\oplus B_2$ and $\pi_2^{-1}(A_2) = B_1\oplus A_2 \leq_{\delta e} B_1\oplus B_2$, by(3) and hence $A_1\oplus A_2 = (A_1\oplus B_2)\cap(B_1\oplus A_2) \leq_{\delta e} B_1\oplus B_2$, by (2).

Now, assume that the result is true for the case when the index set with $n-1$ elements. Now let $\{A_1,A_2,\ldots,A_n\}$ be an independent family and assume that $A_i\leq_{\delta e} B_i$, $\forall\, i = 1, 2,\ldots,n$. By the previous case we have $\bigoplus_{i=1}^{n-1} A_i\leq_{\delta e}\bigoplus_{i=1}^{n-1} B_i$ and $A_n\leq_{\delta e} B_n$, hence we get $\bigoplus_{i=1}^{n} A_i\leq_{\delta e}\bigoplus_{i=1}^{n} B_i$. Finally, let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an independent family of submodules of M and $A_\alpha\leq_{\delta e} B_\alpha$, $\forall\alpha\in\Lambda$. Let $0\neq N \ll_\delta \bigoplus_{\alpha\in\Lambda} B_\alpha$ and let x be a nonzero element in N. So $x = b_1+b_2+\ldots+b_n$, where $b_i\in B_{\alpha_i}$, $\forall\, i = 1,2,\ldots,n$. Hence $N\cap (A_{\alpha 1}+ A_{\alpha 2}+\ldots+ A_{\alpha n})\neq 0$ which implies that $N\cap \bigoplus_{\alpha\in\Lambda} A_\alpha\neq 0$. Thus $\bigoplus_{\alpha\in\Lambda} A_\alpha\leq_{\delta e} \bigoplus_{\alpha\in\Lambda} B_\alpha$.

Note that $\{B_\alpha\}_{\alpha\in\Lambda}$ in proposition (2.6) need not be an independent family. For example: Let M be the Z-module $Z\oplus Z_2$ and let $A_1 = 0\oplus Z_2$, $B_1 = Z\oplus Z_2$, $A_2 = B_2 = Z\oplus \bar{0}$. One can easily show that $A_1\leq_{\delta e}B_1$ and $A_2\leq_{\delta e}B_2$ and $A_1\cap A_2 = \{0\}$ but $B_1\cap B_2 = Z\oplus \bar{0}$. Hence $\{B_1,B_2\}$ is not independent family.

Now, I define the δ-closed submodules and introduce the basic properties of these submodules.

Definition (2.7): Let A be a submodule of an R-module M, we say that A is δ-closed in M (briefly $A\leq_{\delta c} M$) if A has no proper δ-essential extension in M.

Examples and Remarks (2.8).

(1) Consider Z_6 as Z_6-module, $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ are δ-closed submodules of Z_6.
(2) Consider Z_4 as Z-module, $\{\bar{0},\bar{2}\}$ is not δ-closed in Z_4.
(3) Every δ-closed submodule of an R-module M is closed in M. The converse is not true in general. For example, Consider Z_6 as Z- module $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ are closed in Z_6 but not δ-closed in Z_6. When M is δ-hollow, they are equivalent.
(4) It is well known that every direct summand of an R-module M is closed in M. But in case δ-closed there is no relationship with direct summands. For example, Z_6 as Z- module, the nontrivial direct summands of Z_6 are $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ which are not δ-closed in Z_6.

Next, the basic properties of δ-closed submodules are gived.

Proposition (2.9): Let M be an R-module. If A is δ-closed in M, then $\frac{B}{A}\leq_{\delta e}\frac{M}{A}$, whenever $B\leq_{\delta e} M$ with $A\leq B$.

Proof. Suppose that $A\leq B\leq_{\delta e} M$ and let $\frac{L}{A}\ll_{\delta}\frac{M}{A}$ such that $\frac{L}{A}\cap\frac{B}{A}=A$, then $L\cap B = A$. Since $B\leq_{\delta e} M$, then $A\leq_{\delta e} L$, by proposition (2.6). But A is δ-closed in M, therefore $A = L$. Thus $\frac{B}{A}\leq_{\delta e}\frac{M}{A}$.

Proposition (2.10): Let $f: M\rightarrow M'$ be an epimorphism and let A be a submodule of M such that $Ker f\leq A$. If A is δ-closed in M, then $f(A)$ is δ-closed in M'.

Proof. Let K' be a submodule of M' such that $f(A)\leq_{\delta e} K'$, then $f^{-1}(f(A))\leq_{\delta e} f^{-1}(K')$, by proposition (2.6). One can easily show that $f^{-1}(f(A)) = A$, hence $A\leq_{\delta e} f^{-1}(K')$. But A is δ-closed in M, therefore $A = f^{-1}(K')$, and hence $f(A) = K'$. Thus $f(A)$ is δ-closed in M'.

One can easily prove the following corollary.

Corollary (2.11): Let A and B be submodules of an R-module M with $A\leq B$. If B is δ-closed in M, then $\frac{B}{A}$ is δ-closed in $\frac{M}{A}$.

One can easily prove the following proposition.

Proposition (2.12): Let M be an R-module and let A, B be submodules of M with $A \leq B \leq M$. If A is δ-closed in M, then A is δ-closed in B.

The transitive property for δ-closed submodules can not proved. However under certain condition it can be proved this property as we see in the following result.

Recall that an R-module M is called chained module if for each submodules A and B of M either $A \leq B$ or $B \leq A$, see [5].

Proposition (2.13): Let M be a chained R-module and let A and B be submodules of M such that $A \leq B \leq M$. If $A \leq_{\delta c} B \leq_{\delta c} M$, then $A \leq_{\delta c} M$.

Proof. Let K be a submodule of M such that $A \leq_{\delta e} K \leq M$. By our assumption, we have two cases: If $K \leq B$, since A is δ-closed in B, then $A = K$, hence $A \leq_{\delta c} M$. If $B \leq K$, since $A \leq_{\delta e} K$, then $B \leq_{\delta e} K$, by proposition (2.6). But $B \leq_{\delta c} M$, therefore $B = K$, hence $A \leq_{\delta e} B$. But $A \leq_{\delta c} B$, therefore $A = B = K$. Thus A is δ-closed in M.

The following theorem ensures the existences of δ-closed submodules.

Theorem (2.14): Let M be an R-module. Then every submodule is δ-essential in δ-closed submodule of M.

Proof: Let A be a submodule of M. Consider the collection $\Gamma=\{K: K \leq M: A \leq_{\delta e} K\}$. It is clear that Γ is nonemplty set. Let $\{C_\alpha\}_{\alpha \in \Lambda}$ be a chain in Γ. To show that $A \leq_{\delta e} \bigcup_{\alpha \in \Lambda} C_\alpha$, let $0 \neq x \in \bigcup_{\alpha \in \Lambda} C_\alpha$ with $Rx <<_\gamma \bigcup_{\alpha \in \Lambda} C_\alpha$, then there is $\alpha_\circ \in \Lambda$ such that $0 \neq x \in C_{\alpha_\circ}$.But $A \leq_{\delta e} C_\alpha$, $\forall \alpha \in \Lambda$, therefore there exists $r \in R$ such that $0 \neq rx \in A$, hence $A \leq_{\delta e} \bigcup_{\alpha \in \Lambda} C_\alpha$ which means that $\bigcup_{\alpha \in \Lambda} C_\alpha \in \Gamma$. By Zorn's lemma Γ has a maximal element say H. To show that H is δ-closed in M, let B be a submodule of M such that $H \leq_{\delta e} B$, then $A \leq_{\delta e} H \leq_{\delta e} B$ and hence $A \leq_{\delta e} B$, by proposition (2.6). But H is maximal element in Γ. Thus $H = B$.

The following proposition shows that the direct sum of δ-closed submodules is again δ-closed.

Proposition (2.15): Let M_1, M_2 be two R-modules. If $A_1 \leq_{\delta c} M_1$ and $A_2 \leq_{\delta c} M_2$, then $A_1 \oplus A_2 \leq_{\delta c} M_1 \oplus M_2$.

Proof: Assume that $A_1 \oplus A_2 \leq_{\delta e} B_1 \oplus B_2$, $B_1 \leq M_1$ and $B_2 \leq M_2$, let $i_1: M_1 \rightarrow M_1 \oplus M_2$ and $i_2: M_2 \rightarrow M_1 \oplus M_2$ be the inclusion maps. Since $A_1 \oplus A_2 \leq_{\delta e} B_1 \oplus B_2$, then $i_1^{-1}(A_1 \oplus A_2) \leq_{\delta e} i_1^{-1}(B_1 \oplus B_2)$. Note that $i_1^{-1}(A_1 \oplus A_2) = \{x \in M_1: i_1(x) \in (A_1 \oplus A_2)\} = \{x \in M_1: (x,0) \in (A_1 \oplus A_2)\} = A_1 \leq_{\delta e} i_1^{-1}(B_1 \oplus B_2) = B_1$. Similarly, $A_2 \leq_{\delta e} B_2$. But $A_1 \leq_{\delta c} M_1$ and $A_2 \leq_{\delta c} M_2$, therefore $A_1 = B_1$ and $A_2 = B_2$. Thus $A_1 \oplus A_2 \leq_{\delta c} M_1 \oplus M_2$.

An R-module M is called uniform module if every nonzero submodule of M is essential in M, see [1].

Now, I introduce δ-uniform modules as a generalization of uniform modules.

Definition (2.16): An R-module M is called δ-uniform if every nonzero submodule of M is δ-essential in M.

Examples and Remarks (2.17):

(1) If M has no nonzero δ-small submodules, then M is δ-uniform.
(2) Clearly that every uniform module is δ-uniform, hence Q as Z-module and Z as Z-module are δ-uniform modules. The converse is not true in general. For example, Z_6 as Z- module.
(3) Z_6 as Z_6-module is not δ-uniform module.
(4) Let M be a δ-hollow R-module. Then M is uniform if and only if M is δ-uniform.

The following theorem gives a characterization of δ-uniform modules

Theorem (2.18): Let M be an R-module. Then M is δ-uniform if and only if every nonzero δ-small submodule of M is essential in M.

Proof: ($\Rightarrow$) Assume that M is δ-uniform and let A be a nonzero δ-small submodule of M. Assume that there exists a nonzero submodule B of M such that $A \cap B = 0$. Since M is δ-uniform, then $B \leq_{\delta e} M$ and we have A is nonzero δ-small submodule of M, then $A \cap B \neq 0$, which is a contradiction.

($\Leftarrow$) To show that M is δ-uniform, let A be a nonzero submodule of M and assume that A is not δ-essential in M, hence there exists a nonzero δ-small submodule B of M such that $A \cap B = 0$. By our assumption $B \leq_e M$, then $A = 0$, which is a contradiction.

Proposition (2.19): Let $f : M \to M'$ be an R-monomorphism. If M' is δ-uniform, then M is δ-uniform

Proof: Let $f : M \to M'$ be an R-monomorphism and assume that M' is δ-uniform, we have to show that M is δ-uniform, let A be a nonzero submodule of M, then $f(A) \neq 0$, if $f(A) = 0$, then $A \leq Kerf = 0$ which is a contradiction. Since M' is δ-uniform, then $f(A) \leq_{\delta e} M'$ and hence $A \leq_{\delta e} M$. Thus, M is δ-uniform.

Corollary (2.20): A submodule of δ-uniform is again δ-uniform.

The following proposition gives a condition under which a quotient of δ-uniform is δ-uniform.

Proposition (2.21): Let M be a δ-uniform and let A be a δ-closed submodule of M, then $\frac{M}{A}$ is δ-uniform.

Proof: Let $\frac{L}{A}$ be a nonzero submodule of $\frac{M}{A}$, hence L is nonzero submodule of M. But M is δ-uniform, therefore $L \leq_{\delta e} M$. Since A is δ-closed in M, then $\frac{L}{A} \leq_{\delta e} \frac{M}{A}$, by proposition (2.9). Thus $\frac{M}{A}$ is δ-uniform.

Proposition (2.22): Let $M = M_1 \oplus M_2$ be a duo module. If M_1 and M_2 are δ-uniform modules, then M is δ-uniform. Provided that $A \cap M_i \neq 0$, $\forall\, i = 1,2$.

Proof: Let A be a nonzero submodule of M. Since M is duo module, then A is fully invariant and hence $A = (A \cap M_1) \oplus (A \cap M_2)$. Since each of $(A \cap M_1)$ and $(A \cap M_2)$ is a nonzero submodule of M_1 and M_2 respectively, it follows that $(A \cap M_1) \leq_{\delta e} M_1$ and $(A \cap M_2) \leq_{\delta e} M_2$. Then $A \leq_{\delta e} M$, by proposition (2.6).

In similar argument one can easily prove the following proposition.

Proposition (2.23): Let $M = M_1 \oplus M_2$ be a distributive module. If M_1 and M_2 are δ-uniform modules, then M is δ-uniform. Provided that $A \cap M_i \neq 0$, $\forall\, i = 1,2$.

3. δ-extending modules.

In this section, I introduce the concept of δ-extending modules as a generalization of extending modules. I generalize some properties of extending modules to δ-extending modules. Also, I discuss when the submodule of δ-extending module is δ-extending.

Definition (3.1): An R-module M is called δ-extending module if every submodule of M is δ-essential in a direct summand.

Examples and Remarks (3.2):

(1) Clearly that every δ-uniform module is δ-extending. The converse is not true in general. For example, Z_6 as Z_6-module.

(2) Every extending module is δ-extending, example (3.3) shows that the converse does not hold in general. Note that they are equivalent when M is δ-hollow module.

(3) For any prime number p, the Z-module $M = Z_p \oplus Z_{p2}$ is δ-extending.

Example (3.3): Consider the Z-module $M = Z_8 \oplus Z_2$. The submodules of M are:

$A_1 = \{(\bar{1},\bar{0}), (\bar{2},\bar{0}), (\bar{3},\bar{0}), (\bar{4},\bar{0}), (\bar{5},\bar{0}), (\bar{6},\bar{0}), (\bar{7},\bar{0}), (\bar{0},\bar{0})\}$.

$A_2 = \{(\bar{2},\bar{0}), (\bar{4},\bar{0}), (\bar{6},\bar{0}), (\bar{0},\bar{0})\}$.

$A_3 = \{(\bar{4},\bar{0}), (\bar{0},\bar{0})\}$.

$A_4 = \{(\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.

$A_5 = \{(\bar{1},\bar{1}), (\bar{2},\bar{0}), (\bar{3},\bar{1}), (\bar{4},\bar{0}), (\bar{5},\bar{1}), (\bar{6},\bar{0}), (\bar{7},\bar{1}), (\bar{0},\bar{0})\}$.

$A_6 = \{(\bar{2},\bar{1}), (\bar{4},\bar{0}), (\bar{6},\bar{1}), (\bar{0},\bar{0})\}$.

$A_7 = \{(\bar{4},\bar{1}), (\bar{0},\bar{0})\}$.

$A_8 = \{(\bar{2},\bar{0}), (\bar{4},\bar{0}), (\bar{6},\bar{0}), (\bar{2},\bar{1}), (\bar{4},\bar{1}), (\bar{6},\bar{1}), (\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.

$A_9 = \{(\bar{4},\bar{0}), (\bar{4},\bar{1}), (\bar{0},\bar{1}), (\bar{0},\bar{0})\}$.

$A_{10} = \{(\bar{0},\bar{0})\}$.
$A_{11} = M$.
Clearly that $M= A_1 \oplus A_4 = A_1 \oplus A_7 = A_4 \oplus A_5$ and the nonzero δ-small submodules of M are A_2 and A_3. It is enough to check that A_6, A_8 and A_9 satisfy the definition. For A_6, we have $A_6 \cap A_2 = A_3$ and $A_6 \cap A_3 = A_3$. For A_8, we have $A_8 \cap A_2 = A_2$ and $A_8 \cap A_3 = A_3$. Finally, we have $A_9 \cap A_2 = A_3$ and $A_9 \cap A_3 = A_3$. Thus, A_6, A_8 and A_9 are δ-essential in M. Thus M is δ-extending with is not extending.

The following proposition gives a condition under which the δ-extending module and δ-uniform module are equivalent.

Proposition (3.4): Let M be an indecomposable module. Then the following statements are equivalent.

(1) M is δ-uniform.
(2) M is δ-extending.
(3) Every cyclic submodule of M is δ-essential in a direct summand of M.

Proof: (1)$\Rightarrow$(2)$\Rightarrow$(3) It is clear.

(3)$\Rightarrow$(1) Assume that every cyclic submodule of M is δ-essential in a direct summand of M and let A be a nonzero submodule of M, let $0 \neq x \in A$, hence Rx is δ-essential in a direct summand D of M. But M is indecomposable, therefore $D = M$. Since $Rx \leq A \leq M$, then $A \leq_{\delta e} M$. Thus M is δ-uniform.

Now, we give various conditions under which a submodule of a δ-extending module is δ-extending.

Proposition (3.5): Let M be a δ-extending R-module and let A be a submodule of M such that the intersection of A with any direct summand of M is a direct summand of A, then A is a δ-extending module.

Proof: Let $X \leq A \leq M$. Since M is δ-extending, then there exists a direct summand D of M such that $X \leq_{\delta e} D$. By our assumption, $A \cap D$ is a direct summand of A. Hence $X = (X \cap A) \leq_{\delta e} (A \cap D)$, by proposition (2.6). Thus A is δ-extending.

Let M be an R- module. Recall that a submodule A of M is called a fully invariant if $f(A) \leq A$, for every $f \in End(M)$ and M is called duo module if every submodule of M is a fully invariant, see [6].

Proposition (3.6): Every fully invariant submodule of δ-extending module is δ-extending.

Proof. Let M be a δ-extending module and let A be a fully invariant submodule of M. Let X be a submodule of A. Since M is δ-extending, then there exists a direct summand D of M such that $X \leq_{\delta e} D$. Let $M = D \oplus D'$, $D' \leq M$, then $A =$

$(A\cap D)\oplus(A\cap D')$. Since $X\leq_{\delta e} D$, then $X=(X\cap A)\leq_{\delta e}(A\cap D)$.Thus A is δ-extending, by proposition (2.6).

Corollary (3.7): Let M be a duo δ-extending module, then every submodule of M is δ-extending.

Recall that an R- module M is called distributive module if for all submodules A,B and C of M , $A\cap(B+C)=(A\cap B)+(A\cap C)$, see [7].

The next proposition gives another condition under which the submodule of δ-extending module is a δ-extending.

Proposition (3.8): Let M be a distributive δ-extending R-module, then every submodule of M is δ-extending.

Proof: Let A be a submodule of M and let X be a submodule of A. Since M is δ-extending, then there exists a direct summand D of M such that $X\leq_{\delta e} D$, let $M=D\oplus D'$, where $D'\leq M$. But M is distributive, therefore $A=(A\cap D)\oplus(A\cap D')$, then $(A\cap D)$ is a direct summand of A and $X\leq_{\delta e} A\cap D$. Thus A is δ-extending.

Let M be an R-module. Recall that a proper submodule A of M is called a maximal submodule if whenever $A\subset B\leq M$, then $B=M$. Equivalently, A is maximal submodule if $M=Rx+A$, $\forall x\notin A$, see [8].

Proposition (3.9): Let M be a δ-extending module which contains maximal submodules. Then for any maximal submodule A of M, either $A\leq_{\delta e} M$ or $M=A\oplus B$, for some simple submodule B of M.

Proof: Let A be a maximal submodule of M and suppose that A is not δ-essential submodule of M, then there is a nonzero δ-small submodule B of M such that $A\cap B=0$, let $0\neq x\in B$ and $x\notin A$. Since A is maximal submodule of M, then $M=A+Rx\leq A+B$, hence $M=A\oplus B$. Since $B\cong\frac{M}{A}$, so B is simple.

A module M is called local module if it has a largest submodule, i.e, a proper submodule which contains all other proper submodules. For a local module M, Rad(M) is small in M, see [9].

Corollary (3.10): Let M be a local δ-extending module, then $Rad(M)\leq_{\delta e} M$.

Proof: Since M is local module, then $Rad(M)<<M$, hence $Rad(M)$ can not be a direct summand of M. Thus $Rad(M)\leq_{\delta e} M$, by proposition (3.9).

§4: (Characterizations of δ-extending modules)

In this section, I give various characterizations of δ-extending modules. Also, I give some conditions under which the direct sum of δ-extending modules is δ-extending module.

We start by the following theorem.

Theorem (4.1): Let M be an R-module. Then M is δ-extending module if and only if every δ-closed submodule of M is a direct summand.

Proof: ($\Rightarrow$) Suppose that M is δ-extending and let A be a δ-closed in M, then there is a direct summand D of M such that $A \leq_{\delta e} D$. But A is δ-closed in M, therefore $A = D$.

($\Leftarrow$) To show that M is δ-extending, let A be a submodule of M, then there is a δ-closed submodule B of M such that $A \leq_{\delta e} B$, by theorem (2.14). By our assumption, B is a direct summand of M. Thus M is δ-extending module.

Theorem (4.2): Let M be an R-module. Then the following statements are equivalent.

(1) M is δ-extending module.
(2) For every submodule A of M, there is a decomposition $M = D \oplus D'$, such that $A \leq D$ and $D'+A \leq_{\delta e} M$.
(3) For every submodule A of M, there is a decomposition $\frac{M}{A} = \frac{D}{A} \oplus \frac{K}{A}$ such that D is a direct summand of M and $K \leq_{\delta e} M$.

Proof: (1)$\Rightarrow$(2) Let M be a δ-extending and let A be a submodule of M, there is a direct summand D of M such that $A \leq_{\delta e} D$, then $M = D \oplus D'$, $D' \leq M$. Since $\{A, D'\}$ is an independent family, then $A+D' \leq_{\delta e} M$, by proposition (2.6).

(2)$\Rightarrow$(3) Let A be a submodule of M. By (2), there is a decomposition $M = D \oplus D'$, such that $A \leq D$ and $D'+A \leq_{\delta e} M$. Claim that $\frac{M}{A} = \frac{D}{A} \oplus \frac{D'+A}{A}$. Since $M = D \oplus D'$, then $\frac{M}{A} = \frac{D+D'}{A} = \frac{D}{A} + \frac{D'+A}{A}$ and $\frac{D}{A} \cap \frac{D'+A}{A} = \frac{D \cap (D'+A)}{A} = \frac{A+(D \cap D')}{A} = A$, hence $\frac{M}{A} = \frac{D}{A} \oplus \frac{D'+A}{A}$. Take $K = D'+A$, so we get the result.

(3)$\Rightarrow$(1) To show that M is δ-extending, let A be a submodule of M. By (3), there is a decomposition $\frac{M}{A} = \frac{D}{A} \oplus \frac{K}{A}$ such that D is a direct summand of M and $K \leq_{\delta e} M$. It is enough to show that $A \leq_{\delta e} D$. Let $i : D \to M$ be the injection map. Since $K \leq_{\delta e} M$, then $i^{-1}(K) \leq_{\delta e} i^{-1}(M)$, that is $D \cap K \leq_{\delta e} D$. One can easily show that $D \cap K = A$, so M is δ-extending module.

By using [10, Lemma 2], we can prove the following proposition.

Proposition (4.3): Let M be an R-module. Then M is δ-extending module if and only if for each δ-closed submodule A of M, there is a complement B of A in M such that every homomorphism $f : A \oplus B \to M$ can be lifted to a homomorphism $g : M \to M$.

Proposition (4.4): Let M be an R-module. Then M is δ-extending module if and only if for every submodule A of M, there exists an idempotent $f \in \mathrm{End}(M)$ such that $A \leq_{\delta e} f(M)$.

The following proposition gives another characterization of δ-extending module.

Proposition (4.5): Let M be an R-module, then M is δ-extending module if and only if for each direct summand A of the injective hull $E(M)$ of M, there exists a direct summand D of M such that $(A\cap M)\leq_{\delta e} D$.

Proof: Let A be a submodule of M and let B be a complement of A, then $A\oplus B\leq_e M$, by [11]. Since $M\leq_e E(M)$, then $A\oplus B\leq_e E(M)$. Thus $E(A)\oplus E(B) = E(A\oplus B) = E(M)$. By our assumption, there exists a direct summand D of M such that $E(A)\cap M\leq_{\delta e} D$. But $A\leq_e E(A)$, therefore $A = A\cap M\leq_{\delta e} E(A)\cap M\leq_{\delta e} D$, hence $A\leq_{\delta e} D$. Thus M is δ-extending. The converse is clear.

The following proposition shows that the direct summand of δ-extending module is δ-extending.

Proposition (4.6): A direct summand of δ-extending module is δ-extending.

Proof: Let $M = A\oplus B$ be a δ-extending module, where A and B are submodules of M. To show that A is a δ-extending, let X be a δ-closed submodule of A, then $X\oplus B$ is a δ-closed submodule of M, by proposition (2.15). Hence $X\oplus B$ is a direct summand of M, then $M = X\oplus B\oplus Y$, $Y\leq M$. But $X\leq A$, therefore X is a direct summand of A. Thus A is δ-extending module.

The following proposition gives a condition under which a quotient of δ-extending module is a δ-extending.

Proposition (4.7): Let M be a δ-extending module and let A be a δ-closed submodule of M, then $\frac{M}{A}$ is δ-extending module.

Proof: Let M be a δ-extending module and let A be a δ-closed submodule of M, then A is a direct summand of M, let $M = A\oplus A'$, for some submodule A' of M, hence $\frac{M}{A}\cong A'$ is a δ-extending module, by proposition (4.6).

Corollary (4.8): Assume that $f: M\rightarrow M'$ is an R-homomorphism and let $Kerf$ be a δ-closed submodule of M, then $f(M)$ is δ-extending.

Proof: Let $f: M\rightarrow M'$ be an R-homomorphism and let $Kerf$ be a δ-closed submodule of M, then $\frac{M}{Kerf}\cong f(M)$ is δ-extending module.

There are some sufficient conditions under which the direct sum of δ-extending modules is a δ-extending.

Proposition (4.9): Let $M=M_1\oplus M_2$ be a distributive module if M_1 and M_2 are δ-extending, then M is δ-extending.

Proof: Let $M = M_1\oplus M_2$ be a distributive module, M_1 and M_2 are δ-extending and let $A\leq M$. Since M is distributive, then $A = A\cap M=A\cap(M_1\oplus M_2) = (A\cap M_1)\oplus(A\cap M_2)$. Since M_1, M_2 are δ-extending, then there exists a direct summand D_1 of M_1 and direct summand D_2 of M_2 such that $(A\cap M_1)\leq_{\delta e} D_1$ and

$(A \cap M_2) \leq_{\delta e} D_2$. Hence $A = (A \cap M_1) \oplus (A \cap M_2) \leq_{\delta e} (D_1 \oplus D_2)$, by proposition (2.6). Thus M is δ-extending.

Proposition (4.10): Let $M = \underset{i\in I}{\oplus} M_i$ be an R-module, where M_i is a submodule of M, $\forall$ $i \in I$. If M_i is δ-extending, for each $i \in I$ and every δ-closed submodule of M is fully invariant, then M is δ-extending.

Proof: Let A be a δ-closed submodule of M. By our assumption, A is fully invariant and hence $A = \underset{i\in I}{\oplus} (A \cap M_i)$, $\forall\ i \in I$ .Since $A \cap M_i \leq M_i$ and M_i is δ-extending, $\forall\ i \in I$, then there exists direct summands D_i of M_i such that $(A \cap M_i) \leq_{\delta e} D_i$, $\forall\ i \in I$. By proposition (2.6), $A = (\underset{i\in I}{\oplus} (A \cap M_i)) \leq_{\delta e} (\underset{i\in I}{\oplus} D_i)$. Thus M is δ-extending.

Proposition (4.11): Let M_1 and M_2 be δ-extending modules such that $annM_1 + annM_2 = R$, then $M_1 \oplus M_2$ is δ-extending.

Proof: Let A be a submodule of $M_1 \oplus M_2$. Since $annM_1 + annM_2 = R$, then by the same way of the proof of [12, Prop.4.2,CH.1], $A = B \oplus C$, where B is a submodule of M_1 and C is a submodule of M_2. Since M_1 and M_2 are δ-extending, then there exists direct summands D_1 of M_1 and D_2 of M_2 such that $B \leq_{\delta e} D_1$ and $C \leq_{\delta e} D_2$, hence $A = (B \oplus C) \leq_{\delta e} (D_1 \oplus D_2)$, by proposition (2.6). Thus M is δ-extending.

Proposition (4.12): Let $M = M_1 \oplus M_2$ be an R-module with M_1 being δ-extending and M_2 is semisimple. Suppose that for any submodule A of M, $A \cap M_1$ is a direct summand of A. Then M is δ-extending.

Proof: Let A be a submodule of M. Then it is easy to see that $A + M_1 = M_1 \oplus [(A + M_1) \cap M_2]$. Since M_2 is semisimple, then $(A + M_1) \cap M_2$ is a direct summand of M_2 and therefore $A + M_1$ is a direct summand of M. By our assumption $A = (A \cap M_1) \oplus A'$, for some submodule A' of A. Since M_1 is δ-extending, then there is a direct summand D of M_1 such that $A \cap M_1 \leq_{\delta e} D$. Hence $A = (A \cap M_1) \oplus A' \leq_{\delta e} D \oplus A'$. Since $D \oplus A' \leq_{\oplus} A + M_1 \leq_{\oplus} M$, then $D \oplus A'$ is a direct summand of M. Thus M is δ-extending.

Now, we need the following lemma before we give a new result.

Lemma (4.13): [2] Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$, then M_1 is M_2-injective if and only if for each submodule A of M such that $A \cap M_1 = 0$, there exists a submodule A' of M such that $M = M_1 \oplus A'$ and $A \leq A'$

Proposition (4.14): Let $M = M_1 \oplus M_2$ such that M_1 is δ-extending and M_2 is injective module. Then M is δ-extending module if and only if for every submodule A of M such that $A \cap M_2 \neq 0$, there is a direct summand D of M such that $A \leq_{\delta e} D$.

Proof: Suppose that for every submodule A of M such that $A \cap M_2 \neq 0$, there is a direct summand D of M such that $A \leq_{\delta e} D$. Let A be a submodule of M such that $A \cap M_2 = 0$. By lemma (4.13), there is a submodule M' of M containing A such that $M = M' \oplus M_2$. Since $M' \cong \frac{M}{M_2} \cong M_1$ is δ-extending, so there is a direct summand K of M', hence K is a direct summand of M, such that $A \leq_{\delta e} K$. Thus M is δ-extending. The converse is obvious.

Proposition (4.15): Let R be a ring, then the following statements are equivalent:

1- $\underset{I}{\oplus} R$ is δ-extending, for every index set I.

2- Every projective R-module is δ-extending.

Proof: (1)$\Rightarrow$(2) Let M be a projective R-module, then by [8, Corollary (4.4.4), p.89], there exists a free R-module F and an epimorphism $f : F \longrightarrow M$. Since F is free, then $F \cong \underset{I}{\oplus} R$, for some index set I. Now consider the following short exact sequence:

$$0 \longrightarrow Kerf \xrightarrow{i} \underset{I}{\oplus} R \xrightarrow{f} M \longrightarrow 0$$

Where i is the inclusion map. Since M is projective, then the sequence splits .Thus $\underset{I}{\oplus} R = Kerf \oplus M$. Since $\underset{I}{\oplus} R$ is δ-extending, then M is δ-extending, by proposition (4.6).

(2)$\Rightarrow$(1) Clear.

By the same argument, we can prove the following:

Proposition (4.16): Let R be a ring, then the following statements are equivalent:

1- $\underset{I}{\oplus} R$ is δ-extending, for every finite index set I.

2- Every finitely generated projective R-module is δ-extending.

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